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On Toeplitz and Hankel Operators with Oscillatory Symbols Containing Blaschke Products and Applications to the KdV Equation

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To Vladimir Rabinovich on the occasion of his 70th birthday

Abstract. We derive an asymptotic formula for the argument of a Blaschke product in the upper half-plane with purely imaginary zeros. We then use this formula to find conditions for the quotient of two such Blaschke products to be continuous on the real line. These results are applied to certain Hankel and Toeplitz operators arising in the Cauchy problem for the Korteweg-de Vries equation. Our main theorems include certain compactness conditions for Hankel operators and invertibility conditions for Toeplitz operators with oscillating symbols containing such quotients. As a by-product we obtain a well-posedness result for the Korteweg-de Vries equation.

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1. Introduction

The theory of Toeplitz operators on Hardy spaces with symbols having discontinuities of the second kind has been in focus of one of the authors (see, e.g., [2–5], [9], [14, 15] and the literature cited therein). The range of symbols under consideration is quite large and varies from discontinuities with rapidly oscillating

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behavior (oscillations of power, exponential and super-exponential types) to slowly oscillating (e.g., logarithmic). A large variety of generalizations of classical almost periodic symbols has been considered. For example, the so-called α-almost periodic and α-semi-almost periodic symbols have been studied in great detail [3] (see also [4, 6, 7] for matrix-valued analogs). We note that those generalizations are highly non trivial. The main problem is that, as opposed to traditional symbols (continuous or with at most jump discontinuities), the Toeplitz operators with those more general symbols need not be Fredholm, i.e., the kernels and co-kernels may be infinitely dimensional. This raises serious problems: finding criteria for one-sided and generalized invertibility, construction of bases in kernels and co-kernels, to name just two. Addressing these issues has required developing new methods (see monographs [7, 9]). We mention here only the method of the so-called "u-periodic factorizations of symbols". Further development of the theory of Toeplitz and Hankel operators with such symbols would therefore be interesting in its own right due to the nontriviality of its methods.

What is perhaps even more important is that, while the symbols above may look a bit artificially complicated, there are some problems of mathematical physics and partial differential equations where such symbols naturally appear. In particular, a symbol with a cubic oscillation of its argument is a main player in the study of the Cauchy problem for the Korteweg-de Vries (KdV) equation [18–20].

In the present paper we consider Toeplitz and Hankel operators with symbols which besides the cubic oscillation contain quotients of Blaschke products with zeros on the imaginary line. We obtain asymptotics of such Blaschke products and then use them to find some sufficient conditions for continuity of their quotients. We then apply these results to study one-sided invertibility of the corresponding Toeplitz operator and compactness of the Hankel operator. We emphasize that our interest to this circle of problems was stimulated by certain well-posedness issues more related to the Cauchy problem for the KdV equation.

Let us describe our main objects in detail. Consider the Blaschke product in the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$

$$B(z) = \prod_{n=1}^{\infty} \frac{z - i\kappa_n}{z + i\kappa_n}, \tag{1.1}$$

with purely imaginary simple zeros such that

$$\kappa_n > \kappa_{n+1} > 0 \quad \text{and} \quad \lim \kappa_n = 0, \quad n \to \infty. \tag{1.2}$$

Such Blaschke products are of course very specific but they do arise in the spectral and scattering theories for Schrödinger operators (see, e.g., [17]). Typically, $i\kappa_n = \sqrt{E_n}$ where E_n is the (negative) nth bound state of a Schrödinger operator.

It is well known (see [10, 16]) that $B(z)$ is convergent for any $z \in \overline{\mathbb{C}}_+ \setminus \{0\}$ if and only if

$$\sum_{n=1}^{\infty} \kappa_n < \infty. \tag{1.3}$$

Of course $B(z)$ is analytic in any neighborhood of a real point x not containing 0. We are specifically concerned with the asymptotic behavior of suitably defined $\arg B(x)$ as $x \to 0$ and conditions providing continuity at $x = 0$ of

$$Q\,(x) := \frac{B_1(x)}{B_2(x)}, \tag{1.4}$$

where $B_{1,2}(x)$ are two Blaschke products given by (1.1). The results obtained are then applied to the study of Toeplitz and Hankel operators with symbols

$$a(x) = D\,(x)\,Q\,(x)\,, \tag{1.5}$$

where either $D \in H^\infty_+ + C(\dot{\mathbb{R}})$ or $D \in \overline{H^\infty_+} + C(\dot{\mathbb{R}})$. We recall that H^∞_+ stands for the Hardy space of analytic and bounded functions in the upper half-plane $\mathbb{C}_+$ and $C(\dot{\mathbb{R}})$ is the space of functions continuous on the one point compactification of the real axis $\mathbb{R}$. The class of operators with such symbols is quite broad (see (4.10) below) and includes the Hankel and Toeplitz operators arising in the initial value problem for the Korteweg-de Vries (KdV) equation. We use our results on Hankel and Toeplitz operators to describe some subtle properties of solutions to the KdV equation which we believe cannot be achieved by usual PDE methods. We emphasize that although Hankel operators naturally appear in many other (if not every) so-called completely integrable systems of nonlinear PDEs (see, e.g., [1]), not much from the theory of Hankel and Toeplitz operators have been actually used there so far. We believe that the language of Hankel and Toeplitz operators is quite adequate in the setting of completely integrable systems and the theory of those operators will find more useful applications in integrable systems.

This work is organized as follows. In Section 2 we derive an asymptotic formula for the argument of the Blaschke product (1.1). The sufficient conditions of continuity of the function $Q\,(x)$ (1.4) at the point $x = 0$ are given in Section 3. Applications to the theory of Toeplitz and Hankel operators with oscillating symbol are considered in Section 4. In Section 5 we apply our results to the theory of the KdV equation.

2. Argument of Blaschke products

Let $B(x)$ be of the form (1.1)–(1.3) and let the branch of $\arctan x$ be chosen such that $\arctan x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for $x \in \mathbb{R}$. We define the Blaschke product (1.1) under conditions (1.2)–(1.3) such that the function

$$B : \dot{\mathbb{R}} \setminus \{0\} \to \mathbb{C}, \quad x \mapsto B(x)$$

is continuous, $B(\infty) = 1$ and $|B(x)| = 1$ for all $x \in \dot{\mathbb{R}} \setminus \{0\}$. So we can choose a branch of $\arg B$ such that $\arg B(x)$ is continuous on $\dot{\mathbb{R}} \setminus \{0\}$ and $\arg B(\infty) = 0$. The following statement is elementary.

Theorem 2.1. *The function* $\arg B(x)$ *is continuously increasing on* $\mathbb{R}\setminus\{0\}$,

$$\arg B(x) = -2\sum_{n=1}^{\infty}\arctan\frac{\kappa_n}{x}, \quad x\neq 0 \tag{2.1}$$

and

$$\arg B(x) = -\arg B(-x), \quad x\in\mathbb{R}, \tag{2.2}$$

$$\lim_{x\to\pm 0}\arg B(x) = \mp\infty. \tag{2.3}$$

Proof. Since for $\pm x > 0$

$$\arg\frac{x-i\kappa_n}{x+i\kappa_n} = 2\arg(x-i\kappa_n) = -2\arctan\frac{\kappa_n}{x},$$

we immediately have (2.1) and (2.2). The series is convergent due to the Blaschke condition (1.3). It follows from

$$\sum_{n=1}^{\infty}\left|\arctan\frac{\kappa_n}{x}\right| > \sum_{\kappa_n>|x|}\left|\arctan\frac{\kappa_n}{x}\right| > \sum_{\kappa_n>|x|}\frac{\pi}{4}$$

that (2.3) holds. The function $-2\arctan\frac{\kappa_n}{x}$ is clearly increasing on $\mathbb{R}_+ := (0,+\infty)$ and $\mathbb{R}_- := (-\infty,0)$ respectively and so is $\arg B(x)$. □

With each Blaschke product B of the type (1.1) we associate a function f constructed as follows. Fix a point $\kappa_{1/2} > \kappa_1$ and define f

$$f : [1/2,\infty) \to (0,\kappa_{1/2}], \quad x\mapsto f(x)$$

as a continuous strictly decreasing function that interpolates the points $\{(1/2,\kappa_{1/2}),(1,\kappa_1),(2,\kappa_{2,}),\dots\}$. That is

$$f(1/2) = \kappa_{1/2}, \quad f(n) = \kappa_n, \quad n = 1,2,\dots. \tag{2.4}$$

We call such f a function associated with a Blaschke product B of the type (1.1). Similarly, given a continuous suitably decreasing function f, we call a Blaschke product B of the type (1.1) satisfying (2.4) a Blaschke product associated with f.

Hypothesis 2.2. Let $B(z)$ be a Blaschke product of the form (1.1)–(1.3) such that:

i) its zeros $\{i\kappa_n\}$ satisfy

$$\lim_{n\to\infty}\frac{\kappa_n-\kappa_{n+1}}{\kappa_n} = 0; \tag{2.5}$$

ii) there exists a continuously differentiable associated function $f(x)$ such that

$$\lim_{n\to\infty}\sup_{-1/2\le s\le 1/2}\frac{|f(n+s)-f(n)+s(\kappa_n-\kappa_{n+1})|}{\Delta_n} = 0. \tag{2.6}$$

Theorem 2.3. *Under Hypothesis* 2.2

$$\arg B(x) = -2\int_{1/2}^{\infty}\arctan\frac{f(u)}{x}du + o(1), \quad x\to 0. \tag{2.7}$$

Proof. Since the function $\arg B(x)$ is odd, it is enough to consider the case $x > 0$. Let $\epsilon_n(x)$ be the difference

$$\epsilon_n(x) := \arctan \frac{\kappa_n}{x} - \int_{n-1/2}^{n+1/2} \arctan \frac{f(u)}{x} du.$$

It is easy to see that

$$\begin{aligned}
\epsilon_n(x) &= \int_0^{1/2} \left[\left(\arctan \frac{f(n)}{x} - \arctan \frac{f(n+s)}{x} \right) \right. \\
&\quad \left. + \left(\arctan \frac{f(n)}{x} - \arctan \frac{f(n-s)}{x} \right) \right] ds \\
&= \int_0^{1/2} \left[\arctan \frac{x(f(n) - f(n+s))}{x^2 + f(n)f(n+s)} + \arctan \frac{x(f(n) - f(n-s))}{x^2 + f(n)f(n-s)} \right] ds \\
&= \int_0^{1/2} \left[\arctan \delta_n(s,x) + \arctan \delta_n(-s,x) \right] ds,
\end{aligned}$$

where

$$\delta_n(s,x) := \frac{x(f(n) - f(n+s))}{x^2 + f(n)f(n+s)}, \quad s \in [-1/2, 1/2].$$

By a direct computation

$$\epsilon_n(x) = \int_0^{1/2} \arctan \frac{\delta_n(s,x) + \delta_n(-s,x)}{1 - \delta_n(s,x)\delta_n(-s,x)} ds.$$

Since $-\delta_n(s,x)\delta_n(-s,x) > 0$, we have

$$|\epsilon_n(x)| \le \int_0^{1/2} |\delta_n(s,x) + \delta_n(-s,x)| ds.$$

For $s \in [-1/2, 1/2]$, we set

$$\begin{aligned}
\Delta_n &:= \kappa_n - \kappa_{n+1}, \\
\Delta_n^{(1)}(s) &:= f(n+s) - f(n), \\
\Delta_n^{(2)}(s) &:= \Delta_n^{(1)}(s) - s\Delta_n^{(1)}(1).
\end{aligned}$$

Note that $\Delta_n^{(1)}(1) = -\Delta_n$, $\Delta_n^{(2)}(1) = 0$ and

$$\Delta_n^{(2)}(s) = f(n+s) - f(n) + s(\kappa_n - \kappa_{n+1}). \tag{2.8}$$

Let us evaluate now

$$\begin{aligned}\delta_n(s,x)+\delta_n(-s,x) =& -x\left\{\frac{\Delta_n^{(2)}(s)}{x^2+f(n)f(n+s)}+\frac{\Delta_n^{(2)}(-s)}{x^2+f(n)f(n-s)}\right.\\ &\left.+\frac{s\Delta_n(f(n+s)-f(n-s))f(n)}{(x^2+f(n)f(n+s))(x^2+f(n)f(n-s))}\right\}\\ =& -x\left\{\frac{\Delta_n^{(2)}(s)}{x^2+f(n)f(n+s)}+\frac{\Delta_n^{(2)}(-s)}{x^2+f(n)f(n-s)}\right.\\ &\left.+\frac{s\Delta_n(\Delta_n^{(1)}(s)-\Delta_n^{(1)}(-s))f(n)}{(x^2+f(n)f(n+s))(x^2+f(n)f(n-s))}\right\}.\end{aligned}$$

Consider two cases: $f(n)\ge x$ and $f(n)<x$. If $f(n)\ge x$, then

$$\begin{aligned}|\delta_n(s,x)+\delta_n(-s,x)| \le& \frac{x\left|\Delta_n^{(2)}(s)\right|}{f(n)f(n+s)}+\frac{x\left|\Delta_n^{(2)}(-s)\right|}{f(n)f(n-s)}\\ &+\frac{x\,|s|\,\Delta_n(|\Delta_n^{(1)}(s)|+|\Delta_n^{(1)}(-s)|)}{f(n)f(n+s)f(n-s)}.\end{aligned}$$

Since for $s\in[-1/2,1/2]$

$$f(n+s)>f(n+1)$$

and

$$|\Delta_n^{(1)}(|s|)|<\Delta_n,\quad \Delta_n^{(1)}(-|s|)<\Delta_{n-1},$$

one has

$$|\delta_n(s,x)+\delta_n(-s,x)|<\left\{\frac{|\Delta_n^{(2)}(s)|+|\Delta_n^{(2)}(-s)|}{\Delta_n}+\frac{\Delta_n+\Delta_{n-1}}{2f(n)}\right\}\frac{x\Delta_n}{f(n)f(n+1)}.$$

Recalling (2.8), it follows from (2.6) and (2.5) that[1]

$$|\delta_n(s,x)+\delta_n(-s,x)|\lesssim \alpha_n\frac{x\Delta_n}{f(n)f(n+1)},$$

where α_n is independent of s, and $\lim_{n\to\infty}\alpha_n=0$.

If $f(n)<x$ then

$$\begin{aligned}&|\delta_n(s,x)+\delta_n(-s,x)|\\ &\lesssim \frac{|\Delta_n^{(2)}(s)|}{x}+\frac{|\Delta_n^{(2)}(-s)|}{x}+\frac{s\,\Delta_n\left\{|\Delta_n^{(1)}(s)|+|\Delta_n^{(1)}(-s)|\right\}f(n)}{x^3}.\end{aligned}$$

It follows from (2.6) that

$$\sup_{-1/2\le s\le 1/2}\frac{|\Delta_n^{(1)}(s)|}{\Delta_n}=\sup_{-1/2\le s\le 1/2}\frac{|\Delta_n^{(2)}(s)-s\Delta_n|}{\Delta_n}$$

[1]We write $f\lesssim g$ if $f\le Cg$ with some $C>0$ independent of arguments of the functions f and g.

is bounded with respect to n and hence

$$\left\{|\Delta_n^{(1)}(s)| + |\Delta_n^{(1)}(-s)|\right\} \frac{sf(n)\Delta_n}{x^3} \lesssim \frac{|\Delta_n^{(1)}(s)| + |\Delta_n^{(1)}(-s)|}{\Delta_n} \left(\frac{\Delta_n}{x}\right)^2 \lesssim \left(\frac{\Delta_n}{x}\right)^2.$$

Therefore

$$|\delta_n(s,x) + \delta_n(-s,x)| \lesssim \frac{|\Delta_n^{(2)}(s)| + |\Delta_n^{(2)}(-s)|}{\Delta_n} \cdot \frac{\Delta_n}{x} + \left(\frac{\Delta_n}{x}\right)^2 \lesssim \beta_n \frac{\Delta_n}{x},$$

where β_n is independent of s and $\lim\limits_{n\to\infty} \beta_n = 0$, and we finally have

$$|\epsilon_n(x)| \lesssim \begin{cases} \alpha_n \dfrac{x\Delta_n}{f(n)f(n+1)}, & f(n) \geq x \\ \beta_n \dfrac{\Delta_n}{x}, & f(n) < x. \end{cases}$$

We now estimate the remainder $\delta(x) := \arg B(x) + 2 \int\limits_{1/2}^{\infty} \arctan \frac{f(u)}{x}\, du$ for $x > 0$ small enough. We have

$$\begin{aligned} |\delta(x)| &\lesssim \sum_{n=1}^{\infty} |\epsilon_n(x)| \leq \left\{ \sum_{f(n)\geq\sqrt{x}} + \sum_{x\leq f(n)<\sqrt{x}} + \sum_{f(n)<x} \right\} |\epsilon_n(x)| \\ &\lesssim \sum_{f(n)\geq\sqrt{x}} x \left\{\frac{1}{f(n+1)} - \frac{1}{f(n)}\right\} + \sum_{x\leq f(n)<\sqrt{x}} \sigma_1(x) x \left\{\frac{1}{f(n+1)} - \frac{1}{f(n)}\right\} \\ &\quad + \sum_{f(n)<x} \frac{\sigma_2(x)}{x} \{f(n) - f(n+1)\}, \end{aligned}$$

where

$$\sigma_1(x) := \sup\left\{\alpha_n : x \leq f(n) < \sqrt{x}\right\}, \quad \sigma_2(x) := \sup\left\{\beta_n : f(n) < x\right\}.$$

Thus, we have

$$\begin{aligned} |\delta(x)| &\lesssim x\left(\frac{1}{f(n_1+1)} - \frac{1}{f(1)}\right) \\ &\quad + x\sigma_1(x)\left(\frac{1}{f(n_2+1)} - \frac{1}{f(n_1+1)}\right) + \frac{\sigma_2(x)}{x} f(n_2+1), \end{aligned}$$

where

$$n_1 = \max\left\{n : f(n) \geq \sqrt{x}\right\}, \quad n_2 = \max\left\{n : f(n) \geq x\right\}.$$

It is easy to see that

$$\lim_{x\to 0} \sigma_1(x) = \lim_{x\to 0} \sigma_2(x) = \lim_{x\to 0} \frac{x}{f(n_1+1)} = 0$$

and

$$\lim_{x\to 0} \frac{x}{f(n_2+1)} = 1.$$

Hence $\lim_{x\to 0} \delta(x) = 0$, and the theorem is proved. □

Theorem 2.4. *Under Hypothesis* 2.2

$$\arg B(x) = \frac{\pi}{2} \operatorname{sgn}(x) - 2x \int_0^1 \frac{f^{-1}(v)}{x^2+v^2} dv + o(1), \quad x \to 0, \tag{2.9}$$

where $f^{-1} : (0, \kappa_{1/2}] \to [1/2, \infty)$ *is the inverse function of* f.

Proof. As above we may assume $x > 0$. By Theorem 2.3 for $x \to 0$ one has

$$\begin{aligned}\arg B(x) &= -2 \int_{1/2}^{\infty} \arctan \frac{f(u)}{x} du + o(1) \\ &= -2u \arctan \frac{f(u)}{x}\bigg|_{1/2}^{\infty} + 2x \int_{1/2}^{\infty} \frac{uf'(u)du}{x^2+f^2(u)} + o(1) \\ &= \arctan \frac{\kappa_{1/2}}{x} + 2x \int_{1/2}^{\infty} \frac{ud\,f(u)}{x^2+f^2(u)} + o(1).\end{aligned}$$

Here we have used

$$\lim_{u\to\infty} u \arctan\left(\frac{f(u)}{x}\right) = 0, \tag{2.10}$$

that can be easily shown by contradiction. If (2.10) does not hold, then there exists a sequence of positive numbers $\{u_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} u_n = \infty \tag{2.11}$$

and

$$u_n f(u_n) \geq \delta > 0. \tag{2.12}$$

By the definition of the function $f(u)$, the integral

$$I := \int_{u_1}^{\infty} f(u)\, du$$

is finite. Since $f(u)$ is a continuous strictly decreasing function it follows from (2.12) that

$$
\begin{aligned}
I &= \sum_{n=1}^{\infty} \int_{u_n}^{u_{n+1}} f(u)\, du \geq \delta \sum_{n=1}^{\infty} \frac{u_{n+1}-u_n}{u_{n+1}} \\
&\gtrsim \delta \sum_{n=1}^{\infty} \left| \ln \left(1 - \frac{u_{n+1}-u_n}{u_{n+1}} \right) \right| \\
&= \delta \ \ln \prod_{n=1}^{\infty} \frac{u_{n+1}}{u_n}.
\end{aligned}
$$

Thus the last infinite product is convergent, and hence

$$
\lim_{N\to\infty} \prod_{n=1}^{N} \frac{u_{n+1}}{u_n} = \lim_{N\to\infty} \frac{u_{N+1}}{u_1}
$$

must be finite, which contradicts (2.11). Thus (2.10) holds true.

Changing the variable $v = f(u)$ we continue

$$
\begin{aligned}
\arg B(x) &= \arctan \frac{\kappa_{1/2}}{x} - 2x \int_0^{\kappa_{1/2}} \frac{f^{-1}(v)dv}{x^2+v^2} + o(1) \\
&= -2x \int_0^1 \frac{f^{-1}(v)dv}{x^2+v^2} + 2x \int_{\kappa_{1/2}}^1 \frac{f^{-1}(v)dv}{x^2+v^2} + \arctan\left(\frac{\kappa_{1/2}}{x}\right) + o(1).
\end{aligned}
$$

Due to $\lim\limits_{x\to 0} \arctan \dfrac{\kappa_{1/2}}{x} = \dfrac{\pi}{2}$ and $\sup\left\{f^{-1}(v) : v \in [\kappa_{1/2}, 1]\right\} < \infty$ we have

$$
2x \left| \int_{\kappa_{1/2}}^1 \frac{f^{-1}(v)dv}{x^2+v^2} \right| \lesssim x \left| \int_{\kappa_{1/2}}^1 \frac{dv}{x^2+v^2} \right| = \left| \arctan \frac{1}{x} - \arctan \frac{\kappa_{1/2}}{x} \right|.
$$

That is

$$
\lim_{x\to 0} 2x \int_{\kappa_{1/2}}^1 \frac{f^{-1}(v)}{x^2+v^2} dv = 0
$$

and (2.9) follows. □

In place of Hypothesis 2.2 we can state somewhat stronger.

Hypothesis 2.5. Let $B(z)$ be a Blaschke product of the form (1.1)–(1.3) that has an associated function $f(x)$ such that $|f'(x)|$ is decreasing and

$$
\lim_{n\to\infty} \frac{f^{(l)}(n) - f^{(l)}(n+1)}{f^{(l)}(n)} = 0, \quad l = 0, 1. \tag{2.13}
$$

Hypothesis 2.5 implies Hypothesis 2.2. For $l = 0$ condition (2.13) is the same as i) of Hypothesis 2.2 and one only needs to show that (2.13) for $l = 1$ implies (2.6). Indeed, it is easy to see that

$$\Delta_n^{(2)}(s) = f'(n_0)s - f'(n_1)s = (f'(n_0) - f'(n_1))s,$$

with some n_0 and n_1 from $[n, n+1]$ and $[n, n+s]$ respectively. One has

$$\begin{aligned}\left|\frac{\Delta_n^{(2)}(s)}{\Delta_n}\right| &= \left|\frac{f'(n_0) - f'(n_1)}{f'(n_1)}\right| s \le \left|\frac{f'(n-1) - f'(n+1)}{f'(n+1)}\right| \\ &\le \left|\frac{f'(n-1) - f'(n)}{f'(n-1)}\right| \left|\frac{f'(n-1)}{f'(n+1)}\right| + \left|\frac{f'(n) - f'(n+1)}{f'(n)}\right| \left|\frac{f'(n)}{f'(n+1)}\right|.\end{aligned}$$

Since $f'(x)$ satisfies (2.13) we immediately conclude that (2.6) holds.

Hypothesis 2.5 is of course much easier to verify and a simple example is in order.

Example. Take

$$f(x) = x^{-\alpha} \ln^{\beta} x, \tag{2.14}$$

where β is any real number if $\alpha > 1$ and $\beta < -1$ if $\alpha = 1$. It follows from

$$f'(x) = -\frac{f(x)}{x}\left(\alpha - \frac{\beta}{\ln x}\right),$$

that $f(x)$ is continuous and decreasing for x large enough. Moreover for some $n_0 \in [n, n+1]$

$$\left|\frac{f(n) - f(n+1)}{f(n)}\right| = \left|\frac{f'(n_0)}{f(n)}\right| \to 0$$

and condition (2.13) for $l = 0$ holds. Similarly using the second derivative of $f(x)$ one verifies that (2.13) holds also for $l = 1$. Therefore any Blaschke product associated with the function (2.14) satisfies Hypothesis 2.5.

Let us demonstrate now how Theorem 2.4 applies in the case of (2.14) with $\alpha > 1$ and $\beta = 0$.

Example. Take

$$f(x) = x^{-\alpha}, \quad \alpha > 1,$$

then $f^{-1}(v) = v^{-1/\alpha}$ and by (2.9) for $x > 0$ we have

$$\arg B(x) = \frac{\pi}{2} - 2x\int_0^1 \frac{v^{-1/\alpha}}{x^2 + v^2}\,dv + o(1) = \frac{\pi}{2} - 2x^{-1/\alpha}\int_0^{1/x} \frac{u^{-1/\alpha}\,du}{1+u^2} + o(1), \quad x \to 0.$$

Due to the symmetry of $\arg B(x)$ we finally obtain

$$\arg B(x) = \left(\frac{\pi}{2} - c|x|^{-\frac{1}{\alpha}}\right)\operatorname{sgn}(x) + o(1), \quad x \to 0,$$

where $c := 2\int_0^{\infty} \frac{u^{-1/\alpha}}{1+u^2}\,du.$

3. Quotient of Blaschke products

In this section we consider the continuity of the quotient $Q(x) = B_1(x)/B_2(x)$ of Blaschke products $B_{1,2}(x)$ subject to Hypothesis 2.2. More specifically, we study conditions on $B_{1,2}$ providing continuity of $\arg Q(x)$ as $x = 0$. The following statement is the main result of this section.

Theorem 3.1. *Let $B_{1,2}$ be subject to Hypothesis 2.2 and $f_{1,2}$ be associated with $B_{1,2}$ functions. Set*

$$r(v) := f_1^{-1}(v) - f_2^{-1}(v).$$

The function $\arg Q(x)$ is continuos at $x = 0$ if at least one of the following holds:

i) $\lim_{v\to 0} r(v)$ *exists;*

ii) *there exists $c_1 \in \mathbb{C}$, such that $\int_0^v r(s)ds - c_1 v = o(v)$.*

Proof. Let $x > 0$ and assume condition i). Then by Theorem 2.4 we have

$$\arg Q(x) = -2x \int_0^1 \frac{r(v)}{x^2+v^2} dv + o(1), \quad x \to 0.$$

Introduce a function $O_1(v) := r(v) - r_0$ where $r_0 = \lim_{v\to 0} r(v)$. Then, by i), we get

$$\begin{aligned}\arg Q(x) &= -2r_0 \int_0^1 \frac{x\,dv}{x^2+v^2} - 2x \int_0^1 \frac{O_1(v)dv}{x^2+v^2} + o(1) \\ &= -2r_0 \arctan\frac{1}{x} - 2x \int_0^1 \frac{O_1(v)dv}{x^2+v^2} + o(1).\end{aligned}$$

Estimate the integral in the last equation:

$$\begin{aligned}\left| x \int_0^1 \frac{O_1(v)}{x^2+v^2} dv \right| &\lesssim \alpha(x) \int_0^{\sqrt{x}} \frac{x\,dv}{x^2+v^2} + \int_{\sqrt{x}}^1 \frac{x\,dv}{x^2+v^2} \\ &= \alpha(x) \arctan x^{-1/2} + (\arctan x^{-1} - \arctan x^{-1/2}),\end{aligned}$$

where $\alpha(x) = \sup\{|O_1(v)| : v \in [0, \sqrt{x}]\}$. Thus we obtain

$$\lim_{x\to 0} x \int_0^1 \frac{O_1(v)dv}{x^2+v^2} = 0$$

and the theorem is proven under condition i).

Assume that ii) is satisfied. Denoting $F(v) := \int_0^v r(s)ds$ we have

$$\begin{aligned}\arg Q(x) &= -2x \int_0^1 \frac{d\,F(v)}{x^2+v^2} + o(1) \\ &= -2x\,\frac{F(v)}{x^2+v^2}\bigg|_{\nu=0}^1 - 4x\int_0^1 \frac{v\,F(v)dv}{(x^2+v^2)^2} + o(1) \\ &= -\frac{2x\,F(1)}{x^2+1} - 4xr_1\int_0^1 \frac{v^2\,dv}{(x^2+v^2)^2} - 4x\int_0^1 \frac{v\,O_2(v)\,dv}{(x^2+v^2)^2} + o(1),\end{aligned}$$

where $O_2(v) := F(v) - r_1 v$ and $r_1 = \lim_{v\to 0}\int_0^v r(s)ds$. Consider the last integrals:

$$x\int_0^1 \frac{v^2\,dv}{(x^2+v^2)^2} = \int_0^{1/x} \frac{s^2 ds}{(1+s^2)^2} = \int_0^{\infty} \frac{s^2 ds}{(1+s^2)^2} + o(1), \tag{3.1}$$

and

$$\begin{aligned}\left| x\int_0^1 \frac{vO_2(v)\,dv}{(x^2+v^2)^2}\right| &\lesssim |x|\left\{\beta(x)\int_0^{\sqrt{x}} \frac{v^2 dv}{(x^2+v^2)^2} + \int_{\sqrt{x}}^1 \frac{v^2 dv}{(x^2+v^2)^2}\right\} \\ &\lesssim \beta(x)\int_0^{1/\sqrt{x}} \frac{s^2\,ds}{(1+s^2)^2} + \int_{1/\sqrt{x}}^{1/x} \frac{s^2\,ds}{(1+s^2)^2},\end{aligned}$$

where $\beta(x) = \sup\left\{\frac{|O_2(v)|}{v} : v \in (0,\sqrt{x})\right\}$. Hence we have

$$\lim_{x\to 0} x\int_0^1 \frac{vO_2(v)dv}{(x^2+v)^2} = 0. \tag{3.2}$$

Taking into account (3.1) and (3.2), the assertion is proven under condition ii). □

Example. Consider a Blaschke product B_1 satisfying Hypothesis 2.2. Let f_1 be a function associated with B_1 set $f_1 =: f$. Next let $\alpha(x)$ be a continuous function such that $\lim_{x\to\infty} \alpha(x) = 0$ and $\beta(x) := x + \alpha(x)$ is monotonically increasing. Define $f_2(x) := f(\beta(x)+c)$, where c is a real constant. Then $f_2^{-1}(v) = f^{-1}(v) - c - \alpha_1(v)$, where $\lim_{v\to 0}\alpha_1(v) = 0$, and hence

$$r(v) = f^{-1}(v) - f_2^{-1}(v) = c + \alpha_1(v) \to c, \quad v \to 0.$$

By Theorem 3.1 $\lim_{x\to 0} \arg B_1(x)/B_2(x)$ exists.

Example. Consider a more delicate case of Theorem 3.1 (part ii)). Let $\beta(x) = x + p(x)$, where $p(x)$ is a periodic continuous function such that $\beta(x)$ is increasing on $[1/2, \infty)$. Then the inverse function has the form

$$\beta^{-1}(v) = v - q(v),$$

where $q(v)$ is a periodic continuous function. As in the previous example, let us construct two Blaschke products B_1 and B_2 with the associated functions f_1 and f_2. Let $f_1(x) = f(x)$, where f satisfies Hypothesis 2.2 and such that $f'(x)$ is monotonic function and $f''(x)$ is bounded. Set $f_2(x) = f(\beta(x))$. Then $f_2^{-1}(v) = \beta^{-1}(f^{-1}(v)) = f^{-1}(v) - q(f^{-1}(v))$ and $r(v) = q(f^{-1}(v))$. Consider

$$F(v) = \int_0^v q(f^{-1}(u))du = \int_0^v q_0 du + \int_0^v q_1(f^{-1}(u))du,$$

where q_0 is the zero Fourier coefficient of $q(v)$ and $q_1(v) = q(v) - q_0$. Then

$$F(v) = q_0 v - \int_{f^{-1}(v)}^{\infty} q_0(u) f'(u)du.$$

Let $F_1(v)$ be an antiderivative of $q_1(v)$. That is $F_1'(v) = q_1(v)$. Then

$$\begin{aligned} F(v) &= q_0 v - F_1(u)f'(u)\Big|_{f^{-1}(v)}^{\infty} + \int_{f^{-1}(v)}^{\infty} F_1(u)f''(u)du \\ &= q_0 v + \frac{F_1(f^{-1}(v))}{(f^{-1}(v))'} + \int_{f^{-1}(v)}^{\infty} F_1(u)f''(u)du. \end{aligned}$$

Since $f'(f^{-1}(v)) = \dfrac{1}{(f^{-1}(v))'}$ and $F_1(f^{-1}(v))$ is bounded, one has

$$|F(v) - q_0 v| \lesssim \left(\left| \frac{v}{v\,(f^{-1}(v))'} \right| + \left| \int_{f^{-1}(v)}^{\infty} f''(u)du \right| \right).$$

Let $v = f(x)$, then

$$|F(v) - q_0 v| \lesssim \left(v \left| \frac{f'(x)}{f(x)} \right| + \left| f'(x) \right| \right) \lesssim v \left| \frac{f'(x)}{f(x)} \right|.$$

Condition (2.5) and the monotonicity of the function $f'(x)$ imply the condition ii) of Theorem 3.1 and $\arg Q(x)$ approaches a finite limit as $x \to 0$.

Theorem 3.1 has a consequence which will be crucial in the last section.

Corollary 3.2. *Let*

$$B_1(z) = \prod_{n=1}^{\infty} \frac{z - i\nu_n}{z + i\nu_n} \quad \textit{and} \quad B_2(z) = \prod_{n=1}^{\infty} \frac{z - i\kappa_n}{z + i\kappa_n}$$

be two Blaschke products subject to Hypothesis 2.2 *with interlacing zeros* (*i.e.,* $\kappa_n > \nu_n > \kappa_{n+1}$ *for any* $n \in \mathbb{N}$) *and associated functions* f_1 *and* f_2. *If there exists a real continuously differentiable function* f *such that*

$$f(2x - 1) = f_1(x), \qquad f(2x) = f_2(x),$$

and

$$f(n) = \begin{cases} \kappa^{(1)}_{\frac{n+1}{2}}, & n \text{ is odd} \\ \nu^{(2)}_{\frac{n}{2}}, & n \text{ is even} \end{cases},$$

then $\arg B_1(x)/B_2(x)$ *is continuous on the real line.*

Proof. Indeed

$$f_1^{-1}(v) - f_2^{-1}(v) = \frac{f^{-1}(v) + 1}{2} - \frac{f^{-1}(v)}{2} = \frac{1}{2}$$

and Theorem 3.1 now applies. □

4. Toeplitz and Hankel operators

Let $H^2_\pm$ be the usual Hardy space of the upper and lower half-planes. By the Paley-Wiener theorem

$$H^2_\pm = \left\{ f : f(x) = \int_0^\infty g(t) e^{\pm itx} dt, \quad x \in \mathbb{R}, \quad g \in L_2(\mathbb{R}_+) \right\}.$$

Let $P^\pm$ be the orthogonal projector of $L_2(\mathbb{R})$ onto $H^2_\pm(\mathbb{R})$. The operators $P^\pm$ can be written as follows

$$P^\pm = \frac{1}{2}(I \pm S),$$

where

$$(Sf)(x) := \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\tau)}{\tau - x} d\tau \ : \ L_2(\mathbb{R}) \to L_2(\mathbb{R}),$$

with the singular integral understood in the sense of the Cauchy principal value.

The Toeplitz operator with a symbol[2] $a(x) \in L_\infty(\mathbb{R})$ is defined by

$$T(a)f := P^+ af : H^2_+ \to H^2_+. \tag{4.1}$$

Let

$$(Jf)(x) = f(-x) : L_2(\mathbb{R}) \to L_2(\mathbb{R}) \tag{4.2}$$

[2] $L_\infty(\mathbb{R})$ is the usual space of functions essentially bounded on $\mathbb{R}$.

be the reflection operator. The Hankel operator with the symbol a is given by the formula

$$(\mathbb{H}(a)f)(x) := (JP^{-}af)(x) : H_+^2 \to H_+^2. \quad (4.3)$$

The theory of Toeplitz and Hankel operators is given, e.g., in [8, 12, 13]. Recall a few more definitions.

Definition 4.1. A bounded linear operator A acting in a Banach space B is called left (right) invertible if there exists a bounded in B operator A_ℓ^{-1} (A_r^{-1}) such that

$$A_\ell^{-1}A = I \quad (AA_r^{-1} = I),$$

where I is the identity operator on B.

Definition 4.2. A bounded linear operator A is called Fredholm if

$$\operatorname{Im} A = \overline{\operatorname{Im} A}, \quad \dim\ker A < \infty, \text{ and } \dim(B/\operatorname{Im} A) < \infty.$$

The number

$$\operatorname{ind}(A) := \dim\ker A - \dim(B/\operatorname{Im} A)$$

is called the index of the operator A.

Define the distance between a function $a \in L_\infty(\mathbb{R})$ and a subset $M \subset L_\infty(\mathbb{R})$ as

$$\operatorname{dist}(a, M) := \inf_{m \in M} \operatorname*{ess\,sup}_{x \in \mathbb{R}} |a(x) - m(x)|.$$

Introduce

$$H_+^\infty + C(\dot{\mathbb{R}}) := \{f + g : f \in H_+^\infty, \quad g \in C(\dot{\mathbb{R}})\}.$$

This space is a closed subspace (and even a closed subalgebra) of $L_\infty(\mathbb{R})$ and is particularly important in the theory of Toeplitz and Hankel operators. We will use the following well-known results.

Theorem 4.3 (Widom-Devinatz, see [8], p. 59). *Let $a(x)$ be a unimodular function (that is $|a(x)| = 1$ for almost all $x \in \mathbb{R}$). Then the operator $T(a)$ defined by* (4.1)

i) *is left invertible if and only if* $\operatorname{dist}(a, H_+^\infty) < 1$*;*
ii) *is right invertible if and only if* $\operatorname{dist}(a, \overline{H_+^\infty}) < 1$;
iii) *is invertible if and only if* $\operatorname{dist}(a, GH_+^\infty) < 1$*,*
where $GH_+^\infty \subset H_+^\infty$ is the set of all invertible in H_+^∞ elements.

Theorem 4.4 (I. Gohberg, see [8], [12, 13]). *Let $a(x) \in C(\dot{\mathbb{R}})$, then the operator $T(a)$ is Fredholm if and only if $a(x) \neq 0$ for all $x \in \mathbb{R}$. Moreover*

$$\operatorname{ind}(T(a)) = -\operatorname{wind} a,$$

where $\operatorname{wind} a$ *is the number of rotations which the point $z = a(x)$ makes around the origin in the complex plane (when x moves along $\mathbb{R}$ from $-\infty$ to $+\infty$).*

Theorem 4.5 ([8], Ch.2, [9], Theorem 2.7). *Let $a(x) \in L_\infty(\mathbb{R})$ and* $\operatorname{ess\,inf}\{|a(x)| : x \in \mathbb{R}\} > 0$*. Then*

i) *if $a(x) \in H_+^\infty + C(\dot{\mathbb{R}})$ but $1/a(x) \notin H_+^\infty + C(\dot{\mathbb{R}})$ then $T(a)$ is left invertible;*

ii) *if* $a(x) \in \overline{H_+^\infty} + C(\dot{\mathbb{R}})$ *but* $1/a(x) \notin \overline{H_+^\infty} + C(\dot{\mathbb{R}})$ *then* $T(a)$ *is right invertible;*
iii) *if* $a(x) \in (H_+^\infty + C(\dot{\mathbb{R}})) \cap (\overline{H_+^\infty} + C(\dot{\mathbb{R}}))$ *then* $T(a)$ *is Fredholm.*

Theorem 4.6 ([8], [12, 13]). *Let* $a(x) \in L_\infty(\mathbb{R})$*. Then*

$$\| \mathbb{H}(a) \| \leq \| a \|_{L_\infty}$$

and the Hankel operator (4.3) *is compact if and only if*

$$a(x) \in H_+^\infty + C(\dot{\mathbb{R}}).$$

Note that if $h(x) \in H_+^\infty$ then $\mathbb{H}(h) = 0$ and consequently

$$\mathbb{H}(a) = \mathbb{H}(a - h). \tag{4.4}$$

Consider now

$$a(x) = D\,(x)\, B_1(x)/B_2(x), \tag{4.5}$$

where $D\,(x)$ is a unimodular function and $B_{1,2}(x)$ are Blaschke products satisfying the conditions of Theorem 3.1. Then Theorems 3.1, 4.4 and 4.5 imply the following result.

Theorem 4.7. *Let a have the form* (4.5)*.*

i) *If* $D \in H_+^\infty + C(\dot{\mathbb{R}})$ $(D \in \overline{H_+^\infty} + C(\dot{\mathbb{R}}))$ *and* $1/D \notin H_+^\infty + C(\dot{\mathbb{R}})$ $(1/D \notin \overline{H_+^\infty} + C(\dot{\mathbb{R}}))$ *then* $T(a)$ *is left* (*right*) *invertible.*
ii) *If* $D \in (H_+^\infty + C(\dot{\mathbb{R}})) \cap \overline{(H_+^\infty + C(\dot{\mathbb{R}}))}$ *then* $T(a)$ *is Fredholm.*
iii) *If* $D \in C(\dot{\mathbb{R}})$ *then* $a \in C(\dot{\mathbb{R}})$ *and* $T(a)$ *is Fredholm and*

$$\operatorname{ind}(T(a)) = -\operatorname{wind} a(x).$$

We will also need

Theorem 4.8. *Let a function a have the form* (4.5) *with some* $D \in H_+^\infty + C(\dot{\mathbb{R}})$ *and* $1/D \notin H_+^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$*. Then the Hankel operator* $\mathbb{H}(a)$ *is compact,* $\| \mathbb{H}(a) \| < 1$ *and hence the operator* $I + \mathbb{H}(a)$ *is invertible.*

Proof. The compactness of the operator $\mathbb{H}(a)$ is a direct consequence of Theorem 4.6. Turn to the invertibility of $I + \mathbb{H}(a)$. By Theorem 4.7, the operator $T(a)$ is left invertible and thus by Theorem 4.3 (i) there exists a function $h(x)$ from H_+^∞ such that $\| a - h \|_{L_\infty} < 1$. By (4.4), $\mathbb{H}(a) = \mathbb{H}(a - h)$ and hence by Theorem 4.6

$$\| \mathbb{H}(a) \| \leq \| a - h \|_{L_\infty} < 1 \tag{4.6}$$

and operator $I + \mathbb{H}(a)$ is invertible. □

The symbol

$$\phi(x) = e^{i\left(tx^3 + cx\right)} D\,(x)\,, \quad t > 0, c \in \mathbb{R} \tag{4.7}$$

arises in the inverse scattering transform method for the Korteweg-de Vries (KdV) equation (see [18, 19]). The form of the unimodular function $D\,(x)$ depends on the properties of the initial data in the Cauchy problem for the KdV equation. In certain particular cases discussed in the next section the function $D(x)$ is of the form

$$D(x) = \frac{B_1(x)}{B_2(x)} I(x), \tag{4.8}$$

where $B_{1,2}(x)$ are Blaschke products with zeros converging to 0 along the imaginary axis and $I(x)$ is an inner function ($I(x) \in H_+^\infty$ and $|I(x)| = 1$ a.e. on $\mathbb{R}$). To apply Theorem 4.8 to the case of (4.7)–(4.8) we need one result from [3, 9].

Definition 4.9. Let Δ be a real-valued function defined for all sufficiently large $x > 0$. The function Δ is called regular if it is strictly monotonically increasing, twice continuously differentiable and satisfies

$$\liminf_{x\to\infty} \frac{x\Delta''(x)}{\Delta'(x)} > -2,$$
$$\lim_{x\to\infty} \frac{x\Delta''(x)}{\Delta'(x)^2} = 0,$$
$$\lim_{x\to\infty} \frac{\sqrt{x}\Delta''(x)}{\Delta'(x)^{3/2}} = 0.$$

Theorem 4.10 ([3], [9], Ch. 5). *If the homeomorphism $\delta(x) : \mathbb{R} \to \mathbb{R}$ is a regular function and $\delta(-x) = -\delta(x)$ for sufficiently large $x > 0$, then*

$$\exp\{i\xi\delta(x)\} \in H_+^\infty + C(\dot{\mathbb{R}})$$

for all $\xi > 0$. Moreover the following representation holds

$$\exp\{i\xi\delta(x)\} = B_\xi(x)\, C_\xi(x), \tag{4.9}$$

where $B_\xi(x)$ is a Blaschke product with an infinite number of zeros with no accumulation points at a finite distance and $C_\xi(x)$ is a unimodular function from $C(\dot{\mathbb{R}})$.

The following theorem is one of the main results of this paper.

Theorem 4.11. *Let $B_{1,2}(x)$ be Blaschke products of the form* (1.1) *with zeros satisfying the conditions of Theorem* 2.4 *and Theorem* 3.1 *and let $I(x)$ be an inner function. Consider*

$$\phi(x) = e^{i\left(tx^3+cx\right)} \frac{B_1(x)}{B_2(x)} I(x), \quad t > 0, \quad c \in \mathbb{R}. \tag{4.10}$$

Then the Toeplitz operator $T(\phi) : H_+^2 \to H_+^2$ is left invertible, the Hankel operator $\mathbb{H}(\phi) : H_+^2 \to H_+^2$ is compact and the operator $I + \mathbb{H}(\phi) : H_+^2 \to H_+^2$ is invertible.

Proof. By Theorem 3.1

$$Q(x) = \frac{B_1(x)}{B_2(x)} \in C(\dot{\mathbb{R}}).$$

It follows from Theorem 4.10 that

$$e^{i\left(tx^3+cx\right)} \in H_+^\infty + C(\dot{\mathbb{R}})$$

(it is easy to check that function $\delta(x) := tx^3 + cx$ is regular). Since the set $H_+^\infty + C(\dot{\mathbb{R}})$ is an algebra we have

$$\phi(x) \in H_+^\infty + C(\dot{\mathbb{R}}). \tag{4.11}$$

It remains to demonstrate that

$$1/\phi(x) \notin H_+^\infty(\mathbb{R}) + C(\dot{\mathbb{R}}). \tag{4.12}$$

To this end consider

$$1/\phi(x) = \overline{B_\xi(x)}\; d(x),$$

where $B_\xi(x)$ is as in (4.9) $d(x) \in C(\dot{\mathbb{R}})$ and $|d(x)| = 1$ for all $x \in \mathbb{R}$. Since the Blaschke product $B_\xi(x)$ has an infinite number of zeros, we conclude that $\dim\ker T(1/\phi) = \infty$ (see, e.g., [9], p. 24) and hence the operator $T(1/\phi)$ cannot be Fredholm. On the other hand if (4.12) doesn't hold, i.e., $1/\phi \in H_+^\infty + C(\dot{\mathbb{R}})$ (and (4.11) also holds), then ([8, 12, 13]) $T(1/\phi)$ must be Fredholm. This contradiction proves (4.12). □

5. Applications to the Korteweg-de Vries equation

In this section we apply the results obtained in the previous sections to soliton theory (see, e.g., the book [1] by Ablowitz-Clarkson). We do not assume that the reader is familiar with this theory and therefore present here some background information. Consider the initial value (Cauchy) problem for the Korteweg-de Vries (KdV) equation

$$\frac{\partial u(x,t)}{\partial t} - 6u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} = 0, \quad t \geq 0, x \in \mathbb{R}. \tag{5.1}$$

$$u(x,0) = q(x). \tag{5.2}$$

This equation is arguably the most celebrated nonlinear partial differential equations. It was derived by Korteweg and de Vries in 1895 as a model for describing shallow water but remained essentially unused until the 50s when it was found to be particularly important in plasma physics. In 1955, Fermi, Pasta, and Ulam took a chain of harmonic oscillators coupled with a quadratic nonlinearity and investigated how the energy in one mode would spread to the rest. (One of the first dynamics calculations carried out on a computer.) They found that the system cycled periodically and never came to the rest. This was a striking phenomenon which back then had no explanation. Although Fermi, Pasta, and Ulam never published their observation, the equation drew attention of mathematicians and theoretical physicists. The breakthrough occurred in the mid 60s when Gardner, Greene, Kruskal, and Miura found a truly ingenious way to linearize it. Their method, now called the inverse scattering transform (IST), is a major achievement of the 20th century mathematics and with its help we have learned an incredible amount about the KdV equation and physical systems described by it[3]. We have given here only a small part of the fascinating story behind the KdV equation. The interested reader can learn more about the history in [1] or any other book on soliton theory.

Conceptually, the IST is similar to the Fourier transform and consists, as the standard Fourier transform method, of the following three steps:

1. the direct transform mapping the (real) initial data $q(x)$ to a new set of variables S_0 in which (5.1) turns into a very simple first-order linear ordinary equation for $S(t)$ with the initial condition $S(0) = S_0$;
2. solve then this linear ordinary differential equation for $S(t)$;
3. apply the inverse transform to find $u(x,t)$ from $S(t)$.

In its original edition due to Gardner-Greene-Kruskal-Miura (see, e.g., [1]), S_0 was the set of the so-called scattering data associated with the pair of Schrödinger operators $H_0 = -d^2/dx^2$ and $H_q = -d^2/dx^2 + q\,(x)$ on $L_2\,(\mathbb{R})$. Moreover, this procedure comes with a beautiful formula

$$u\,(x,t) = -2\frac{\partial^2}{\partial x^2}\log\det\left(I + \mathbb{M}_{x,t}\right), \tag{5.3}$$

where $\mathbb{M}_{x,t} : L_2\,(0,\infty) \to L_2\,(0,\infty)$ is a two parametric family of integral operators

$$\left(\mathbb{M}_{x,t} f\right)(y) = \int_0^\infty M_{x,t}\,(y+s)\, f\,(s)\, ds, \quad f \in L_2\,(0,\infty)\,, \tag{5.4}$$

explicitly constructed in terms of $S(t)$.

One immediately sees that the operator defined by (5.4) is Hankel. We describe this operator following [18, 19]. The operator (5.4) is unitary equivalent to

$$\mathbb{H}_{x,t} := \mathbb{H}^{(1)}_{x,t} + \mathbb{H}^{(2)}_{x,t}. \tag{5.5}$$

The first operator on the right-hand side of (5.5) is the Hankel operator defined by (4.3) with the symbol $R_{x,t}$ given by

$$R_{x,t}\,(\lambda) = e^{2i\lambda\left(4\lambda^2 t - x\right)} R\,(\lambda)\,,$$

where $R\,(\lambda)$ is the so-called reflection coefficient corresponding to the pair of Schrödinger operators H_0, H_q. We can easily do without presenting its formal definition by stating its properties. For a.e. real λ

$$R\,(-\lambda) = \overline{R\,(\lambda)}, \quad |R\,(\lambda)| \le 1. \tag{5.6}$$

Note that (5.6) implies that $\mathbb{H}\,(R(x,t))$ is self-adjoint.

The other operator $\mathbb{H}^{(2)}_{x,t}$ on the right-hand side of (5.5) is also a Hankel operator corresponding to the measure

$$d\rho_{x,t}(\alpha) := e^{2\alpha\left(4\alpha^3 t - x\right)} d\rho(\alpha),$$

[3]Similar methods have also been developed for many other physically important evolution nonlinear partial differential equations (PDE), which are typically referred to as completely integrable.

where $\rho(\alpha)$ is a measure subject to

$$\operatorname{Supp}\rho \subseteq [0,a], \quad d\rho \geq 0, \quad \int_0^a d\rho < \infty. \tag{5.7}$$

The measure ρ is related to the negative spectrum of H_q but its explicit expression in terms of H_q is not essential in our consideration. What we need is the following relation between the support of ρ and the negative spectrum of H_q:

$$\alpha \in \operatorname{Supp}\,\rho \iff -\alpha^2 \in \operatorname{Spec}(H_q) \cap \mathbb{R}_-.$$

More specifically, the operator $\mathbb{H}^{(2)}_{x,t}$ is unitarily equivalent to $\chi_{\mathbb{R}_+}\widehat{\rho}_{x,t}\mathcal{F}$, where $\chi_{\mathbb{R}_+}$ is the Heaviside function of $\mathbb{R}_+$, $\mathcal{F}$ is the Fourier operator

$$(\mathcal{F}f)(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x} f(x)dx,$$

and $\widehat{\rho}_{x,t}$ is the Fourier transform of the measure[4] $\rho_{x,t}$.

The pair of functions $(R_{x,t}, \rho_{x,t})$ is called the scattering data and we view $\mathbb{H}_{x,t}$ as the Hankel operator associated with $(R_{x,t}, \rho_{x,t})$.

It is quite easy to see that the Hankel operator $\chi_{\mathbb{R}_+}\widehat{\rho}_{x,t}\mathcal{F}$ is (self-adjoint) non-negative. The operator $\mathbb{H}^{(2)}_{x,t}$ then is also non-negative for any real x and $t \geq 0$. That is

$$\mathbb{H}^{(2)}_{x,t} \geq 0 \tag{5.8}$$

and it is all we can say so far about $\mathbb{H}_{x,t}$ based upon (5.6) and (5.7). Besides the full line Schrödinger operator H_q, introduce $H_q^D = -d^2/dx^2 + q(x)$ defined on $L_2(\mathbb{R}_-)$ with the Dirichlet boundary condition $u(0) = 0$. We label quantities related to H_q^D with a superscript D. We are now able to state the main result of this section.

Theorem 5.1. *Assume that the initial profile $q(x)$ in* (5.2) *is real, locally integrable, supported on $(-\infty, 0)$ and such that*

$$\inf \operatorname{Spec}(H_q) = -a^2 > -\infty. \tag{5.9}$$

Then the Cauchy problem for the KdV equation (5.1)–(5.2) *has a unique solution $u(x,t)$ which is a meromorphic function in x on the whole complex plane with no real poles for any $t > 0$ if at least one of the following conditions holds:*

1. *The operator H_q^D has a non-empty absolutely continuous spectrum;*
2. *The set $i\operatorname{Supp}\rho$ is a set of uniqueness of H_+^∞ functions;*
3. *The sets* $\operatorname{Supp}\rho^D = \{\nu_n\}_{n\geq 1}$ *and* $\operatorname{Supp}\rho = \{\kappa_n\}_{n\geq 1}$ *satisfy the Blaschke condition and the corresponding Blaschke products are subject to the conditions of Corollary* 3.2.

[4]We recall $\widehat{\mu}(\lambda) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x} d\mu(x)$.

Proof. Under conditions 1 and/or 2, the theorem is already proven in [18, 19] and it remains to show that the conclusion of theorem also holds under condition 3. Moreover, the arguments of [18, 19] (see also [20]) based upon (5.3) can be easily adjusted to handle condition 3 if we prove that the operator $I+\mathbb{H}_{x,t}$ is invertible under this condition.

Without loss of generality, we may assume that the operator H_q^D has an empty absolutely continuous spectrum (otherwise we are under condition 1). The structure of the reflection coefficient $R(\lambda)$ is studied in [17] where it is shown that $R(\lambda)$ admits the following factorization

$$R(\lambda)=\lim_{m\to\infty}\left\{\left(\prod_{n=1}^{m}\frac{\lambda-i\nu_n}{\lambda+i\nu_n}\right)\left(\prod_{n=1}^{m}\frac{\lambda-i\kappa_n}{\lambda+i\kappa_n}\right)^{-1}\right\}S(\lambda),\quad \lambda\in\mathbb{C}_+, \tag{5.10}$$

where $S\in H_+^\infty$ and S is contractive on $\mathbb{C}_+$ (i.e., $|S(\lambda)|\le 1,\quad \lambda\in\mathbb{C}_+$) and the sequence $\{\nu_n\}_{n\ge1}$ is such that

$$\{-\nu_n^2\}_{n\ge1}=\operatorname{Spec}\left(H_q^D\right)\cap\mathbb{R}_-,$$

(the negative spectrum of the half-line Dirichlet Schrödinger operator), and the sequence $\{\kappa_n\}_{n\ge1}$ is such that

$$\{-\kappa_n^2\}_{n\ge1}=\operatorname{Spec}(H_q)\cap\mathbb{R}_-,$$

(the negative spectrum of the full-line Schrödinger operator). Moreover these sequences are interlacing, i.e.,

$$\kappa_n>\nu_n>\kappa_{n+1}\text{ for any }n\in\mathbb{N}. \tag{5.11}$$

Since we have assumed that the operator H_q^D has no absolutely continuous spectrum, $|S(\lambda)|=1$ for a.e. real λ (see, e.g., [17]) and hence $S(\lambda)=I(\lambda)$ where $I(\lambda)$ is an inner function of $\mathbb{C}_+$.

Note next that

$$\lim_{m\to\infty}\left\{\left(\prod_{n=1}^{m}\frac{\lambda-i\nu_n}{\lambda+i\nu_n}\right)\left(\prod_{n=1}^{m}\frac{\lambda-i\kappa_n}{\lambda+i\kappa_n}\right)^{-1}\right\}=\left(\prod_{n=1}^{\infty}\frac{\lambda-i\nu_n}{\lambda+i\nu_n}\right)\left(\prod_{n=1}^{\infty}\frac{\lambda-i\kappa_n}{\lambda+i\kappa_n}\right)^{-1}$$
$$=: B_1(\lambda)\,B_2(\lambda)^{-1},$$

where

$$B_1(\lambda)=\prod_{n=1}^{\infty}\frac{\lambda-i\nu_n}{\lambda+i\nu_n}\quad\text{and}\quad B_2(\lambda)=\prod_{n=1}^{\infty}\frac{\lambda-i\kappa_n}{\lambda+i\kappa_n}.$$

We have thus arrived at the factorization

$$R(\lambda)=\frac{B_1(\lambda)}{B_2(\lambda)}I(\lambda),\quad \lambda\in\mathbb{C}_+,$$

and hence for every $x\in\mathbb{R}$ and $t>0$ the function

$$R_{x,t}(\lambda)=e^{2i\lambda\left(4\lambda^2t-x\right)}R(\lambda),$$

by Corollary 3.2, satisfies the conditions of Theorem 4.11 and hence

$$\|\mathbb{H}(R_{x,t})\| < 1.$$

This immediately implies that

$$\left\|\mathbb{H}_{x,t}^{(1)}\right\| = \|\mathbb{H}(R_{x,t})\| < 1.$$

Therefore $I + \mathbb{H}_{x,t}^{(1)} \geq 0$ and is boundedly invertible. Due to (5.8)

$$I + \mathbb{H}_{x,t}^{(1)} + \mathbb{H}_{x,t}^{(2)} = I + \mathbb{H}_{x,t}^{(1)} \geq 0$$

is also boundedly invertible and the theorem is proven. □

Note that Theorem 5.1 represents an existence and uniqueness result for the KdV equation in a very strong sense. We refer the interested reader to [18, 19] for detailed discussions of statements like Theorem 5.1 and the extensive recent literature on the subject cited therein.

Let us discuss what the conditions of Theorem 5.1 actually mean in terms of the initial profile $q(x)$ in (5.2). Condition (5.9) means that the spectrum of H_q is bounded from below, which (see, e.g., [11]) is satisfied if

$$\sup_x \int_{x-1}^{x} \max(-q, 0) < \infty. \tag{5.12}$$

The condition (5.12) becomes also necessary for (5.9) if q is negative. Note that (5.9) imposes no restriction on the positive part $\max(q, 0)$ of $q(x)$ (e.g., it can grow arbitrarily fast at $-\infty$ or look like the stock market) but H_q still satisfies (5.9).

Condition 1 means that $q(x)$ has a certain pattern of behavior at $-\infty$. The precise statement is rather complicated but particular examples are easy. Condition 1 is satisfied if, for example, q is quasi-periodic on $(-\infty, 0)$ or approaches a constant as $x \to -\infty$ sufficiently fast.

Condition 2 means that the negative spectrum of H_q is, in a way, rich enough. Condition 2 holds if, loosely speaking, $\max(-q, 0)$ (the negative part of q) is large. A typical example would be $q(x) \to -c^2$ as $x \to -\infty$ for some real c (so-called step like initial profiles).

Condition 3 is much trickier as the problem of the negative spectrum distribution for the Schrödinger operator is notoriously difficult. In fact, besides the Lieb-Thirring estimate [21]

$$\sum_{n \geq 1} \kappa_n \lesssim \int_{\mathbb{R}} \max(-q, 0), \tag{5.13}$$

nothing is known about the distribution of $\{\kappa_n\}$ in general. The reason for that is a poor understanding of how individual eigenvalues $-\kappa_n^2$ of H_q depend on q and even (5.13) was a good open problem for quite some time. By the same token constructing a nontrivial explicit example of $q(x)$ subject to condition 3 but not condition 1 appears to be a real challenge. Note that one can always start with a desired spectrum and then work backwards to an essentially non-computable (and quite pathological) $q(x)$ via the Gelfand-Levitan-Marchenko inverse method.

The following statement is important.

Corollary 5.2. *The conclusions of Theorem* 5.1 *hold if* $q(x)$ *in* (5.2) *is real, locally integrable, supported on* $\mathbb{R}_-$ *and such that*

$$\int_{-\infty}^{0} |x| \max(-q(x), 0)\, dx < \infty. \tag{5.14}$$

Proof. The condition (5.14) clearly implies (5.12). Furthermore, it is well known that the negative spectra of H_q and H_q^D are finite under the condition (5.14). Hence $\{\kappa_n\}$ and $\{\nu_n\}$ are also finite and Corollary 3.2 clearly applies. We are then under Condition 3. □

We emphasize that even Corollary 5.2 is new and nontrivial as it cannot be achieved by usual PDEs techniques. We however conjecture that the condition (5.9) alone will be sufficient for Theorem 5.1 to hold. We are not sure if condition (5.9) implies that $I + \mathbb{H}(R_{x,t})$ is boundedly invertible but there are some strong reasons to believe that $I + \mathbb{H}_{x,t}$ has this property.

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