The Wiener-Hopf integral equation on a finite interval: asymptotic solution for large intervals with an application to acoustics

S.M.Grudsky, S.S.Mikhalkovich,
E. Ramires de Arelano

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$$(W_L(a)f)(x) = \int_0^L K(x-t)f(t)dt, \quad x \in (0, L)$$

$$a(\mu) = (\mathcal{F}K)(\mu) \quad \text{— symbol}$$

I

$$a(\mu) = \frac{\sqrt{1-\mu^2}}{\nu + \sqrt{1-\mu^2}}, \quad \mu_1 = 1, \quad \mu_2 = -1$$

A problem of sound propagation in an air half-space over the earth surface with motorway

II

$$a(\mu) = 1 + c\sqrt{1-\mu^2}, \quad c \in \mathbb{R}, \quad |a(\mu)| \sim c|\mu|$$

A problem of sound propagation in the water with the surface partially covered by ice

III

$$a(\mu) = -ic\mu + c_1[2 - (1+i\mu)^\nu - (1-i\mu)^\nu], \quad \nu \in [0, 2)$$

The theory of barrier options

$$\mathrm{Re}\,(Af, f) \geq 0, \quad \forall f \in H \quad \Longleftrightarrow \quad A \text{ is semisectorial}$$

$$\varepsilon = \inf_{\|f\|=1} \mathrm{Re}(Af, f)$$

$$\varepsilon > 0 \quad \Longleftrightarrow \quad A \text{ is sectorial}$$

$$a(\mu) \in L_\infty(\mathbb{R}), \quad (\chi_{(0,L)} f)(x) = \begin{cases} f(x), & x \in (0,L); \\ 0, & x \in \mathbb{R} \setminus (0,L) \end{cases}$$

$$W_L(a) = \chi_{(0,L)} \mathcal{F}^{-1} a \mathcal{F}\Big|_{L_2(0,L)} : L_2(0,L) \to L_2(0,L) \tag{1}$$

$\mathcal{F}$ is Fourier transform

$$\widehat{W}_L(a) = \mathcal{F} W_L(a) \mathcal{F}^{-1} : E_{2,L} \to E_{2,L} \tag{2}$$

$$\Phi_L^+(\mu) \in E_{2,L} \Longleftrightarrow \Phi_L^+(\mu) = \int_0^L f(x) e^{i\mu x} dx, \quad f(x) \in L_2(0,L)$$

$$E_{2,L} \subset H_2$$

$$\Phi^+(\mu) \in H_2 \Longleftrightarrow \Phi^+(\mu) = \int_0^\infty f(x) e^{i\mu x} dx, \quad f(x) \in L_2(0,\infty)$$

$$\widehat{W}_L(a) = P_L a P_L \tag{3}$$

$$P_L = \mathcal{F} \chi_{(0,L)} \mathcal{F}^{-1} : L_2(\mathbb{R}) \to E_{2,L}$$

$P_L^2 = P_L$ is projector

$$(\widehat{W}_L(a) X_L^+)(\mu) = (P_L a X_L^+)(\mu) = \Phi_L^+(\mu) \tag{4}$$

$$X_L^+(\mu), \Phi_L^+(\mu) \subset E_{2,L}$$

$$H = L_2(0,L) \quad \Longleftrightarrow \quad (f,g)_H = \int_0^L f(x)\overline{g}(x)dx$$

$$\begin{aligned}(W_L(a)f,f)_H &= (\chi_{(0,L)}\mathcal{F}^{-1}a\mathcal{F}f,f)_H = (\mathcal{F}^{-1}a\mathcal{F}f,f)_{L_2(\mathbb{R})} \\ &= (a\mathcal{F}f,\mathcal{F}f)_{L_2(\mathbb{R})} = \int_{-\infty}^{\infty} a(\mu)|(\mathcal{F}f)(\mu)|^2 d\mu\end{aligned}$$

$$\mathcal{F} = \mathcal{F}^*$$

$$\operatorname{Re} a \geq 0 \quad \Rightarrow \quad W_L(a) \text{ is semisectorial}$$

$$\operatorname{Re} a \geq \varepsilon > 0 \Rightarrow W_L(a) \text{ is sectorial}$$

$$\operatorname{Re}(W_L(a)f,f) \geq \varepsilon(\mathcal{F}f,\mathcal{F}f) = \varepsilon(f,f)$$

Theorem 1 (Brown and Halmos). *Let* $a \in L_\infty(\mathbb{R})$ *and suppose that* $\operatorname{Re} a \geq \varepsilon > 0$. *Then* $W_L(a)$ *is invertible and for all* $L > 0$ *we have*

$$\|W_L^{-1}(a)\| \leq \frac{1}{\varepsilon}\left(1 + \sqrt{1 - \frac{\varepsilon^2}{\|a\|_\infty^2}}\right) \leq \frac{2}{\varepsilon}$$

where

$$\|a\|_\infty = \operatorname{ess\,sup}_{\mu\in\mathbb{R}} |a(\mu)|.$$

Theorem 2 (A. Böttcher, 1994). *Let*

$$W_\infty(a) = \chi_{(0,\infty)}\mathcal{F}^{-1}a\mathcal{F}\big|_{L_2(0,\infty)} : L_2(0,\infty) \to L_2(0,\infty)$$

be a convolution operator on semiaxis and symbol $a(\mu)$ *is piecewise continuous on* $\dot{\mathbb{R}}$ *function such that* $W_\infty(a)$ *is invertible. Then for* L *large enough* $W_L(a)$ *is invertible and*

$$\lim_{L\to\infty} \|W_L^{-1}(a)\| = \|W_\infty^{-1}\|.$$

Semisectoriality

$$\tilde{a}(\mu) = a(\mu) + e^{iL\mu}h^+(\mu) + e^{-iL\mu}h^-(\mu)$$

$$h^+(\mu) \in H_2, \qquad h^-(\mu) \in \overline{H}_2$$

$$e^{iL\mu}h^+(\mu) = \int_L^\infty f_+(x)e^{i\mu x}dx$$

$$e^{-iL\mu}h^-(\mu) = \int_{-\infty}^{-L} f_-(x)e^{i\mu x}dx$$

$$\Downarrow$$

$$W_L(a) = W_L(\tilde{a})$$

If $\tilde{a}$ is sectorial $\Rightarrow W_L(a)$ is invertible

$$u(\mu) := \operatorname{Re} a(\mu), \qquad v := \operatorname{Im} a(\mu)$$

DF. $\mu_j \in \mathbb{R}$ is said to be a (finite) zero of u if $\operatorname{ess\,inf}\{u(\mu) : |\mu - \mu_j| < \delta\} = 0$ for each $\delta > 0$.

$$\frac{1}{w_j(L)} := \operatorname{ess\,inf}\{u(\mu) : 1/L < |\mu - \mu_j| < \delta\}$$

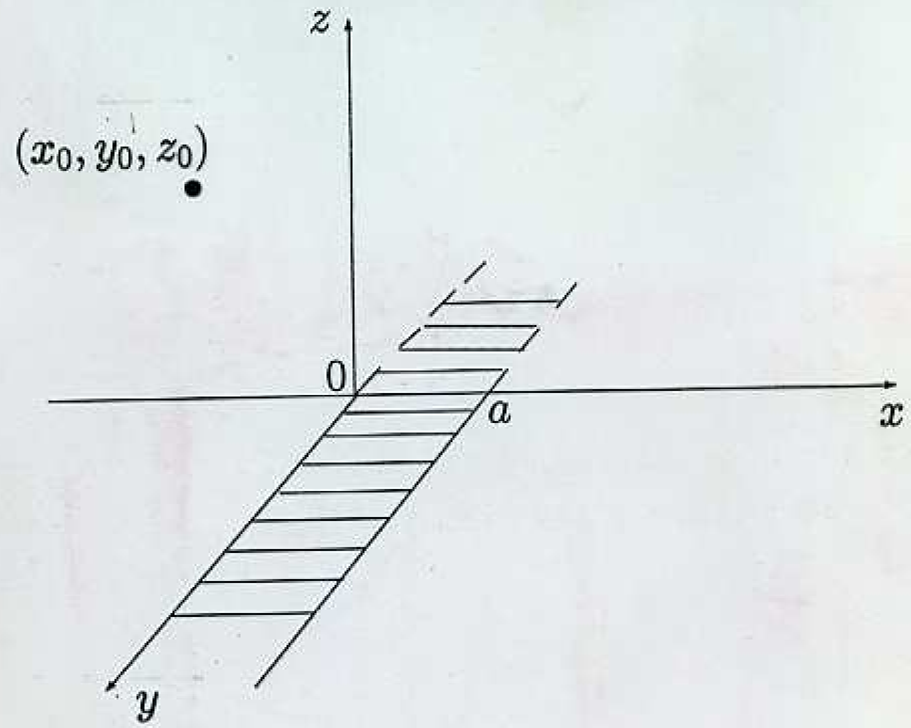
$$\lim_{L\to\infty} w_j(L) = \infty$$

Upper estimates

Theorem 3. *Let $a \in L_\infty(\mathbb{R})$. Suppose $u \geq 0$ a.e. and u has exactly $n \geq 1$ finite zeros $\mu_1, \ldots, \mu_n$ on $\mathbb{R}$. Put $w(L) = \max(w_1(L), \ldots, w_N(L))$. Then for all L large enough the operator $W_L(a)$ is invertible and*

$$\|W_L^{-1}(a)\| \leq 8(\|v\|_\infty + 1)w(9L/(2\pi)).$$

Sound propagation in an air half-space with motorway



$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = -\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \qquad (5)$$

$$\frac{\partial p}{\partial z}(x, y, 0) = 0, \quad (x, y) \in (0, a) \times \mathbb{R} \quad \text{— motorway} \qquad (6)$$

$$\frac{\partial p}{\partial z}(x, y, 0) + \nu p(x, y, 0) = 0, \quad (x, y) \in (\mathbb{R} \setminus (0, a)) \times \mathbb{R} \quad \text{— ground} \qquad (7)$$

$$\operatorname{Re} \nu > 0$$

$$k^2 = k_0^2 + i\varepsilon \quad (k_0 > 0, \varepsilon > 0) \quad \text{— wave number}$$

LAP (Limit Absorption Principle) (8)

$$\varepsilon > 0 \quad \Rightarrow \quad p_\varepsilon(x, y, z)$$

$$\lim_{\varepsilon \to 0} p_\varepsilon(x, y, z) = p_0(x, y, z)$$

$$
\begin{aligned}
p(x,y,z) &= p_\delta(x,y,z) \\
&+ \frac{k_0^2 \nu}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{X_L^+(\mu,\beta)}{\nu + \gamma(\mu,\beta)} \exp(-ik_0(\mu x + \beta y - \gamma(\mu,\beta) z))\, d\mu d\beta
\end{aligned} \tag{9}
$$

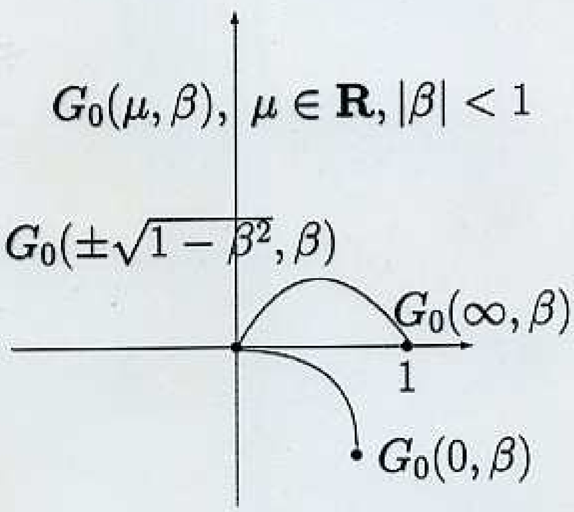
$$
\gamma(\mu,\beta) = \sqrt{n^2 - \mu^2 - \beta^2}, \quad L = k_0 a, \quad n^2 = \frac{k^2}{k_0^2} = 1 + i\frac{\varepsilon}{k_0^2}
$$

$$
(\widehat{W}(G_\varepsilon) X_{\varepsilon,L}^+)(\mu,\beta) = \Phi_L(\mu,\beta) \tag{10}
$$

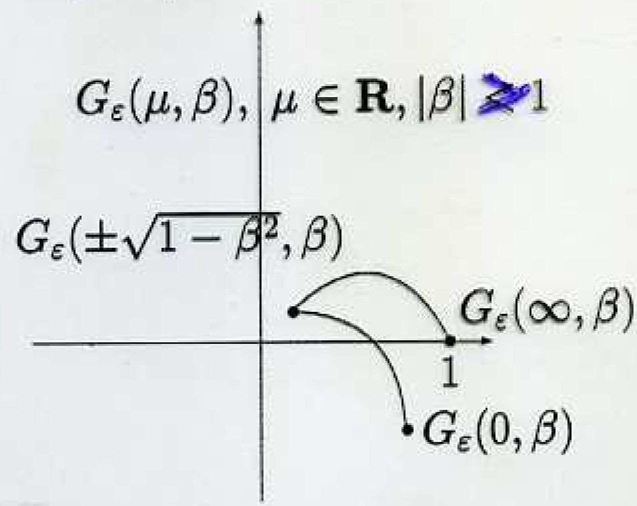
$$
\Phi_L(\mu,\beta) = P_L \left\{ \frac{1}{2\pi i k_0} \cdot \frac{\exp(ik_0(\mu x_0 + \gamma(\mu,\beta) z_0))}{\nu + \gamma(\mu,\beta)} \right\}
$$

$$
G_\varepsilon(\mu,\beta) = \frac{\gamma(\mu,\beta)}{\nu + \gamma(\mu,\beta)} \quad \text{— symbol} \tag{11}
$$

$$
G_0(\mu,0) = \frac{\sqrt{1-\mu^2}}{\nu + \sqrt{1-\mu^2}}
$$



semisectorial



sectorial

Theorem 6. *Let $L \in (0, \infty)$ and $\varepsilon \in [0, \varepsilon_0]$ where $\varepsilon_0 > 0$ and small enough. Then equation (10) has a unique solution in the space $E_{2,L}$ for every $\beta \in \mathbb{R}$. Moreover for each $L \in (0, \infty)$ there exists an indepentent of $\varepsilon \in [0, \varepsilon_0]$ constant M_L such that*

$$\|X^+_{\varepsilon,L}(\mu, \beta)\|_{L_2(\mathbb{R}^2)} \le M_L$$

and

$$\lim_{\varepsilon \to 0} \|X^+_{\varepsilon,L}(\mu, \beta) - X^+_{0,L}(\mu, \beta)\|_{L_2(\mathbb{R}^2)} = 0.$$

DF. $f(x, y, z) \in ML_2 \iff$

1. $f(x, y, z) \in C^2(\mathbb{R}^2 \times (0, \infty))$;
2. $\xi(z), \xi_1(z) \in C[0, \infty]$

$$\xi(z) := \|f(\cdot, \cdot, z)\|_{L_2(\mathbb{R}^2)}, \quad \xi_1(z) := \left\|\frac{\partial f}{\partial z}(\cdot, \cdot, z)\right\|_{L_2(\mathbb{R}^2)},$$

and

$$\lim_{z \to 0} \|f(\cdot, \cdot, z) - f(\cdot, \cdot, 0)\|_{L_2(\mathbb{R}^2)} = 0,$$
$$\lim_{z \to 0} \left\|\frac{\partial f}{\partial z}(\cdot, \cdot, z) - \frac{\partial f}{\partial z}(\cdot, \cdot, 0)\right\|_{L_2(\mathbb{R}^2)} = 0;$$

3. $\lim_{z \to \infty} \xi(z) = 0$, $\lim_{z \to \infty} \xi_1(z) = 0$.

Justification of LAP

Theorem 7. *The problem (5)–(8) for arbitrary* $\varepsilon > 0$ *has a unique solution of the form (9)*

$$p_\varepsilon(x, y, z) = p_{\varepsilon,\delta}(x, y, z) + \varphi_\varepsilon(x, y, z)$$

where $\varphi_\varepsilon(x, y) \in ML_2$ *and for every* $(x, y, z) \in \mathbb{R}^2 \times (0, \infty)$ *there exist*

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon(x, y, z) = \varphi_0(x, y, z),$$
$$\lim_{\varepsilon \to 0} p_{\varepsilon,\delta}(x, y, z) = p_0(x, y, z).$$

1. Along the motorway

Theorem 8. *Let the coordinates* (x_0, y_0, z_0) *of a source be fixed and fix the coordinates* x *and* z *of the receiver* (x, y, z). *Then*

$$\varphi_0(x, y, z) = c\frac{e^{ik_0(y-y_0)}}{(y-y_0)^2} + o\left(\frac{1}{(y-y_0)^2}\right) \quad as \quad |y| \to \infty$$

where

$$c = \frac{1 - ik_0 z\nu}{\nu} X^{+}_{0,L}(0,1) + 2i\nu c_0 b_0.$$

2. Outside the motorway

Theorem 9. *Let* $\psi \in (-\pi, \pi] \setminus \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$. *Then for fixed* z *and* (x_0, y_0, z_0) *we have*

$$\varphi_0(R\cos\psi, R\sin\psi) = \frac{1 - ik_0 z\nu}{\nu} X^{+}_{0,L}(-\cos\psi, -\sin\psi)\frac{e^{ik_0 R}}{R^2} + o\left(\frac{1}{R^2}\right)$$

where $R = \sqrt{x^2 + y^2} \to \infty$.

Asymptotics by $L \to \infty$
Two-dimensional problem

$$\frac{\partial^2 p(x,z)}{\partial x^2} + \frac{\partial^2 p(x,z)}{\partial z^2} + k^2 p(x,z) = -\delta(x - x_0)\delta(z - z_0), \qquad (12)$$

$$(x,z) \in \mathbb{R} \times (0,\infty)$$

$$\frac{\partial p}{\partial z}(x,0) = 0, \quad x \in (0,a) \qquad (13)$$

$$\frac{\partial p}{\partial z}(x,0) + i\nu p(x,0) = 0, \quad x \in \mathbb{R} \setminus (0,a) \qquad (14)$$

$$\mathrm{Re}\ \nu > 0 \qquad (15)$$

$$\text{LAP} \qquad (16)$$

$$p(x,z) = p_\delta(x,z) + \varphi_L(x,z)$$

$$\varphi_L(x,z) = \frac{k_0 \nu}{2\pi} \int_{\mathbb{R}} \frac{X_L^+(\mu)}{\nu + \gamma(\mu)} e^{-ik(\mu x - \gamma(\mu) z)} d\mu \qquad (17)$$

$$\gamma(\mu) = \sqrt{1 - \mu^2}$$

$$(\widehat{W}_L G)(\mu) X_L^+(\mu) = (P_L G)(\mu) X_L^+(\mu) = f_L(\mu) \qquad (18)$$

$$G(\mu) = \frac{\gamma(\mu)}{\nu + \gamma(\mu)}, \quad f_L(\mu) = P_L \left(\frac{e^{ik_0(x_0\mu + z_0\gamma(\mu))}}{2\pi k(\nu + \gamma(\mu))} \right)$$

$$X_L^+(\mu) \in E_{2,L,\varrho_s}$$

$$\varrho_s(\mu) = (|\mu| + 1)^{-2s} |1 - \mu|^s |1 + \mu|^s, \quad s \in (-1,1)$$

$$P^+ = P_{(0,\infty)}, \quad T(G) = P^+GP^+$$

$$J := J(f(x)) = f(-x), \quad W_L = e^{iLx}JP_L$$

$$G(\mu) = G_+(\mu)G_-(\mu) \tag{19}$$

$$G_+(\mu) = \frac{\sqrt{1+\mu}}{a_+(\mu)}, \quad G_-(\mu) = \frac{\sqrt{1-\mu}}{a_-(\mu)}$$

$$\nu + \sqrt{1-\mu^2} = a_+(\mu)a_-(\mu)$$

$$T^{-1}(G) = G_+^{-1}P^+G_-^{-1} : P^+L_2(\mathbb{R}, \varrho_{s+1}) \to P^+L_2(\mathbb{R}, \varrho_s)$$

$$K(G) = T^{-1}(G) - T(G^{-1})$$

$$B_L(G) = P_LT^{-1}(G)P_L + W_L(K(G))W_L \tag{20}$$

Theorem 10. *Let $\varepsilon = 0$. Then for arbitrary $L > 0$ operator $\widehat{W}_L(G) = P_LGP_L : L_2(\mathbb{R}, \varrho_s) \to L_2(\mathbb{R}, \varrho_s)$, $s \in (-1, 1)$, is invertible and for $L \geq 1$ the following evaluation holds*

$$\|T_L^{-1}(G)\|_{L_2(\mathbb{R},\varrho_s)} \leq c_1L^{1/2}$$

where c_1 does not dependent on L.

$$\|X_L^+(\mu) - (B_L(G)f_L)(\mu)\|_{L_2(\mathbb{R},\varrho_s)} \leq c_2 L^{-s/2}, \quad s \in (0,1)$$

$$\varphi_\infty(x,z) = \frac{k_0\nu}{2\pi}\int_{\mathbb{R}} \frac{B_L(G)f_L(\mu)}{\nu+\gamma(\mu)} e^{-ik(\mu x - \gamma(\mu)z)} d\mu$$

$$|\varphi_L(x,z) - \varphi_\infty(x,z)| \leq c_3 L^{-s/2}$$

where ε_3 does not dependent on $L \geq 1$ and $(x,z) \in \mathbb{R} \times (0,\infty)$

$$B_L(G) \sim P^+(T^{-1}(G) + T^{-1}(\widetilde{G}) - G^{-1})P^+$$