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Estimates for singular numbers of Hausdorff–Zhu operators and applications

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We continue the study of the so-called Hausdorff–Zhu operators. In this paper, we study the behavior of the singular values of such operators. We prove the general fact that the sequence of singular numbers tends to zero, as well as we prove power-type estimates for such behavior under additional conditions on the kernel of the operator. We give application of these results to the boundedness of Hausdorff–Zhu operators in general classes of analytic functions in the unit disc and also in some special classes of analytic functions defined in terms of conditions on Taylor or Fourier coefficients.

KEYWORDS
Bergman and Hardy spaces, Hausdorff operator, Hausdorff–Zhu operator, integral operator

MSC CLASSIFICATION
47G10, 47B38, 46E30

1 | INTRODUCTION

Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on the unit disc $\mathbb{D}$. Let also $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$, $z, w \in \mathbb{D}$ stand for involutive Möbius automorphism of the unit disc $\mathbb{D}$. Let Φ be a function (kernel) whose properties will be specified below in the text. We continue the study of the class of integral operators

$$\mathcal{H}_\Phi f(z) := \int_{\mathbb{D}} \Phi(w) f(\varphi_w(z)) dA(w), \; z \in \mathbb{D}. \tag{1.1}$$

The operator (1.1) appeared in the paper[1] under the name of the Hausdorff–Zhu operators. Note that in the paper,[2] the operators $\mathcal{H}_\Phi$ with a general measure $d\mu$ instead of dA were studied in several spaces of analytic functions in $\mathbb{D}$. Therefore, our research is in continuation of the recent study in earlier studies.[1,2]

However, the general idea of the investigation of such objects is inspired by the well-known studies of special classes of integral operators with one or another symmetry or invariance in the integral kernel. A class of such operators is the class of Hausdorff operators, intensively studied during the last two decades; see, for example, earlier studies[3–5] and also the paper.[6] The operator (1.1) is of the Hausdorf type, and that is how the name Hausdorff enters into the title of this paper. On the other side, the operator $\mathcal{H}_1$ where $\Phi \equiv 1$, that is,

$$\mathcal{H}_1 f(z) := \int_{\mathbb{D}} f(\varphi_w(z))\mathrm{dA}(w),\ z \in \mathbb{D} \tag{1.2}$$

appeared and was studied in the paper by Kehe Zhu[7] in the context of operators related to Berezin transform $\mathbb{B}f(z) = \int_{\mathbb{D}} f(\varphi_z(w))\mathrm{dA}(w)$; see Karapetyants and Mirotin[1] for more details. That is how the name Zhu appears in the title as well.

In fact, the connection of our operators with such classical operators of mathematical physics as the Berezin transform, and others (see also Karapetyants et al[8] for the study of the so-called Hausdorff-Berezin operators), determines the interest in this topic. We do not intend to dwell on this issue in detail and direct the reader to the paper[1] in which this issue is stated in more detail.

Let us dwell only on the fact that an operator from a given class, with certain kernels, admits an explicit expression on the basis elements, which allows us to speak of singular numbers for this operator. Here we can talk about an arbitrary space of analytic functions on the unit disc in which the elements $e_n = z^n$ form a basis. Namely, for the factorized kernel $\Phi(w) = \Phi_1(|w|^2)w^m$, we have

$$\mathcal{H}_\Phi e_n = \lambda_{\Phi_1,m}(n)e_{m+n}, \quad m, n \in \mathbb{Z}_+.$$

In the case of a radial kernel $\Phi(w) = \Phi_1(|w|^2)$, the numbers $\lambda_{\Phi_1,0}(n)$ form a point spectrum of the corresponding operator. In the general case of a factorized kernel, we refer the reader to the Remark 1 below for the explanation why we continue to call these numbers $(\lambda_{\Phi_1,m}(n))$ singular numbers of the operator $\mathcal{H}_\Phi$.

As is already clear, the behavior of $\lambda_{\Phi_1,m}(n)$ when $n \to \infty$ defines many important properties of the initial operator, such as compactness and boundedness in various spaces of analytic functions, in which the elements e_n form a basis. This paper is devoted to the study of such asymptotic behavior for singular numbers of Hausdorff–Zhu operator with the factorized kernel $\Phi(w) = \Phi_1(|w|^2)w^m$, $m \in \mathbb{Z}_+$.

We start with the general case in Theorem 3.3 which, under minimal conditions, allows us further in Theorem 3.4 to prove that $\lambda_{\Phi_1,m}(n) \to 0$, when $n \to \infty$. The proof of Theorem 3.3 is quite technical and uses a very fine technique of the saddle-point method, but at the same time, in our opinion, it allows us to characterize the behavior as accurately as possible. Later, under additional requirements on the kernel in Corollaries 3.5 and 3.6 and Theorems 3.7 and 3.8, we obtain concrete power-type estimates for the rate of decay of the singular numbers when $n \to \infty$. The results described form the Section 3 of the present paper.

Applications of these results are in Section 5 where we give some sufficient conditions for the boundedness and compactness of the operator with factorized kernel $\Phi(w) = \Phi_1(|w|^2)w^m$, $m \in \mathbb{Z}_+$. Here we use the abovementioned specific power-type estimates and the general facts of the theory of multipliers to obtain a result within the framework of certain general analytic spaces.

Further, we give applications to compactness in more specific classes of analytic functions. Namely, they are given to compactness in the class ℓ^p_A ($p \in [1, \infty]$) of analytic functions in a disc for which the Taylor coefficients form a ℓ^p sequence and to compactness in the class $\mathcal{A}^{q;X}(\mathbb{D})$ of analytic functions with mixed norm in the unit disc, defined in terms of conditions on the Fourier coefficients.

In Section 4, we provide an interesting representation of singular numbers in terms of well-known Jacobi polynomials. This connects the integral representations for singular numbers with classical objects and subsequently makes it possible to further study the asymptotic in more detail. However, Proposition 4.9 and Example 4.10 do not improve on the initially obtained results, but instead, they are confirmed in a slightly weaker form. This supports our thesis about the accuracy of the results from Section 3.

2 | PRELIMINARIES

We equip Lebesgue spaces $L^p(\mathbb{D}) = L^p(\mathbb{D}, \mathrm{dA})$ with the norm

$$\|f\|_p = \left(\int_{\mathbb{D}} |f(z)|^p \mathrm{dA}(z)\right)^{\frac{1}{p}},\ 1 \leqslant p < \infty, \tag{2.1}$$

and we treat the case $p = \infty$ as usual: $\|f\|_\infty = \operatorname{esssup}_{z\in\mathbb{D}}|f(z)|$. Let $H(\mathbb{D})$ be the set of functions f, analytic in $\mathbb{D}$, equipped with the topology defined by the countable set of norms

$$\|f\|_m = \sup_{|z|<1-\frac{1}{m+1}} \left| \sum_{m=0}^{\infty} f_m z^m \right|, \; m = 1, 2, \ldots, \tag{2.2}$$

where f_m are the Taylor coefficients at 0 of a function $f \in H(\mathbb{D})$, that is $f_m = \frac{f^{(m)}(0)}{m!}$. As is well-known, the *Bergman space* $\mathcal{A}^p(\mathbb{D})$ $(0 < p < \infty)$ is the subspace $H(\mathbb{D}) \cap L^p(\mathbb{D})$ of analytic functions in $L^p(\mathbb{D})$. For $0 < p < \infty$, the *Hardy space* $H^p(\mathbb{D})$ consists of analytic functions f in the unit disc $\mathbb{D}$ such that

$$\|f\|^p_{H^p(\mathbb{D})} := \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Then $\|\cdot\|_{H^p(\mathbb{D})}$ is a norm for $p \in [1, \infty)$.

Definition 2.1 ([1]). Given a measurable function Φ the Hausdorff–Zhu operator is introduced by

$$\mathcal{H}_\Phi f(z) = \int_{\mathbb{D}} \Phi(w) f(\varphi_w(z)) \mathrm{dA}(w), \; z \in \mathbb{D}.$$

As it was noted in Section 1, this is a special case of Hausdorff operator over the unit disc introduced in Mirotin.[2] In the case $\Phi \equiv 1$, such an operator appeared and was studied by K. Zhu.[7]

The next theorem from Karapetyants and Mirotin[1] provides us with the explicit form of the singular numbers of an operator $\mathcal{H}_\Phi$.

Theorem 2.2 ([1]). *Let $\Phi(w) = \Phi_1(|w|^2)w^m$, where $\Phi_1 \in L^1(0,1)$, and $m \in \mathbb{Z}_+$. Then for each monomial $e_n(z) = z^n$ $(n \in \mathbb{Z}_+)$, we have*

$$\mathcal{H}_\Phi e_n = \lambda_{\Phi_1,m}(n) e_{m+n}, \quad m, n \in \mathbb{Z}_+,$$

where

$$\lambda_{\Phi_1,m}(n) = \int_0^1 \Phi_1(t) Q_{m,n}(t) dt,$$
$$Q_{m,n}(t) = \frac{n}{m+n} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-1}{k} \binom{m+n}{n-k} t^{m+k}(1-t)^{n-k} \tag{2.3}$$

for $n \in \mathbb{N}$, and

$$\lambda_{\Phi_1,m}(0) = \begin{cases} 0, & m \in \mathbb{N}; \\ \int_{\mathbb{D}} \Phi_1(|w|^2) dA(w), & m = 0. \end{cases} \tag{2.4}$$

Remark 1. Let operator $\mathcal{H}_\Phi$ be bounded on a Hilbert space $\mathfrak{H}(\mathbb{D})$ of analytic functions on $\mathbb{D}$ with orthonormal basis $\{e_n/\|e_n\|_{\mathfrak{H}(\mathbb{D})}\}$. Then by Theorem 2.2, we have

$$\langle \mathcal{H}_\Phi^* \mathcal{H}_\Phi e_k, e_j \rangle = \langle \mathcal{H}_\Phi e_k, \mathcal{H}_\Phi e_j \rangle = \lambda_{\Phi_1,m}(k) \overline{\lambda_{\Phi_1,m}(j)} \delta_{kj}.$$

It follows that $\mathcal{H}_\Phi^* \mathcal{H}_\Phi e_k = |\lambda_{\Phi_1,m}(k)|^2 e_k$ and so $(\mathcal{H}_\Phi^* \mathcal{H}_\Phi)^{1/2} e_k = |\lambda_{\Phi_1,m}(k)| e_k$ for all k. Thus, $|\lambda_{\Phi_1,m}(k)|$ are singular numbers of an operator $\mathcal{H}_\Phi$.

3 | HAUSDORFF–ZHU OPERATORS WITH FACTORIZED KERNELS $\Phi(Z) = \Phi_1(|z|^2)z^m$ where $\Phi_1 \in L^1(0,1)$ and $m \in \mathbb{Z}_+$

The following result provides the estimate for the numbers $\lambda_{\Phi_1,m}(n)$ under minor condition $\Phi_1 \in L^1(0,1)$.

Theorem 3.3. *Let $\Phi(w) = \Phi_1(|w|^2)w^m$, where $\Phi_1 \in L^1(0,1)$, and $m \in \mathbb{Z}_+$. Then the following estimate holds*

$$|\lambda_{\Phi_1,m}(n)| \leqslant C\int_{\alpha(n)}^{1}|\Phi_1(\rho^2)|\rho^{m+1}d\rho + \frac{C}{\sqrt{n}}\int_0^{\alpha(n)}|\Phi_1(\rho^2)|(1-\rho)^{\frac{1}{4}}\rho^{m+1}d\rho + \frac{C}{n}\int_0^{\alpha(n)}\frac{\Phi_1(\rho^2)}{(1-\rho)^{\frac{1}{2}}}\rho^{m+1}d\rho,$$

where $\alpha(n) = 1 - \frac{1}{2}\left(\frac{M}{n}\right)^2$, the number $M > 0$ is a fixed number, and the constant C does not depend on n.

Proof. Assume that $n \in \mathbb{N}$. Passing to polar coordinates, we have

$$\mathcal{H}_\Phi e_n(z) = \frac{1}{\pi}\int_0^1 \Phi_1(\rho^2)\rho^{m+1}d\rho\int_0^{2\pi}\frac{(\rho e^{i\theta}-z)^n}{(1-\rho e^{-i\theta}z)^n}e^{im\theta}d\theta.$$

In particular, if $z = 1$, then

$$\lambda_{\Phi_1,m}(n) = \mathcal{H}_\Phi e_n(z)|_{z=1} = \frac{1}{\pi}\int_0^1 \Phi_1(\rho^2)\rho^{m+1}d\rho\int_0^{2\pi}\frac{(\rho e^{i\theta}-1)^n}{(1-\rho e^{-i\theta})^n}e^{im\theta}d\theta. \tag{3.1}$$

Denote

$$I_{n,m}(\rho) = \int_{-\pi}^{\pi}\frac{(\rho e^{i\theta}-1)^n}{(1-\rho e^{-i\theta})^n}e^{im\theta}d\theta.$$

It can be seen that (Dybin and Grudsky[9] page 64)

$$\frac{\rho e^{i\theta}-1}{1-\rho e^{-i\theta}} = e^{i\theta}\frac{\rho - e^{-i\theta}}{1-\rho e^{-i\theta}} = \exp\left\{i\left(2\arctan\left[\frac{1-\rho}{1+\rho}\cot\frac{\theta}{2}\right]+\theta\right)\right\}.$$

Denote

$$\varepsilon = \frac{1-\rho}{1+\rho}, \quad S(\theta,\varepsilon) = 2\arctan\left[\varepsilon\cot\frac{\theta}{2}\right] + \theta\lambda,$$

where λ is arbitrary number in $[1, 1+\lambda_0]$, where $\lambda_0 > 0$ is a fixed number. Here we note that given $m \in \mathbb{Z}_+$, there exists N_m such that $1 + \frac{m}{n} \in [1, 1+\lambda_0]$ starting $n \geqslant N_m$.

Let us consider the integral

$$\mathcal{I}_{n,\lambda}(\rho) = \int_{-\pi}^{\pi} e^{inS(\theta,\varepsilon)}d\theta.$$

We note that

$$I_{n,m}(\rho) = \mathcal{I}_{n,1+\frac{m}{n}}(\rho);$$

therefore, our aim is to obtain asymptotic behavior of $\mathcal{I}_{n,\lambda}(\rho)$ when $\rho \to 1$ which is uniform in $\lambda \in [1, 1+\lambda_0]$. To find critical points of the function $S(\theta,\varepsilon)$, calculate

$$S'_\theta(\theta,\varepsilon) = \lambda - \frac{\varepsilon}{\sin^2\frac{\theta}{2} + \varepsilon^2\cos^2\frac{\theta}{2}},$$

and hence,

$$S'_\theta(\theta,\varepsilon) = 0 \Leftrightarrow \theta_\pm(\varepsilon) = \pm 2\arcsin\sqrt{\frac{\varepsilon\left(\frac{1}{\lambda}-\varepsilon\right)}{1-\varepsilon^2}} = \pm\frac{2}{\sqrt{\lambda}}\varepsilon^{\frac{1}{2}} + O\left(\varepsilon^{\frac{3}{2}}\right), \ \varepsilon \to 0,$$

and the asymptotic is uniform in $\lambda \in [1, m]$. Here and in what follows, without loss of generality, we assume that

$$\varepsilon = \frac{1-\rho}{1+\rho} \in \left[0, \frac{1}{2\lambda}\right].$$

Denote

$$\delta = \theta_+(\varepsilon), \ \Lambda = \delta n, \ \text{ and } \ \tilde{S}(t,\varepsilon) = \frac{2}{\delta} \arctan\left[\varepsilon \cot \frac{\delta t}{2}\right] + t\lambda.$$

We have

$$\mathcal{I}_{n,\lambda}(\rho) = \delta \int_{-\pi/\delta}^{\pi/\delta} e^{i\Lambda \tilde{S}(t,\varepsilon)} dt. \tag{3.2}$$

We will regard Λ as a large parameter, so let us fix arbitrary $M > 0$ and suppose from now on that

$$\Lambda > M, \ \text{that is} \ \delta > M/n. \tag{3.3}$$

In terms of ε and ρ, it suffices to assume that

$$\varepsilon = \frac{1-\rho}{1+\rho} \geqslant \frac{1}{2}\left(\frac{M}{n}\right)^2.$$

We suppose that n is a sufficiently big number such that

$$\frac{1}{2}\left(\frac{M}{n}\right)^2 \leqslant \varepsilon = \frac{1-\rho}{1+\rho} \leqslant \frac{1}{2\lambda}.$$

Note that integral (3.2) has two critical points $t_\pm = \pm 1$. Apply the decomposition of the unity:

$$\chi_0 + \chi_{-1} + \chi_1 = 1$$

such that

1. χ_j are $C_0^\infty(\mathbb{R})$ functions, $j = 0, \pm 1$;
2. $\operatorname{supp} \chi_0 = \left(-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right) \setminus [-2, 2]$, $\operatorname{supp} \chi_1 = (-\Delta, 2+\Delta)$, $\operatorname{supp} \chi_{-1} = (-2-\Delta, \Delta)$, where Δ is a positive number.

Split integral (3.2) accordingly to the above decomposition:

$$\mathcal{I}_{n,\lambda}(\rho) = \delta I_0(\rho) + \delta I_1(\rho) + \delta I_{-1}(\rho). \tag{3.4}$$

Consider $I_0(\rho)$. Note that the critical points are located outside the $\operatorname{supp} \chi_0$, so we can calculate by using integration by parts:

$$\begin{aligned}
I_0(\rho) &= \int_{\left[-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right] \setminus (-2,2)} \chi_0(t) e^{i\Lambda \tilde{S}(t,\varepsilon)} dt = \frac{1}{i\Lambda} \int_{\left[-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right] \setminus (-2,2)} \chi_0(t) \frac{d e^{i\Lambda \tilde{S}(t,\varepsilon)}}{\tilde{S}_t'(t,\varepsilon)} \\
&= \frac{1}{i\Lambda} \int_{\left[-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right] \setminus (-2,2)} e^{i\Lambda \tilde{S}(t,\varepsilon)} \left(\frac{d}{dt} \frac{\chi_0(t)}{\tilde{S}_t'(t,\varepsilon)} \right) dt \\
&= \frac{1}{i\Lambda} \int_{\left[-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right] \setminus (-2,2)} \frac{e^{i\Lambda \tilde{S}(t,\varepsilon)}}{\left(\tilde{S}_t'(t,\varepsilon)\right)^2} \left(\chi_0'(t) \tilde{S}_t'(t,\varepsilon) - \chi_0(t) \tilde{S}_{tt}''(t,\varepsilon) \right) dt.
\end{aligned}$$

Recall that

$$\tilde{S}_t'(t,\varepsilon) = \lambda - \frac{\varepsilon}{\sin^2 \frac{\delta t}{2} + \varepsilon^2 \cos^2 \frac{\delta t}{2}},$$

$$\tilde{S}_{tt}''(t,\varepsilon) = \frac{\varepsilon \delta (1-\varepsilon^2) \sin \frac{\delta t}{2} \cos \frac{\delta t}{2}}{\left(\sin^2 \frac{\delta t}{2} + \varepsilon^2 \cos^2 \frac{\delta t}{2}\right)^2},$$

and that

$$\inf\left\{|\tilde{S}_t'(t,\varepsilon)| \,:\, \varepsilon\in\left[\frac{1}{2}\left(\frac{M}{n}\right)^2,\frac{1}{2\lambda}\right],\, t\in\left[-\frac{\pi}{\delta},\frac{\pi}{\delta}\right]\setminus(-2,2)\right\}>0.$$

Hence, there exists $C>0$ such that

$$|\delta I_0(\rho)|\leqslant C\frac{\delta}{\Lambda}\operatorname{mes}\left\{\left(-\frac{\pi}{\delta},\frac{\pi}{\delta}\right)\setminus[-2,2]\right\}\leqslant 2\pi\frac{C}{\Lambda},$$

uniformly in $\varepsilon\in\left[\frac{1}{2}\left(\frac{M}{n}\right)^2,\frac{1}{2\lambda}\right]$. Repeating the same procedure, we have that for arbitrary $k=1,2,\dots,$ there exists $C>0$ such that

$$|\delta I_0(\rho)|\leqslant \frac{C}{\Lambda^k},\ \text{uniformly in } \varepsilon\in\left[\frac{1}{2}\left(\frac{M}{n}\right)^2,\frac{1}{2\lambda}\right]. \tag{3.5}$$

Note that for the purpose of proving this theorem, we only need the estimate with $k=1$. But for the subsequent results, the estimate with $k>1$ is essential.

Consider now $I_1(\rho)$. The interval $(-\Delta,2+\Delta)$ contains critical stationary phase point $t_1=1$, so we will apply Theorem 1.6 from Fedoryuk.[10] We note that the function $\tilde{S}(t,\varepsilon)$ satisfies the following conditions:

1. $\tilde{S}(t,\varepsilon)\in C([-\Delta,2+\Delta]\times(0,1))\cap C^\infty((-\Delta,2+\Delta)\times(0,1))$;
2. For each fixed $\varepsilon\in\left(0,\frac{1}{2\lambda}\right)$, the function $\tilde{S}(t,\varepsilon)$ has unique critical point $t_1=1\in(-\Delta,2-\Delta)$, and at this point there exists ϵ_0 such that

$$\inf\{|\tilde{S}_{tt}''(1,\varepsilon)| \,:\, \varepsilon\in(0,\epsilon_0)\}>0.$$

Indeed, let us prove the last assertion. Straightforward calculus gives

$$\begin{aligned}\tilde{S}_{tt}''(1,\varepsilon)&=\varepsilon\delta\frac{(1-\varepsilon^2)\sin\frac{\delta}{2}\cos\frac{\delta}{2}}{\left(\sin^2\frac{\delta}{2}+\varepsilon^2\cos^2\frac{\delta}{2}\right)^2}=\frac{\frac{\varepsilon\delta^2}{2}(1+O(\delta^2))}{\frac{\delta^4}{16}(1+O(\delta^2))}\\&=2\lambda+O(\delta^2)=2\lambda+O(\varepsilon),\quad \text{if } \varepsilon\to 0.\end{aligned}$$

Hence, according to the abovementioned Theorem 1.5 from Fedoryuk,[10] we have

$$I_1(\rho)=\sqrt{\frac{2\pi}{\Lambda|\tilde{S}_{tt}''(1,\varepsilon)|}}\left(\exp\left\{i\Lambda\tilde{S}(1,\varepsilon)+i\frac{\pi}{4}\right\}+O\left(\frac{1}{\Lambda^{3/2}}\right)\right),$$

where

$$\begin{aligned}\tilde{S}(1,\varepsilon)&=\lambda+\frac{2}{\delta}\arctan\varepsilon\cot\frac{\delta}{2}\\&=\lambda+\frac{2}{\delta}\left(\varepsilon\frac{2}{\delta}+O(\delta^3)\right)\\&=2\lambda+O(\varepsilon),\quad\text{if } \varepsilon\to 0.\end{aligned}$$

Similarly, in view of Theorem 1.5 from Fedoryuk,[10] we have

$$I_{-1}(\rho)=\sqrt{\frac{2\pi}{\Lambda|\tilde{S}_{tt}''(-1,\varepsilon)|}}\left(\exp\left\{i\Lambda\tilde{S}(-1,\varepsilon)-i\frac{\pi}{4}\right\}+O\left(\frac{1}{\Lambda^{3/2}}\right)\right),$$

where

$$\begin{aligned}\tilde{S}(-1,\varepsilon)&=-\lambda-\frac{2}{\delta}\arctan\varepsilon\cot\frac{\delta}{2}\\&=-2\lambda+O(\varepsilon),\quad\text{if } \varepsilon\to 0.\end{aligned}$$

We note that

$$|\tilde{S}_{tt}''(-1,\varepsilon)|=|\tilde{S}_{tt}''(1,\varepsilon)|\ \text{ and }\ \tilde{S}(-1,\varepsilon)=-\tilde{S}(1,\varepsilon).$$

Gathering the above estimates, we have

$$
\begin{aligned}
\mathcal{I}_{n,\lambda}(\rho) &= \frac{C_k}{\Lambda^k} + 2\delta\sqrt{\frac{2\pi}{\Lambda|\tilde{S}''_{tt}(1,\varepsilon)|}}\left(\cos\left\{\Lambda\tilde{S}(1,\varepsilon)+\frac{\pi}{4}\right\} + O\left(\frac{1}{\Lambda^{3/2}}\right)\right) \\
&= \frac{C_k}{\Lambda^k} + 2\sqrt{\frac{\delta}{n}}\sqrt{\frac{2\pi}{|\tilde{S}''_{tt}(1,\varepsilon)|}}\cos\left\{\Lambda\tilde{S}(1,\varepsilon)+\frac{\pi}{4}\right\} \\
&+ O\left(\frac{1}{\delta^{1/2}n^{3/2}}\right),
\end{aligned}
\tag{3.6}
$$

where $k = 1, 2, \dots$, the constant C_k is independent on $\varepsilon \in \left[\frac{1}{2}\left(\frac{M}{n}\right)^2, \frac{1}{2\lambda}\right]$, but depends on k.

We recall that

$$
\begin{aligned}
\Lambda &= \delta n, \\
\varepsilon &= \frac{1-\rho}{1+\rho} = \frac{1}{2}(1-\rho) + O((1-\rho)^2), \ \ \rho \to 1, \\
\delta &= 2\arcsin\sqrt{\frac{\varepsilon\left(\frac{1}{\lambda}-\varepsilon\right)}{1-\varepsilon^2}} = \frac{2}{\sqrt{\lambda}}\varepsilon^{\frac{1}{2}} + O\left(\varepsilon^{\frac{3}{2}}\right) = \sqrt{\frac{2}{\lambda}}(1-\rho)^{\frac{1}{2}} + O(1-\rho), \ \ \rho \to 1.
\end{aligned}
$$

Let us return to formula (3.1). Denoting

$$
\alpha(n) = 1 - \frac{1}{2}\left(\frac{M}{n}\right)^2,
$$

we split the following:

$$
\begin{aligned}
\lambda_{\Phi_1,m}(n) &= \frac{1}{\pi}\int_0^1 \Phi_1(\rho^2)I_{n,m}(\rho)\rho^{m+1}d\rho = \frac{1}{\pi}\int_0^{\alpha(n)} \Phi_1(\rho^2)I_{n,m}(\rho)\rho^{m+1}d\rho \\
&+ \frac{1}{\pi}\int_{\alpha(n)}^1 \Phi_1(\rho^2)I_{n,m}(\rho)\rho^{m+1}d\rho \equiv J_1(n) + J_2(n).
\end{aligned}
$$

Since $\left|\frac{\rho e^{i\theta}-1}{1-\rho e^{-i\theta}}\right| = 1$ for the second integral, we have

$$
|J_2(n)| \leqslant 2\int_{\alpha(n)}^1 |\Phi_1(\rho^2)|\rho^{m+1}d\rho \equiv B_{\Phi_1}(1-\alpha^2(n)),
\tag{3.7}
$$

where we denoted

$$
B_{\Phi_1}(s) = \int_{1-s}^1 t^{\frac{m}{2}}|\Phi_1(t)|dt.
\tag{3.8}
$$

To estimate the integral $J_1(n)$, we apply the above made calculus taking into account that δ is a function of ρ. We choose $k = 1$ in (3.6). Hence,

$$
\begin{aligned}
J_1(n) &= \frac{C_1}{n}\int_0^{\alpha(n)} \frac{\Phi_1(\rho^2)}{\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}}}\rho^{m+1}d\rho \\
&+ 2\sqrt{\frac{2}{\pi n}}\int_0^{\alpha(n)} \frac{\Phi_1(\rho^2)\delta^{\frac{1}{2}}(\rho)}{|\tilde{S}''_{tt}\left(1,\frac{1-\rho}{1+\rho}\right)|^{\frac{1}{2}}}\cos\left\{\delta(\rho)n\tilde{S}\left(1,\frac{1-\rho}{1+\rho}\right)+\frac{\pi}{4}\right\}\rho^{m+1}d\rho \\
&+ O\left(\frac{1}{n^{\frac{3}{2}}}\int_0^{\alpha(n)} \frac{\Phi_1(\rho^2)}{\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}}}\rho^{m+1}d\rho\right), \ \ n \to \infty.
\end{aligned}
\tag{3.9}
$$

Here the constant C_1 is uniform in $\varepsilon = \frac{1-\rho}{1+\rho} \in \left[\frac{1}{2}\left(\frac{M}{n}\right)^2, \frac{1}{2\lambda}\right]$, that is, it is uniform in $\alpha(n) = 1 - \frac{1}{2}\left(\frac{M}{n}\right)^2 \to 1$, when $n \to \infty$.

Note that the third term in (3.9) is bigger then the first term in the same formula, so we will not include the third term of (3.9) into the final estimate. However, we will have to keep it in mind later on when we want to obtain a better asymptotic under some additional conditions (see Theorem 3.7).

Now gathering (3.7) and (3.9), it finishes the proof. □

As an immediate corollary, we have the following result.

Theorem 3.4. *Let $\Phi(w) = \Phi_1(|w|^2)w^m$, where $\Phi_1 \in L^1(0,1)$, and $m \in \mathbb{Z}_+$. Then*

$$\lim_{n\to\infty} \lambda_{\Phi_1,m}(n) = 0, \;\; m \in \mathbb{Z}_+.$$

Proof. In view of Theorem 3.3, we have

$$\begin{aligned}|\lambda_{\Phi_1,m}(n)| \leqslant J_1(n) + J_2(n) &\leqslant B_{\Phi_1}(1-\alpha^2(n)) \\ &+ \frac{C}{\sqrt{n}}\int_0^{\alpha(n)} |\Phi_1(\rho^2)|(1-\rho)^{\frac{1}{4}}\rho d\rho \\ &+ \frac{C}{\sqrt{n}}\int_0^{\alpha(n)} \frac{|\Phi_1(\rho^2)|}{n^{\frac{1}{2}}(1-\rho)^{\frac{1}{2}}}\rho d\rho,\end{aligned}$$

where B_{Φ_1} is given by (3.8). Note that

$$1-\alpha^2(n) = \left(\frac{m}{n}\right)^2 - \frac{1}{4}\left(\frac{m}{n}\right)^4 \leqslant \left(\frac{m}{n}\right)^2, \; B_{\Phi_1}(1-\alpha^2(n)) \leqslant B_{\Phi_1}\left(\left(\frac{m}{n}\right)^2\right).$$

Recall the assumption made in (3.3) and below this formula. We have

$$n^{-\frac{1}{2}}(1-\rho)^{-\frac{1}{2}} \leqslant \frac{C}{\sqrt{\delta n}} \leqslant \frac{C}{\sqrt{M}},$$

and hence,

$$|\lambda_{\Phi_1,m}(n)| \leqslant B_{\Phi_1}\left(\left(\frac{m}{n}\right)^2\right) + \frac{C_1}{\sqrt{n}}\int_0^1 |\Phi_1(t)|dt,$$

and the statement now is obvious. Note that the constant C_1 does not depend on n, m. □

We also note the following corollaries.

Corollary 3.5. *Let $\Phi(w) = \Phi_1(|w|^2)w^m$, where $\Phi_1 \in L^1(0,1)$ and $m \in \mathbb{Z}_+$. Let also*

$$B_{\Phi_1}(s) = O\left(s^{\frac{1}{4}}\right), \;\; s \to 0.$$

Then

$$|\lambda_{\Phi_1,m}(n)| \leqslant \frac{C}{\sqrt{n}}, \;\; n \to \infty, \;\; m \in \mathbb{Z}_+.$$

The constant C does not depend on n, m.

Proof. It follows from the proof of Theorem 3.4. □

Corollary 3.6. *Let $\Phi(w) = \Phi_1(|w|^2)w^m$, where $\Phi_1 \in L^1(0,1)$ and $m \in \mathbb{Z}_+$. Let also*

$$\Phi_1(s) = O((1-s)^{-\alpha}), \;\; s \to 1, \;\; \textit{where} \;\; \alpha \leqslant \frac{3}{4}.$$

Then

$$|\lambda_{\Phi_1,m}(n)| \leqslant \frac{C}{\sqrt{n}}, \; n \to \infty, \; m \in \mathbb{Z}_+.$$

The constant C does not depend on n, m.

Proof. It follows from the proof of Theorem 3.4. □

Theorem 3.7. *Let $\Phi(w) = \Phi_1(|w|^2)w^m$, where $\Phi_1 \in C^1[0,1]$ and $m \in \mathbb{Z}_+$. Then*

$$\lambda_{\Phi_1,m}(n) = O\left(\frac{1}{n^{\frac{3}{2}}}\right), \quad n \to \infty, \; m \in \mathbb{Z}_+.$$

Proof. Let us use Theorem 3.3 and notations in this theorem. We split again $\lambda_{\Phi_1,m}(n) = J_1(n) + J_2(n)$ and we see that $\left(\text{recall that } \alpha(n) = 1 - \frac{1}{2}\left(\frac{M}{n}\right)^2\right)$

$$|J_2(n)| \leqslant \left|\int_{\alpha^2(n)}^{1} t^{\frac{m}{2}}\Phi_1(t)dt\right| = O\left(\frac{1}{n^2}\right), \; n \to \infty, \; m \in \mathbb{Z}_+.$$

For the estimate of $J_1(n)$, let us use the formula of type (3.9) but in the modified form, that is, when in (3.6), we choose $k = 2$:

$$\begin{aligned} J_1(n) = & \frac{C_2}{n^2}\int_0^{\alpha(n)} \frac{\Phi_1(\rho^2)}{\frac{1-\rho}{1+\rho}}\rho^{m+1}d\rho \\ & + 2\sqrt{\frac{2}{\pi n}}\int_0^{\alpha(n)} \frac{\Phi_1(\rho^2)\delta^{\frac{1}{2}}(\rho)}{|\tilde{S}''_{tt}\left(1, \frac{1-\rho}{1+\rho}\right)|^{\frac{1}{2}}}\cos\left\{\delta(\rho)n\tilde{S}\left(1, \frac{1-\rho}{1+\rho}\right) + \frac{\pi}{4}\right\}\rho^{m+1}d\rho \\ & + O\left(\frac{1}{n^{\frac{3}{2}}}\int_0^{\alpha(n)} \frac{\Phi_1(\rho^2)}{\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}}}\rho^{m+1}d\rho\right) \equiv J_{1,0} + J_{1,1}(n) + J_{1,2}(n). \end{aligned}$$

We have

$$\begin{aligned} J_{1,2}(n) = & O\left(\frac{1}{n^{\frac{3}{2}}}\int_0^1 \frac{\Phi_1(\tau)}{\left(\frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right)^{\frac{1}{2}}}\tau^{\frac{m}{2}}dt\right) = O\left(\frac{1}{n^{\frac{3}{2}}}\int_0^1 \frac{\Phi_1(\tau)}{\left(1-\sqrt{\tau}\right)^{\frac{1}{2}}}\tau^{\frac{m}{2}}d\tau\right) \\ & = O\left(\frac{1}{n^{\frac{3}{2}}}\right), \; n \to \infty, \; m \in \mathbb{Z}_+. \end{aligned}$$

Passing to the estimate of $J_{1,1}(n)$, we have:

$$\begin{aligned} J_{1,1}(n) = & \frac{\sqrt{2}}{\sqrt{\pi n}}\int_0^{\alpha^2(n)} \frac{\Phi_1(\tau)\delta^{\frac{1}{2}}\left(\sqrt{\tau}\right)}{|\tilde{S}''_{tt}\left(1, \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right)|^{\frac{1}{2}}}\cos\left\{\delta\left(\sqrt{\tau}\right)n\tilde{S}\left(1, \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right) + \frac{\pi}{4}\right\}\tau^{m/2}d\tau \\ = & \frac{\sqrt{2}}{\sqrt{\pi}\,n^{\frac{3}{2}}}\int_0^{\alpha^2(n)} F_1(\tau)F_2(\tau)\tau^{\frac{m}{2}}dF_3(\tau), \end{aligned}$$

where

$$F_1(\tau) = \frac{\Phi_1(\tau)\delta^{\frac{1}{2}}\left(\sqrt{\tau}\right)}{|\tilde{S}''_{tt}\left(1, \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right)|^{\frac{1}{2}}}$$

$$F_2(\tau) = \left(\frac{d}{d\tau}\delta\left(\sqrt{\tau}\right)\tilde{S}\left(1, \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right)\right)^{-1}$$

$$F_3(\tau) = \sin\left\{\delta\left(\sqrt{\tau}\right)n\tilde{S}\left(1, \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right) + \frac{\pi}{4}\right\}.$$

The functions F_1 and F_2, as functions of τ, are $C^1(0,1)$ and at the endpoints of the interval $(0,1)$, we have the following representations:

$$F_1(\tau) = F_1(0) + C_1\sqrt{\tau} + O(\tau), \quad \tau \to 0,$$

$$F_1(\tau) = C_2\left(1-\sqrt{\tau}\right)^{\frac{1}{2}} + O\left(\left(1-\sqrt{\tau}\right)^{\frac{3}{4}}\right), \quad \tau \to 1,$$

$$F_2(\tau) = C_3\sqrt{\tau} + O(\tau), \quad \tau \to 0,$$

$$F_2(\tau) = C_4\left(1-\sqrt{\tau}\right)^{\frac{1}{2}} + O(\left(1-\sqrt{\tau}\right), \quad \tau \to 1,$$

for every $m \in \mathbb{Z}_+$, where the constants $C_1, \ldots, C_4$ do not depend on n and τ. Hence,

$$J_{1,1}(n) = \frac{\sqrt{2}}{\sqrt{\pi}n^{\frac{3}{2}}}\Big[F_1(\alpha^2(n))F_2(\alpha^2(n))\alpha^m(n)F_3(\alpha^2(n)) - F_1(0)F_2(0)\tau^{\frac{m}{2}}\Big|_{\tau=0}F_3(0) - \int_0^{\alpha^2(n)}\left(\frac{d}{d\tau}F_1(\tau)F_2(\tau)\tau^{\frac{m}{2}}\right)F_3(\tau)d\tau\Big].$$

In view of the abovementioned relations, the terms outside the integral are bounded. It suffices to note that

$$\frac{d}{d\tau}F_1(\tau)F_2(\tau)\tau^{\frac{m}{2}} = O\left(\frac{1}{\sqrt{\tau}}\right), \quad \tau \to 0,$$

$$\frac{d}{d\tau}F_1(\tau)F_2(\tau)\tau^{\frac{m}{2}} = O(1), \quad \tau \to 1.$$

Hence,

$$|J_{1,1}(n)| \leqslant \frac{C}{n^{\frac{3}{2}}}, \quad n \to \infty, \quad m \in \mathbb{Z}_+,$$

where the constant C does not depend on n.

Regarding the term $J_{1,0}(n)$, we have

$$|J_{1,0}(n)| \leqslant \frac{2C_2}{n^2}\int_0^{\alpha^2(n)}\frac{|\Phi_1(\tau)|}{1-\sqrt{\tau}}\tau^{\frac{m}{2}}d\tau \leqslant \frac{C}{n^2}\int_0^{\alpha^2(n)}\frac{d\tau}{1-\tau} = O\left(\frac{\ln n}{n^2}\right), \quad n \to \infty, \quad m \in \mathbb{Z}_+,$$

where the constant C does not depend on n.

Gathering all the above made estimates, we arrive to the desired conclusion. □

In fact, the same result can be obtained under some weakening of the conditions, namely, under differentiability inside the interval and local Hölder property at point 1.

Theorem 3.8. *Let $\Phi(w) = \Phi_1(|w|^2)w^m$, $m \in \mathbb{Z}_+$. Let the value $\Phi_1(1)$ exist and $\Phi_1(s) - \Psi(1) = (1-s)^\mu \Psi(s)$, $\mu \in (0,1)$, for all $s \in [0,1]$, where $\Psi \in C^1[0,1]$. Then*

$$\lambda_{\Phi_1,m}(n) = O\left(\frac{1}{n^{\frac{3}{2}}}\right), \quad n \to \infty, \quad m \in \mathbb{Z}_+.$$

Proof. In the representation (3.1) for the numbers $\lambda_{\Phi_1,m}(n)$, let us write

$$\begin{aligned}\lambda_{\Phi_1,m}(n) &= \frac{1}{\pi}\int_0^1 \Phi_1(\rho^2)\rho^{m+1} I_{n,m}(\rho)d\rho \\ &= \frac{1}{\pi}\int_0^1 (\Phi_1(\rho^2) - \Phi(1))\rho^{m+1} I_{n,m}(\rho)d\rho \\ &+ \frac{\Phi_1(1)}{\pi}\int_0^1 \rho^{m+1} I_{n,m}(\rho)d\rho.\end{aligned}$$

To the second integral, we apply Theorem 3.7 with $\Phi_1 \equiv 1$, so it behaves at least as $O\left(n^{-\frac{3}{2}}\right)$ when $n \to \infty$.

Regarding the first integral, let us denote $\tilde{\Phi}_1(s) = \Phi_1(s) - \Phi_1(1)$; therefore, we need to deal with the asymptotic of the numbers $\lambda_{\tilde{\Phi}_1,m}(n)$, when $n \to \infty$.

Let us use Theorem 3.3 and notations in this theorem. As in Theorem 3.3, we split again $\lambda_{\tilde{\Phi}_1,m}(n) = J_1(n) + J_2(n)$. But now, we keep in mind that now the elements $J_1(n)$ and $J_2(n)$ are constructed with the use of the function $\tilde{\Phi}_1$.

Let us estimate $J_2(n)$. Since $\tilde{\Phi}_1$ is locally Hölder at $t = 1$, we have $\left(\text{recall that } \alpha(n) = 1 - \frac{1}{2}\left(\frac{M}{n}\right)^2\right)$ the following:

$$\begin{aligned}|J_2(n)| &\leqslant \left|\int_{\alpha^2(n)}^1 t^{\frac{m}{2}}\tilde{\Phi}_1(t)dt\right| \\ &\leqslant C\int_{\alpha^2(n)}^1 (1-t)^\mu dt = O\left(\frac{1}{n^{2(\mu+1)}}\right), \quad n \to \infty, \quad m \in \mathbb{Z}_+.\end{aligned}$$

For the estimate of $J_1(n)$, as in Theorem 3.7, let us use the formula of type (3.9) but in the modified form, that is, when in (3.6), we choose $k = 2$:

$$\begin{aligned}J_1(n) &= \frac{C_2}{n^2}\int_0^{\alpha(n)} \frac{\tilde{\Phi}_1(\rho^2)}{\frac{1-\rho}{1+\rho}}\rho^{m+1}d\rho \\ &+ 2\sqrt{\frac{2}{\pi n}}\int_0^{\alpha(n)} \frac{\tilde{\Phi}_1(\rho^2)\delta^{\frac{1}{2}}(\rho)}{|\tilde{S}''_{tt}\left(1, \frac{1-\rho}{1+\rho}\right)|^{\frac{1}{2}}}\cos\left\{\delta(\rho)n\tilde{S}\left(1, \frac{1-\rho}{1+\rho}\right) + \frac{\pi}{4}\right\}\rho^{m+1}d\rho \\ &+ O\left(\frac{1}{n^{\frac{3}{2}}}\int_0^{\alpha(n)} \frac{\tilde{\Phi}_1(\rho^2)}{\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}}}\rho^{m+1}d\rho\right) \equiv J_{1,0} + J_{1,1}(n) + J_{1,2}(n).\end{aligned}$$

It can be easily seen that

$$\begin{aligned}J_{1,0}(n) &= O\left(\frac{1}{n^2}\right), \quad n \to \infty, \quad m \in \mathbb{Z}_+, \\ J_{1,2}(n) &= O\left(\frac{1}{n^{\frac{3}{2}}}\right), \quad n \to \infty, \quad m \in \mathbb{Z}_+.\end{aligned}$$

It remains to estimate the middle term $J_{1,1}(n)$. Let us consider the decomposition of unity $1 = \chi_1 + \chi_2$, $\chi_{1,2} \in C^\infty[0,1]$, such that supp$\chi_1 \in [1-\sigma, \alpha(n)]$ and supp$\chi_2 \in \left[0, 1-\frac{\sigma}{2}\right]$, with $\sigma > 0$ being some sufficiently small fixed number. In accordance with this decomposition, let us split the integral in $J_{1,1}(n)$ into two parts: $J_{1,1}(n) = \mathcal{I}_1(n) + \mathcal{I}_2(n)$.

The term $\mathcal{I}_2(n)$ admits the following representation:

$$\mathcal{I}_2(n) = 2\sqrt{\frac{2}{\pi n}} \int_0^{1-\frac{\sigma}{2}} D_1(\rho) \cos\{nD_2(\rho)\} d\rho,$$

where

$$D_{1,2} \in C^1 \left[0, 1 - \frac{\sigma}{2}\right], \text{ and } D_2'(\rho) \neq 0, \ \rho \in \left[0, 1 - \frac{\sigma}{2}\right].$$

Integrating by parts in the integral for $\mathcal{I}_2(n)$ and neglecting the boundary terms, we obtain the following:

$$|\mathcal{I}_2(n)| \leqslant \frac{C}{n^{\frac{3}{2}}}, \ n \to \infty,$$

where C is independent of n.

Now, considering the integral $\mathcal{I}_1(n)$, we split it again into two parts $\mathcal{I}_1(n) = \mathcal{I}_{1,1}(n) - \mathcal{I}_{1,2}(n)$, where in $\mathcal{I}_{1,1}(n)$, we integrate over $[1-\sigma, 1]$ and in $\mathcal{I}_{1,2}(n)$, we integrate over $[\alpha(n), 1]$. For the second term $\mathcal{I}_{1,2}(n)$, we have

$$\begin{aligned}\mathcal{I}_{1,2}(n) &\leqslant \frac{C}{\sqrt{n}} \int_{\alpha(n)}^1 (1-\rho^2)^{\mu} \delta^{\frac{1}{2}}(\rho) d\rho \leqslant \frac{C_1}{\sqrt{n}} \int_{\alpha(n)}^1 (1-\rho^2)^{\mu+\frac{1}{4}} d\rho \\ &\leqslant \frac{C_2}{n^{\frac{3}{2}+2\mu}}, \ n \to \infty,\end{aligned}$$

where the constants in the above formula do not depend on n.

Finally, let us estimate the first term $\mathcal{I}_{1,1}(n)$. We set

$$E(\rho) = \delta(\rho)\tilde{S}\left(1, \frac{1-\rho}{1+\rho}\right) = C_0(1-\rho)^{\frac{1}{2}} + O(1-\rho), \ \rho \to 1, \tag{3.10}$$

where $C_0 \neq 0$. Making change of the variable $E(\rho) = t$ in the integral for $\mathcal{I}_{1,1}(n)$, we obtain

$$\begin{aligned}\mathcal{I}_{1,1}(n) &= 2\sqrt{\frac{2}{\pi n}} \int_{1-\sigma}^1 \frac{\tilde{\Phi}_1(\rho^2)\delta^{\frac{1}{2}}(\rho)}{|\tilde{S}''_{tt}\left(1, \frac{1-\rho}{1+\rho}\right)|^{\frac{1}{2}}} \cos\left\{nE(\rho) + \frac{\pi}{4}\right\} \chi_1(\rho)\rho^{m+1} d\rho \\ &= -2\sqrt{\frac{2}{\pi n}} \int_0^{\Delta} \frac{\tilde{\Phi}_1((E^{-1}(t))^2)\delta^{\frac{1}{2}}(E^{-1}(t))}{|\tilde{S}''_{tt}\left(1, \frac{1-E^{-1}(t)}{1+E^{-1}(t)}\right)|^{\frac{1}{2}}} \cos\left\{nt + \frac{\pi}{4}\right\} \\ &\quad \times \chi_1(E^{-1}(t))(E^{-1}(t))^{m+1}(E^{-1}(t))' dt,\end{aligned}$$

where $\Delta = E(1-\sigma)$. From (3.10), we observe that

$$E^{-1}(t) = 1 - \frac{t^2}{C_0^2} + O(t^4), \quad t \to 0.$$

Hence,

$$\begin{aligned}\delta^{\frac{1}{2}}(E^{-1}(t)) &= C_1\sqrt{t} + O\left(t^{\frac{3}{2}}\right), \ t \to 0, \\ \tilde{\Phi}_1((E^{-1}(t))^2) &= C_2 t^{2\mu} + O(t^{2\mu+1}), \ t \to 0, \\ (E^{-1}(t))' &= -\frac{2t}{C_0^2} + O(t^3), \ t \to 0.\end{aligned}$$

Therefore, we can represent the integral $\mathcal{I}_{1,1}(n)$ as

$$\mathcal{I}_{1,1}(n) = -2\sqrt{\frac{2}{\pi n}} \int_0^{\Delta} t^{2\mu+\frac{3}{2}} G(t)\chi_1(E^{-1}(t)) \cos\left\{nt + \frac{\pi}{4}\right\} dt,$$

where $G(t) \in C^1[0, \Delta]$. From this in view of Erdélyi,[11] pages 47–49, it follows that

$$\mathcal{I}_{1,1}(n) = \frac{C}{n^{3+2\mu}} + o\left(\frac{C}{n^{3+2\mu}}\right), \quad n \to \infty.$$

This finished the proof. □

4 | REPRESENTATION OF SINGULAR NUMBERS WITH THE USE OF JACOBI POLYNOMIALS

Let us turn again to formula (2.3). In this section, we present a representation for the expression $Q_{m,n}(t)$ in terms of Jacobi polynomials, which, firstly, will allow us to obtain some variant of the previously obtained results, and secondly, and more importantly will open up new possibilities for calculating the numbers $\lambda_{\Phi_1,m}(n)$ for the Hausdorff–Zhu operator with a factorized kernel.

Since $\binom{n-1}{n} = 0$, we have

$$\begin{aligned} Q_{m,n}(t) &= \frac{n}{m+n} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-1}{k} \binom{m+n}{n-k} t^{m+k}(1-t)^{n-k} \\ &= \frac{n}{m+n} t^m \sum_{j=0}^{n} (-1)^{n-j} \binom{n-1}{j} \binom{n+m}{n-j} t^j (1-t)^{n-j}. \end{aligned} \tag{4.1}$$

On the other hand, Jacobi polynomials have the form ($\alpha, \beta \in \mathbb{R}$) (see, e.g., Szegö[12], (4.3.2))

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{1}{2^n} \sum_{j=0}^{n} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} (x-1)^{n-j}(x+1)^j = [x = 1-2t] \\ &= \sum_{j=0}^{n} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} (-1)^{n-j}(1-t)^{n-j} t^j. \end{aligned} \tag{4.2}$$

Thus,

$$Q_{m,n}(t) = \frac{n}{m+n} t^m P_n^{(-1,m)}(1-2t). \tag{4.3}$$

It follows that

$$\lambda_{\Phi_1,m}(n) = \frac{n}{m+n} \int_0^1 \Phi_1(t) t^m P_n^{(-1,m)}(1-2t)dt.$$

Putting here $t = \sin^2 \frac{\theta}{2}$, $\theta \in [0, \pi]$, we get

$$\lambda_{\Phi_1,m}(n) = \frac{n}{m+n} \int_0^{\pi} \Phi_1\left(\sin^2 \frac{\theta}{2}\right) \sin^{2m+1} \frac{\theta}{2} \cos \frac{\theta}{2} P_n^{(-1,m)}(\cos\theta)d\theta. \tag{4.4}$$

Proposition 4.9. *Let $\Phi(w) = \Phi_1(|w|^2)w^m$, where $m \in \mathbb{Z}_+$. Let Φ_1 be supported on a segment $[\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $\Phi_1 \in L^1[\delta, 1-\delta]$. Then*

$$\lambda_{\Phi_1,m}(n) = O\left(n^{-\frac{1}{2}}\right).$$

Proof. Setting $p = 1$ in formula (8.21.12) from Szegö,[12, p. 196] we obtain that for $\theta \in (0, \pi)$ and $m \in \mathbb{R}$

$$P_n^{(-1,m)}(\cos\theta) = O\left(n^{-\frac{1}{2}}\right) \tag{4.5}$$

uniformly for $\theta \in [\varepsilon, \pi - \varepsilon]$ (as it was mentioned in Szegö[12, p. 194] in Section 8.21 of this monograph p is an arbitrary positive integer, ε denotes a fixed number with $0 < \varepsilon < \pi/2$). The rest follows by direct estimate in (4.4). □

As it can be expected, and as we have already seen previously, for some Φ_1, the above asymptotic can be better. We find it important to conclude this section with the following example.

Example 4.10. Let $\Phi(w) = \Phi_1(|w|^2)w^m$, where $m \in \mathbb{Z}_+$. Let $\theta \in (0, \pi/2]$ and

$$\Phi_1(t) = \begin{cases} (1-t)^m t^{-m-1}, & \frac{1-\cos\theta}{2} \leq t \leq \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\lambda_{\Phi_1,m}(n) = O\left(n^{-\frac{3}{2}}\right).$$

Proof. Indeed,

$$\begin{aligned} \lambda_{\Phi_1,m}(n) &= \frac{n}{m+n}\int_0^1 \Phi_1(t)t^m P_n^{(-1,m)}(1-2t)dt \\ &= \frac{n}{m+n}\int_{\frac{1-\cos\theta}{2}}^{\frac{1}{2}} \frac{(1-t)^m}{t^{m+1}} t^m P_n^{(-1,m)}(1-2t)dt = [1-2t=x] \\ &= \frac{n}{m+n}2^{1-m}\int_0^{\cos\theta} (1-x)^{-1}(1+x)^m P_n^{(-1,m)}(x)dx. \end{aligned}$$

But it is known (see, e.g., Luke[13, p. 440, (2)]) that

$$\begin{aligned} &\int_0^{\cos\theta} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x)dx \\ &= \frac{1}{2n}(P_{n-1}^{(\alpha+1,\beta+1)}(0) - (1-\cos\theta)^{\alpha+1}(1+\cos\theta)^{\beta+1}P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)). \end{aligned}$$

Since $P_{n-1}^{(0,m+1)}(\cos\theta) = O(n^{-1/2})$ for fixed $\theta \in (0, \pi/2]$ (see, e.g., Szegö[12, (7.32.6)]), our claim follows. □

5 | APPLICATION TO BOUNDEDNESS AND COMPACTNESS IN SOME SPACES OF ANALYTIC FUNCTION

In this section, based on Theorem 2.2 and the obtained results, we give some sufficient conditions for the boundedness and compactness of operator with factorized kernel $\Phi(w) = \Phi_1(|w|^2)w^m$, $m \in \mathbb{Z}_+$.

We start with classical Bergman and Hardy spaces on the unit disc. We recall that B_{Φ_1} is given by formula (3.8).

Theorem 5.11. *Let $0 < p \leq 2 \leq r < \infty$. Let $\Phi(w) = \Phi_1(|w|^2)w^m$, where $m \in \mathbb{Z}_+$, $\Phi_1 \in L^1(0,1)$. Let $e_n = z^n$, $n \in \mathbb{Z}_+$. Then the following statements are true.*

(1) Let either

$$B_{\Phi_1}(s) = O\left(s^{\frac{1}{4}}\right),$$

or

$$\Phi_1(s) = O((1-s)^{-\frac{3}{4}}).$$

Then for $\frac{1}{2} = 1/p - 1/r$, the operator $\mathcal{H}_\Phi$ extends from the linear span span$\{e_n\}$ of the set $\{e_n\}$ to the bounded operator $T : H^p(\mathbb{D}) \to H^r(\mathbb{D})$. Also, for $\frac{1}{4} = 1/p - 1/r$, the operator $\mathcal{H}_\Phi$ extends from span$\{e_n\}$ to the bounded operator $T : \mathcal{A}^p(\mathbb{D}) \to \mathcal{A}^r(\mathbb{D})$.

(2) Let, additionally to the initial assumption, us assume that Φ_1 is as in Theorem 3.8. Then for $p \leqslant r$, the operator $\mathcal{H}_\Phi$ extends from span$\{e_n\}$ to the bounded operator $T : H^p(\mathbb{D}) \to H^r(\mathbb{D})$, and also, for $\frac{3}{4} = 1/p - 1/r$, the operator $\mathcal{H}_\Phi$ extends from span$\{e_n\}$ to the bounded operator $T : \mathcal{A}^p(\mathbb{D}) \to \mathcal{A}^r(\mathbb{D})$.

Proof. We note that by Theorem 2.2, the operator $\mathcal{H}_\Phi$ on span$\{e_n\}$ can be considered as composition of two operators. The first one is the operator that acts on finite sums as

$$f(z) = \sum_n f_n z^n \mapsto \sum_n \lambda_{\Phi_1,m}(n) f_n z^n,$$

and another one is the operator of multiplication $f \mapsto z^m f$, which is clearly bounded in every Hardy or Bergman space. Therefore, boundedness of the first written operator will imply the same for the operator $\mathcal{H}_\Phi$. Now the statement follows from Corollaries 3.5 and 3.6 and Theorem 3.8, since due to Duren[14] and Duren and Schuster[15, p. 88, Theorem 8] under our conditions, the sequence $\{\lambda_{\Phi_1,m}(n)\}$ is an $(H^p(\mathbb{D}), H^r(\mathbb{D}))$ and $(\mathcal{A}^p(\mathbb{D}), \mathcal{A}^r(\mathbb{D}))$ multiplier, respectively. □

In fact, the proof of the previous theorem shows that we can easily state the fact that our operator is compact in a general Hilbert space.

Theorem 5.12. *Let $\Phi(w) = \Phi_1(|w|^2)w^m$, where $m \in \mathbb{Z}_+$, $\Phi_1 \in L^1(0,1)$. Let $X(\mathbb{D})$ be a Hilbert space of analytic functions on $\mathbb{D}$ with orthogonal basis $(e_n)_{n\in\mathbb{Z}_+}$ (e.g., $X(\mathbb{D}) = \mathcal{D}(\mathbb{D})$, $\mathcal{A}^2(\mathbb{D})$, or $H^2(\mathbb{D})$). Then the operator $\mathcal{H}_\Phi$ is compact on $X(\mathbb{D})$.*

Proof. The proof follows from Theorem 3.4 by the similar arguments to those given in Theorem 5.11. □

Recall (see, e.g., Vinogradov[16]) that the Banach space ℓ^p_A ($p \in [1, \infty]$) consists of functions

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad z \in \mathbb{D},$$

which are analytic in the unit disc and such that

$$\|f\|^p_{\ell^p_A} := \sum_{n=0}^{\infty} |f_n|^p < \infty.$$

In particular, $\ell^2_A = H^2(\mathbb{D})$.

Theorem 2.2 shows that operator $\mathcal{H}_\Phi$ acts in the vector subspace span$\{e_n\}$ of ℓ^p_A and the entries of the matrix $[\mathcal{H}_\Phi] = (a_{jk})$ of this operator with respect to the algebraic basis (e_n) are $a_{jk} = \lambda_{\Phi_1,m}(k-1)\delta_{k+m,j}$. Thus, this matrix has the form (for $m > 1$)

$$[\mathcal{H}_\Phi] = \begin{pmatrix} 0 & \dots & 0 & \dots & \dots \\ \vdots & \ddots & \vdots & \dots & \dots \\ 0 & \dots & 0 & \dots & \dots \\ \lambda_{\Phi_1,m}(0) & 0 & 0 & \dots & \dots \\ 0 & \lambda_{\Phi_1,m}(1) & 0 & \dots & \dots \\ 0 & 0 & \lambda_{\Phi_1,m}(2) & 0 & \dots \\ & & & \ddots & \end{pmatrix}. \tag{5.6}$$

Theorem 5.13. *Let $1 < p < \infty, 1 \le r < \infty$. Let $\Phi(w) = \Phi_1(|w|^2)w^m$, where $m \in \mathbb{Z}_+$, $\Phi_1 \in L^1(0,1)$. Let $e_n = z^n$, $n \in \mathbb{Z}_+$. The following statements hold.*

(i) The operator $\mathcal{H}_\Phi$ extends from span$\{e_n\}$ to a compact operator $K : \ell^p_A \to \ell^r_A$ if $B := \left(\sum_{n=0}^{\infty} |\lambda_{\Phi_1,m}(n)|^r\right)^{1/r} < \infty$. And in this case, $\|K\| \le B$.

(ii) Moreover, under the conditions of Corollary 3.5 or 3.6, statement (i) holds for all $r > 2$, and under the conditions of Theorem 3.8, statement (i) holds for all $r \ge 1$.

Proof. To prove (i), we note that the space ℓ^p_A is isometrically isomorphic to the space ℓ^p, and therefore, one can apply the sufficient condition that an infinite matrix corresponds to a compact operator from Kantorovich and Akilov[17], Corollary 2, p. 322. For our matrix (5.6), this condition looks as $B < \infty$ where

$$B := \left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} |a_{jk}|^q\right)^{r/q}\right)^{1/r} = \left(\sum_{n=0}^{\infty} |\lambda_{\Phi_1,m}(n)|^r\right)^{1/r}$$

$(1/p + 1/q = 1)$. This proves the first statement on the theorem.

To prove (ii), we note that the conditions of Corollary 3.5 or 3.6 $B \le \text{const} \sum_{n=1}^{\infty} n^{-r/2} < \infty$ for $r > 2$, and under the conditions of Theorem 3.8, $B \le \text{const} \sum_{n=1}^{\infty} n^{-3r/2} < \infty$ for $r \ge 1$. □

Now we turn to some related mixed norm spaces of analytic functions in the unit disc. The general theory of such spaces was developed in Karapetyants and Samko,[18] and therefore, we refer to this article to the detailed definitions and proofs; see also papers.[19–22,24]

Given an analytic function on the unit disc f its Fourier coefficients,

$$f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\sigma})e^{-in\sigma}d\sigma, \quad n \in \mathbb{N}_+$$

exist for all $r \in (0,1)$. Let $X \subseteq L^1(0,1)$ denote a Banach space of functions on the interval $(0,1)$ containing step functions, and let $\|\cdot\|_X$ stand for the norm. Recall that the mixed norm Bergman space $\mathcal{A}^{q;X}(\mathbb{D})$, $1 \leqslant q < \infty$, is defined to consist of functions analytic in $\mathbb{D}$ such that the following norm is finite:

$$\|f\|_{\mathcal{A}^{q;X}(\mathbb{D})} = \left(\sum_{n=0}^{\infty}\|f_n\|_X^q\right)^{\frac{1}{q}}.$$

Given a function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in the Bergman space $\mathcal{A}^{q;X}(\mathbb{D})$, its Fourier coefficients $f_n = f_n(r), n \in \mathbb{Z}$, can be represented as

$$f_n(r) = c_n r^n \equiv a_n \|r^n\|_X^{-1} r^n, \quad n \in \mathbb{Z}_+, \tag{5.7}$$

where

$$(a_n)_{n\in\mathbb{Z}_+} \in \ell^q, \quad |a_n| = \|f_n\|_X, \quad n \in \mathbb{Z}_+;$$

moreover,

$$\|f\|_{\mathcal{A}^{q;X}(\mathbb{D})} = \|\{a_n\}_{n\in\mathbb{Z}_+}\|_{\ell^q}.$$

To show a certain connection with the above case, let us consider the following Hadamard product composition operator $\mathcal{D}_X$ which acts on analytic functions $f(z) = \sum_{n=0}^{\infty} c_n z^n$ on the unit disc $\mathbb{D}$ as

$$\mathcal{D}_X f(z) = \sum_{n=0}^{\infty}\|r^n\|_X^{-1} c_n z^n.$$

We refer to Samko et al.[25] (Section 22) for the theory of fractional integrodifferentiation of analytic functions.

Clearly, to stay within the space of all analytic functions in the unit disc, it suffices to impose the following condition

$$\limsup_{n\to\infty}\left(\|r^n\|_X^{-1}\right)^{\frac{1}{n}} \leqslant 1,$$

which we will assume to be satisfied from now on.

Remark 2. The space $\mathcal{A}^{q;X}(\mathbb{D})$ is identified with the space of all analytic functions of the form $f = \mathcal{D}_X g$, where $g \in \ell^q_A$. In particular, the space $\mathcal{A}^{2;X}(\mathbb{D})$ coincides with the range of the operator $\mathcal{D}_X$ over the Hardy space $H^2(\mathbb{D})$: $\mathcal{A}^{2;X}(\mathbb{D}) = \mathcal{D}_X(H^2(\mathbb{D}))$.

Note that in the general case the behavior of numbers $\|r^n\|_X$, when n goes to infinity is of course unknown to us. But in a number of spaces of harmonic analysis, such behavior has a specific description. For example, for the case of Orlicz space $X = L^{\Phi}(0,1)$, the numbers $\|r^n\|_X$ behave as $O\left(\frac{1}{\Phi^{-1}(n)}\right)$, $n \to \infty$, where Φ^{-1} is the inverse function to the Young function Φ; for the case of Morrey space $X = L^{p,\varphi}(0,1)$, the corresponding behavior is as follows: $O\left(\left(n\varphi\left(\frac{1}{n}\right)\right)^{-\frac{1}{p}}\right)$; for the case of grand and small Lebesgue spaces $X = L^{p)}(0,1)$ and $X = L^{(p}(0,1)$, we have $O\left(n^{-\frac{1}{p}}\ln^{-\frac{1}{p}} n\right)$ and $O\left(n^{-\frac{1}{p}}\ln^{1-\frac{1}{p}} n\right)$, respectively. We refer to the already mentioned papers[18–22,24] for the precise statements and details.

Similar to the above case, we observe the following statement.

Theorem 5.14. *Let* $1 < p < \infty, 1 \leq r < \infty$. *Let* $\Phi(w) = \Phi_1(|w|^2)w^m$, *where* $m \in \mathbb{Z}_+$, $\Phi_1 \in L^1(0,1)$. *Let* $e_n = z^n$, $n \in \mathbb{Z}_+$. *Let* $X \subseteq L^1(0,1)$ *be a Banach space of functions on the interval* $(0,1)$ *containing step functions. Let also X be ideal space and lim* $\sup_{n\to\infty}\left(\|r^n\|_X^{-1}\right)^{\frac{1}{n}} \leqslant 1$. *The following statements hold.*

(i) The operator $\mathcal{H}_\Phi$ extends from span$\{e_n\}$ *to a compact operator* $K : \mathcal{A}^{p;X}(\mathbb{D}) \to \mathcal{A}^{r;X}(\mathbb{D})$ *if* $B := \left(\sum_{n=0}^{\infty} |\lambda_{\Phi_1,m}(n)|^r\right)^{1/r} < \infty$. *And in this case,* $\|K\| \leq B$.

(ii) Moreover, under the conditions of Corollary 3.5 or 3.6, statement (i) holds for all $r > 2$, *and under the conditions of Theorem 3.8, statement (i) holds for all* $r \geq 1$.

Proof. Since X is an ideal space containing step functions, $r^n \leqslant 1$, and r^n is measurable on $(0, 1)$ for every $n \in \mathbb{Z}_+$, then $r^n \in X$. Moreover, $\|r^{n+m}\|_X \leqslant \|r^n\|_X$, $n, m \in \mathbb{Z}_+$.

Theorem 2.2 shows that the operator $\mathcal{H}_\Phi$ acts in the vector subspace span$\{e_n\}$ of $\mathcal{A}^{p;X}(\mathbb{D})$ and the entries of the matrix $[\mathcal{H}_\Phi] = (a_{jk})$ of this operator with respect to the algebraic basis (e_n) are

$$a_{jk} = \lambda_{\Phi_1,m}(k-1)\frac{\|r^{n+m}\|_X}{\|r^n\|_X}\delta_{k+m,j}.$$

Now we note that the space $\mathcal{A}^{q;X}(\mathbb{D})$ is isomorphic to ℓ^q_A; indeed, since the sequence $\|r^n\|_X^{-1}$ of positive numbers is nondecreasing and bounded, Remark 3 shows that the map $\mathcal{D}_X$ is an isomorphism of $\mathcal{A}^{q;X}(\mathbb{D})$ and ℓ^q_A. Therefore, we can proceed similarly as in the proof of Theorem 5.13, and we leave details to an interested reader. □

6 | CONCLUSION

As is well-known, the Berezin transform plays an important role in complex analysis and operator theory (see, e.g., earlier studies[26–28]). The Hausdorff–Zhu operators arose as some conjugation to the Berezin transformation. Namely, the operator $\mathcal{H}_1$ from (1.2) differs from the Berezin transformation in that the variables in the Möbius transformation are swapped.

Despite the seemingly insignificant replacement, as it was observed by Kehe Zhu[7] that these operators differ spectacularly from each other. The Berezin transform preserves bounded analytic functions, which can be seen via reproducing property of the Bergman kernel function, while operator (1.2) maps every bounded analytic function f into $f_0 - \frac{f_1}{2}z$ (see Theorem 6 from Zhu[7]), that is, $\mathcal{H}_1$ in $H^\infty(\mathbb{D})$ is two dimensional.

However, such a "small" operator still presents an interest in its own. Moreover, we consider the more general case, the class of operators (1.1), and the presence of the kernel Φ of the operator that significantly changes the picture; see Karapetyants and Mirotin.[1] Depending on the choice of the kernel, the operator $\mathcal{H}_\Phi$ possesses some very interesting properties.

The applied significance of this article is that we managed to show the behavior of the singular values of the operator, which makes it possible to definitely advance in understanding the boundedness and compactness of this operator on a number of spaces of analytic functions and also opens up opportunities for further study of the properties of such operators. Separately, we note a technical but essential detail about the integral representation noted in the article using Jacobi polynomials, which undoubtedly also expands such possibilities.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflict of interest.

DATA AVAILABILITY STATEMENT

The authors confirm that all data generated or analyzed during this study are included in this article.

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