Theory of Toeplitz Operators: THE CONJECTURES CAN BE INCORRECT!

S. Grudsky (México, 11.02.2004)

Dedicated to the fond memory of

OLGA GRUDSKAYA

(16.05.1955 - 10.02.2004)

- *Böttcher A., Grudsky S.M.* On the composition of Muckenhoupt weights and inner functions. J. London Mathematical Society, (2) 58, N 1, 1998, 172–184.
- *Böttcher A., Grudsky S. and Spitkovsky I.* The spectrum is discontinuous on the manifold of Toeplitz operators. Archiv d. Math. (Basel), **75**, N 1, 2000, 46–52.
- *Böttcher A., Grudsky S.M.* Asymptotic spectra of dense Toeplitz matrices are unstable. Numerical Algorithms, **33**, 2003, 105–112.

On the composition of Muckenhoupt weights and inner functions

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$$

$$f \in L^p(\mathbb{T}) \iff \left(\int_{\mathbb{T}} |f(t)|^p |dt|\right)^{1/p} < \infty,\ 1 \le p \le \infty$$

Definition 1 *A measurable function*

$$w : \mathbb{T} \to [0, \infty]$$

is called ***weight*** *if the set* $w^{-1}(\{0, \infty\})$ *has measure* 0.

Definition 2 *A weight* w *is said to belong to the Muckenhoupt class* A_p $(1 < p < \infty)$ *if* $w \in L^p(\mathbb{T})$, $(1/w) \in L^q(\mathbb{T})$, $(1/p + 1/q = 1)$ *and*

$$\sup_I \left(\frac{1}{|I|}\int_I w^p(t)|dt|\right)^{1/p} \left(\frac{1}{|I|}\int_I w^{-q}(t)|dt|\right)^{1/q} < \infty,$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$, *and* $|I|$ *denotes the length of* I.

$H^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : f_n = 0, n < 0\}$, where f_n is n-th Fourier coefficient of f

Definition 3 *A nonconstant function* $u \in H^\infty(\mathbb{T})$ *is called* ***inner function*** *if* $|u(t)| = 1$ *a.e.*

PROBLEM: ***Let*** $\mathbf{w} \in \mathbf{A_p}$ ***and*** $\mathbf{u}$ ***is inner function. Does the superposition***

$$(\mathbf{w} \circ \mathbf{u})(\mathbf{t}) = \mathbf{w}(\mathbf{u}(\mathbf{t}))$$

belong $\mathbf{A_p}$***?***

The conjecture was that the answer is **YES**.

WHY?

1. If $p = 2$ then $(w \circ u)(t) \in A_2$.

2. If $u(t) = \exp\left\{\lambda \frac{t+1}{t-1}\right\}$, $\lambda > 0$, then $(w \circ u)(t) \in A_p$ for $p \in (1, \infty)$.

3.

$$B(t) = \prod_{k=-\infty}^{\infty} \frac{|z_k|}{z_k} \cdot \frac{z_k - t}{1 - \overline{z}_k t}, \quad |z_k| < 1.$$

For several classes of Blaschke product $(w \circ B)(t) \in A_p$ for $p \in (1, \infty)$.

Littlwood's subordination theorem. *If* $f \in L^p(\mathbb{T})(H^p(\mathbb{T}))$ *for arbitrary inner function* u *and for* $p \in (1, \infty)$ *we have*

$$f \circ u \in L^p(\mathbb{T})(H^p(\mathbb{T})).$$

1. Boundedness of the Cauchy singular integral operator in weight spaces

$$(Sf)(t) = \frac{1}{\pi i}\text{v.p.}\int_{\mathbf{T}} \frac{f(\tau)}{\tau - t}d\tau \quad (t \in \mathbb{T})$$

Theorem 1 (Hunt, Muckenhoupt and Wheeden, 1973)
S is bounded operator on the weight Lebesgue space $L^p(\mathbb{T}, w)$ with the norm

$$\|f\|_{p,w} := \left(\int_{\mathbb{T}} |f(t)|^p w^p(t)|dt|\right)^{1/p}$$

if and only if $w \in A_p$ $(1 < p < \infty)$.

Reformulation 1 *Let S is bounded on $L^p(\mathbb{T}, w)$ and u is inner function. Is S bounded on $L^p(\mathbb{T}, w \circ u)$?*

2. *Spectral theory of Toeplitz operators*

$\mathbb{P} := (I + S)/2$ is analytical projector

$$\mathbb{P} : L^p(\mathbb{T}) \to H^p(\mathbb{T}),\ \sum_{n=-\infty}^{\infty} f_n t^n \to \sum_{n=0}^{\infty} f_n t^n$$

$$T(a) := \mathbb{P}a\mathbb{P} : H^p(\mathbb{T}) \to H^p(\mathbb{T})$$

a $(\in L^\infty(\mathbb{T}))$ is symbol of Toeplitz operator $T(a)$

Theorem 2 (Simonenko) *$T(a)$ is invertible on $H^p(\mathbb{T})$ if and only if symbol a is invertible in $L^\infty(\mathbb{T})$ and function $a/|a|$ can be represented in the form*

$$\frac{a}{|a|} = \frac{\overline{h}}{h},$$

where

i) $h \in H^p(\mathbb{T})$, $(1/h) \in H^q(\mathbb{T})$, $1/p + 1/q = 1$, *and*

ii) $|h| \in A_p$.

Reformulation 2 *If operator $T(a)$ is invertible on $H^p(\mathbb{T})$ and u is inner function is operator $T(a \circ u)$ invertible on $H^p(\mathbb{T})$?*

Really,

$$\frac{a \circ u}{|a \circ u|} = \frac{\overline{h \circ u}}{h \circ u}.$$

According to Littlwood Theorem we have i)

$$h \circ u \in H^p(\mathbb{T}) \quad \text{and} \quad \left(\frac{1}{h \circ u}\right) \in H^q(\mathbb{T})$$

$$\boxed{\textbf{But } |\mathbf{h} \circ \mathbf{u}| \overset{?}{\in} \mathbf{A_p}.}$$

The superposition $a \circ u$ generates wide class of oscillating symbols

$$e(t) = \exp\left(\lambda \frac{t+1}{t-1}\right) \Longleftrightarrow \exp\{i\lambda x\}, \quad x \in \mathbb{R}.$$

If $a(t) \in C(\mathbb{T}) \Longrightarrow a(\exp\{i\lambda x\}) \in \Pi_\lambda(\mathbb{R})$.
$\Pi_\lambda(\mathbb{R})$ is class of continuous $\frac{2\pi}{\lambda}$-periodic functions.

ANSWER THE MAIN QUESTION

Theorem 3 (Böttcher, Grudsky) .

1. *If* $w \in A_2$ *and* u *is inner function then* $w \circ u \in A_2$.

2. *Let* $p \in (1,2) \cup (2,\infty)$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Further, let* σ *be any number in interval* $\left(\frac{1}{p}, \frac{1}{q}\right)$ *if* $1 < p < 2$, *OR any number in interval* $\left(-\frac{1}{q}, -\frac{1}{p}\right)$ *if* $2 < p < \infty$. *Then*

$$w(t) := |t-1|^{-\sigma}$$

is a weight in A_p, *but there is Blaschke product* B_M *such that*

$$w(B_M(t)) = |B_M(t) - 1|^{-\sigma}$$

is not a weight in A_p.

Construction of $\mathbf{B_M(t)}$

$$B_M(t) = \prod_{k=-\infty}^{\infty} \frac{|z_k|}{z_k} \cdot \frac{z_k - t}{1 - \overline{z}_k t}, \quad z_k = r_k \exp(i\theta_k)$$

$$\theta_k = \begin{cases} (\operatorname{sign} k) \exp(-|k|), & k \neq 0, \\ -1, & k = 0, \end{cases} \qquad r_k = \frac{1 - \delta_k/M}{1 + \delta_k/M},$$

where

$$\delta_k := \min\left\{ \left(\frac{\Delta_k}{\log \Delta_k}\right)^2, \left(\frac{\Delta_{k-1}}{\log \Delta_k}\right)^2 \right\},$$

and

$$\Delta_k = \begin{cases} \theta_k - \theta_{k+1}, & k = 1, 2, 3, \ldots, \\ \theta_{k-1} - \theta_k, & k = 0, -1, -2, \ldots, \end{cases}$$

M (> 0) is large enough.

For $p = 2$ the answer is **YES**, for $p \neq 2$ the answer is **NO**.

THE SPECTRA IS DISCONTINUITY ON THE MANIFOLD OF TOEPLITZ OPERATORS

H is Hilbert space

$\mathfrak{B}(H)$ is the set of all bounded linear operators on H

$$\operatorname{sp} A = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$$

$$\operatorname{sp}_{ess} A = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}$$

$$A \text{ is Fredholm} \iff \operatorname{im} A = \overline{\operatorname{im} A},$$

$$\dim\ker A < \infty \text{ and } \dim(H/\operatorname{im} A) < \infty$$

$$\{A_n\}_{n\in\mathbb{N}} \in \mathfrak{B}(H)$$

$$\liminf_{n\to\infty} \operatorname{sp} A_n$$
$$= \{\lambda \in \mathbb{C} : \forall n\, \exists\, \lambda_n \in \operatorname{sp} A_n \text{ such that } \lim_{n\to\infty} \lambda_n = \lambda\}$$

$$\limsup_{n\to\infty} \operatorname{sp} A_n$$
$$= \{\lambda \in \mathbb{C} : \exists\, \{n_k\}_{k=1}^{\infty} \text{ such that } \lambda_{n_k} \in \operatorname{sp} A_{n_k} \text{ and } \lim_{k\to\infty} \lambda_{n_k} = \lambda\}$$

$$\liminf_{n\to\infty} \operatorname{sp} A_n \subset \limsup_{n\to\infty} \operatorname{sp} A_n \qquad (1)$$

$$\lim_{n\to\infty} \|A_n - A\|_H = 0 \implies \limsup_{n\to\infty} \operatorname{sp} A_n \subset \operatorname{sp} A \qquad (2)$$

It is well known that in general neither in (1) nor in (2) equality holds.

$$f(x) \in H^2(\mathbb{R}) \iff f(x) = \int_0^\infty g(t)e^{itx}dt, \, g \in L^2(0,\infty)$$

$\mathbb{P} : L^2(\mathbb{R}) \to H^2(\mathbb{R})$ is orthogonal projector
$T(a)f := P(af) : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ is Toeplitz operator
$a \; (\in L^\infty(\mathbb{R}))$ is a symbol of the operator $T(a)$

$$\|a_n - a\|_{L^\infty} \to 0 \quad \Longrightarrow \quad \|T(a_n) - T(a)\|_{H^2} = 0$$

Problem. *Let* $a_n \in L^\infty(\mathbb{R})$ *and* $\|a_n - a\|_{L^\infty} \to 0$. *Does the following equality*

$$\liminf_{n\to\infty} \operatorname{sp} T(a_n) = \operatorname{sp} T(a)$$

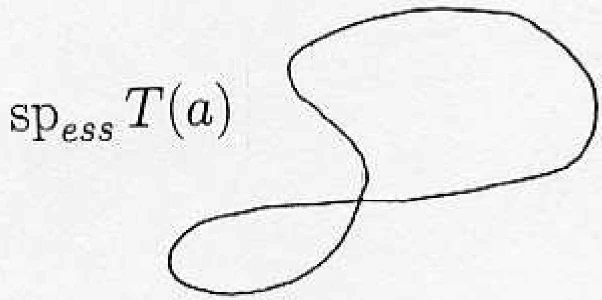
hold?

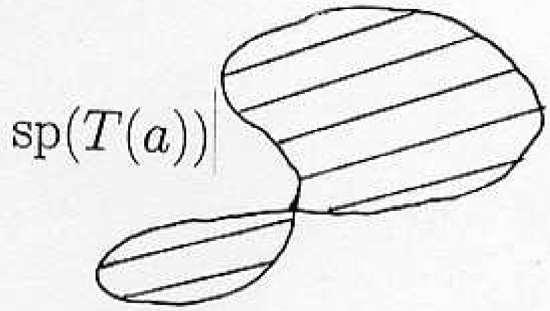
D.R. Fanerick and W.Y. Lee. *Hyponormality and spectra of Toeplitz operators.* Trans. Amer. Math. Soc. **348**, 4153–4174 (1996).

CONJECTURE of Fanerick and Lee: *The answer is* ***YES***.

Particular cases

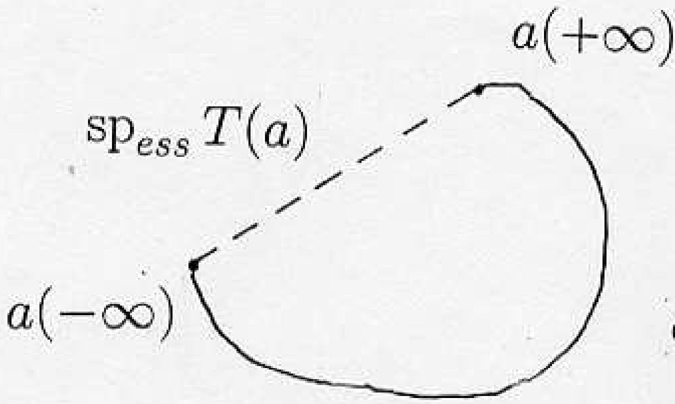
1. $a \in C(\dot{\mathbb{R}}),\ \dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$
   $(a(-\infty) = a(+\infty)$

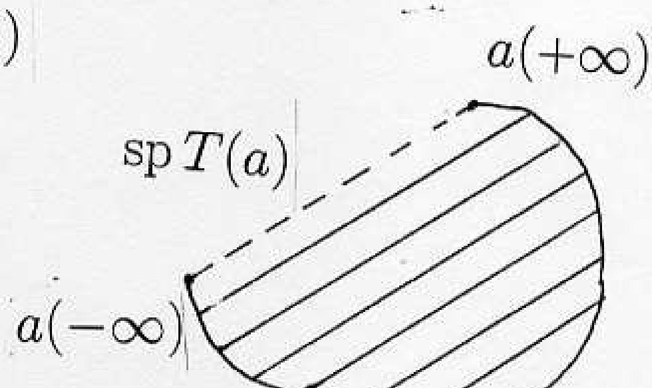




$\mathrm{sp}(T(a)) = \{\lambda \in \mathbb{C} : \inf_{x \in \lambda} |a(x) - \lambda| = 0\} \cup \{\lambda \in \mathbb{C} : \inf_{x \in \lambda} |a(x) - \lambda| > 0 \quad \textit{and}$ $\mathrm{wind}(a(x) - \lambda) \neq 0\}$

2. $a \in C(\overline{\mathbb{R}}),\ \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$
   $(a(-\infty) \neq a(+\infty))$





3. $C(\dot{\mathbb{R}}) + H^\infty(\mathbb{R})$

4. $PQC = ((H^\infty(\mathbb{R}){+}C(\dot{\mathbb{R}}))\cap(\overline{H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})}) \cup C(\overline{\mathbb{R}})$ is class of the *piecewise quasicontinuous ~~periodic~~ functions.*

5.

$$AP(\mathbb{R}) = \left\{ \overline{\sum_{k=1}^{n} c_k e^{i\lambda_k x}, \lambda_k \in \mathbb{R}, c_k \in \mathbb{C}} \right\}$$

is a class of the continuous almost periodic functions.

6. $AP(\mathbb{R}) + C(\dot{\mathbb{R}})$

For all these classes the answer the question by Fanerick and Lee is **YES**.

***BUT: in general case the answer is* NO!**

$$SAP(\mathbb{R}) := AP(\mathbb{R}) \cup C(\overline{R})$$

is class of the semi-almost periodic functions.

$$a \in SAP(\mathbb{R}) \Rightarrow a(x) = u(x)a_r(x) + (1 - u(x))a_l(x) + a_0(x),$$

where

$$a_r(x), a_l(x) \in AP(\mathbb{R}),$$

$$u(x) \in C(\overline{\mathbb{R}})$$

with $u(+\infty) = 1$ and $u(-\infty) = 0$,

$$a_0(x) = C(\dot{\mathbb{R}})$$

with $a_0(\infty) = 0$.

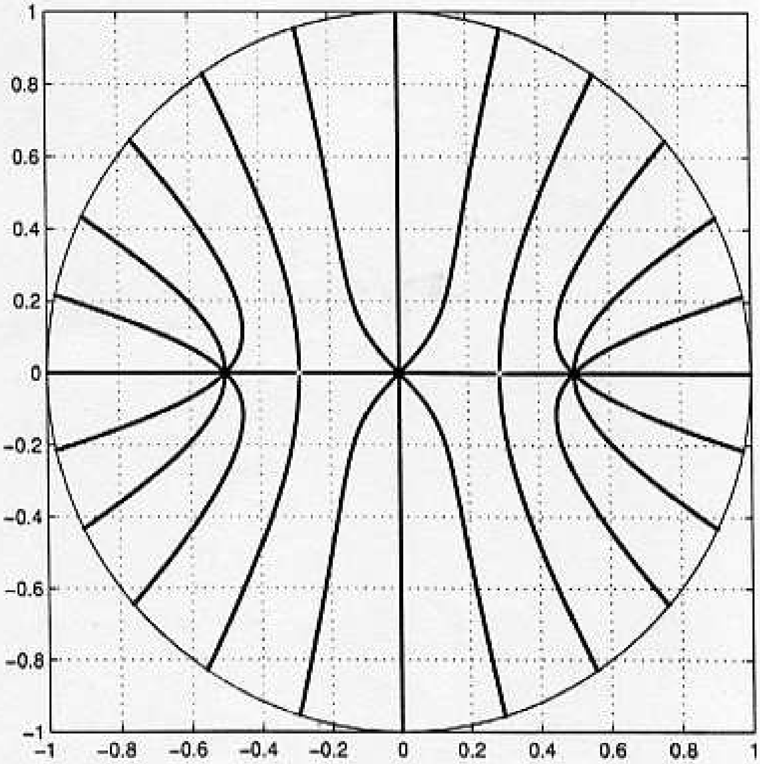


Figure 1: $h(\lambda) = (\lambda - 1/2)^4 \lambda^4 (\lambda + 1/2)^4$

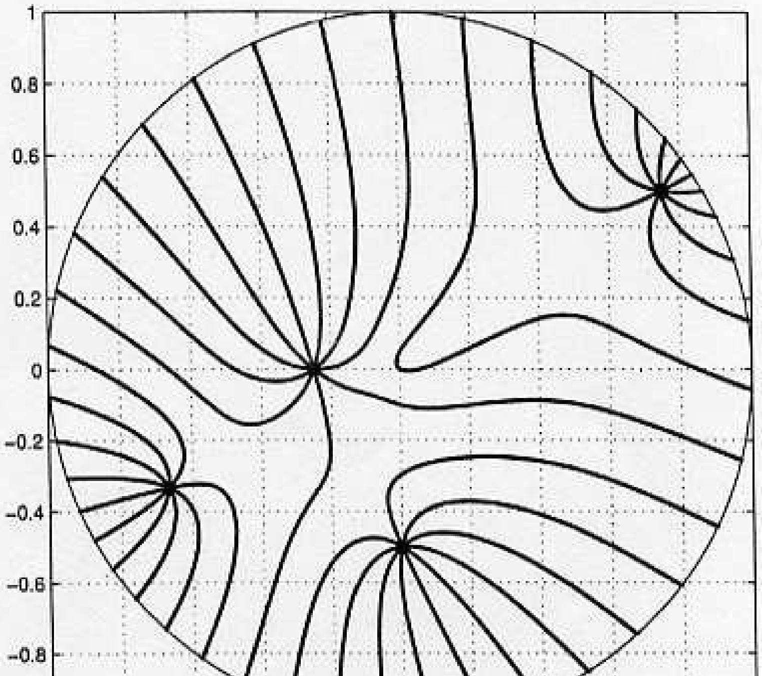
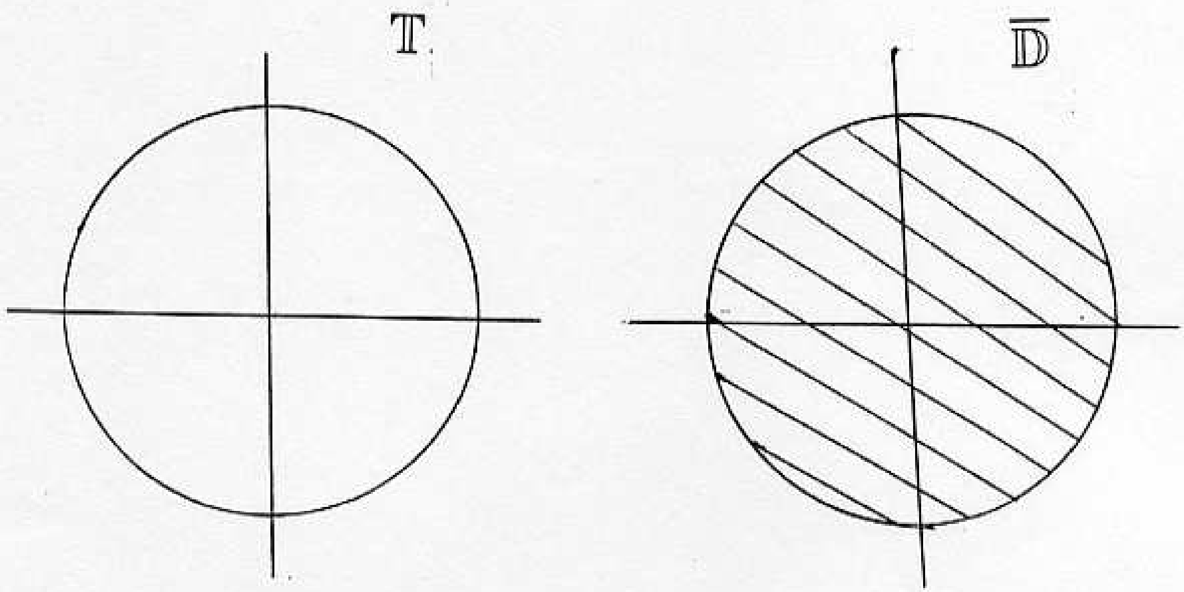


Figure 2: $h(\lambda) = \lambda(\lambda + 1/4)^5(\lambda + i/2)^5(\lambda - 3/4 - i/2)^5(\lambda + 2/3 + i/3)^5$

Theorem 4 (Bö, Gru, Sp) *There exist* $\{a_n\}$ *and* a *in* $SAP(\mathbb{R})$ *such that*

$$\|a_n - a\|_{L^\infty} \to 0$$
$$\operatorname{sp} T(a_n) = \operatorname{sp}_{ess} T(a_n) = \mathbb{T}, \operatorname{sp} T(a) = \operatorname{sp}_{ess} T(a) = \overline{\mathbb{D}}.$$



$$\liminf_{n\to\infty} \operatorname{sp} T(a_n) = \limsup_{n\to\infty} T(a_n) = \mathbb{T}.$$

$$\textbf{BUT! } \operatorname{sp} T(a) = \overline{\mathbb{D}}$$

$$b_{2k} = a_{2k}, \quad b_{2k+1} = a$$

$$\liminf_{n\to\infty} \operatorname{sp} T(b_n) = \mathbb{T} \neq \limsup_{n\to\infty} \operatorname{sp} T(b_n) = \mathbb{D}.$$

Asymptotic spectra of Toeplitz matrices are unstable

$a(t) \in L^\infty(\mathbb{T})$, $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$

$a(t) \sim \sum_{j=-\infty}^{\infty} a_j t^n$ is Fourier series of a

$T(a) = \{a_{j-k}\}_{j,k=0}^{\infty}$ is the infinite Toeplitz matrix

$T_n(a) = \{a_{j-k}\}_{j,k=0}^{n}$ is the finite Toeplitz matrix

$$T(a) = \left(\begin{array}{cccccc|cc}
a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+1} & a_{-n} & a_{-n-1} & \\
a_1 & a_0 & a_{-1} & \cdots & \cdots & a_{-n+1} & a_{-n} & \ddots \\
a_2 & a_1 & a_0 & \ddots & & & a_{-n+1} & \ddots \\
\vdots & & \ddots & \ddots & \ddots & & \vdots & \ddots \\
\vdots & & & \ddots & \ddots & a_{-1} & \vdots & \\
a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & & \\
\hline
 & & & & & & a_{-1} & \\
a_{n+1} & a_n & a_{n-1} & & & a_1 & a_0 & \ddots \\
 & \ddots & \ddots & \ddots & & & \ddots & \ddots
\end{array}\right)$$

$$f \in L^2(\mathbb{T}) \Leftrightarrow \left(\sum_{j=-\infty}^{\infty} |f_j|^2\right)^{1/2} < \infty$$

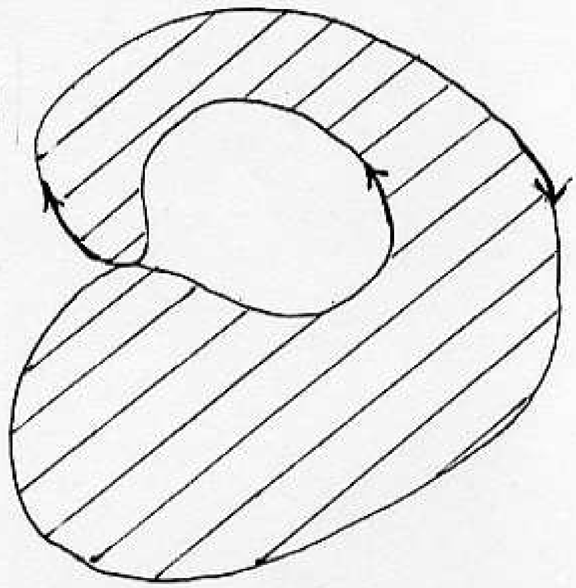
$$\mathbb{P}\left(\sum_{j=-\infty}^{\infty} f_j t^j\right) = \sum_{j=0}^{\infty} f_j t^j, \quad \mathbb{P}_n\left(\sum_{j=-\infty}^{\infty} f_j t^j\right) = \sum_{j=0}^{n} f_j t^j$$

$$T(a) := \mathbb{P} a \mathbb{P} : H^2(\mathbb{T}) \to H^2(\mathbb{T})$$

$$T_n(a) := \mathbb{P}_n a \mathbb{P}_n : H_n^2(\mathbb{T}) \to H_n^2(\mathbb{T})$$

$$a \in C(\mathbb{T}) \Rightarrow spT(a) = \{\lambda \in \mathbb{C} \mid \exists t \in \mathbb{T}, a(t) = \lambda\} \bigcup$$
$$\{\lambda \in \mathbb{C} \mid \min_{t \in \mathbb{T}} |a(t) - \lambda| > 0 \, and \, wind \, (a(t) - \lambda) \neq 0\}$$

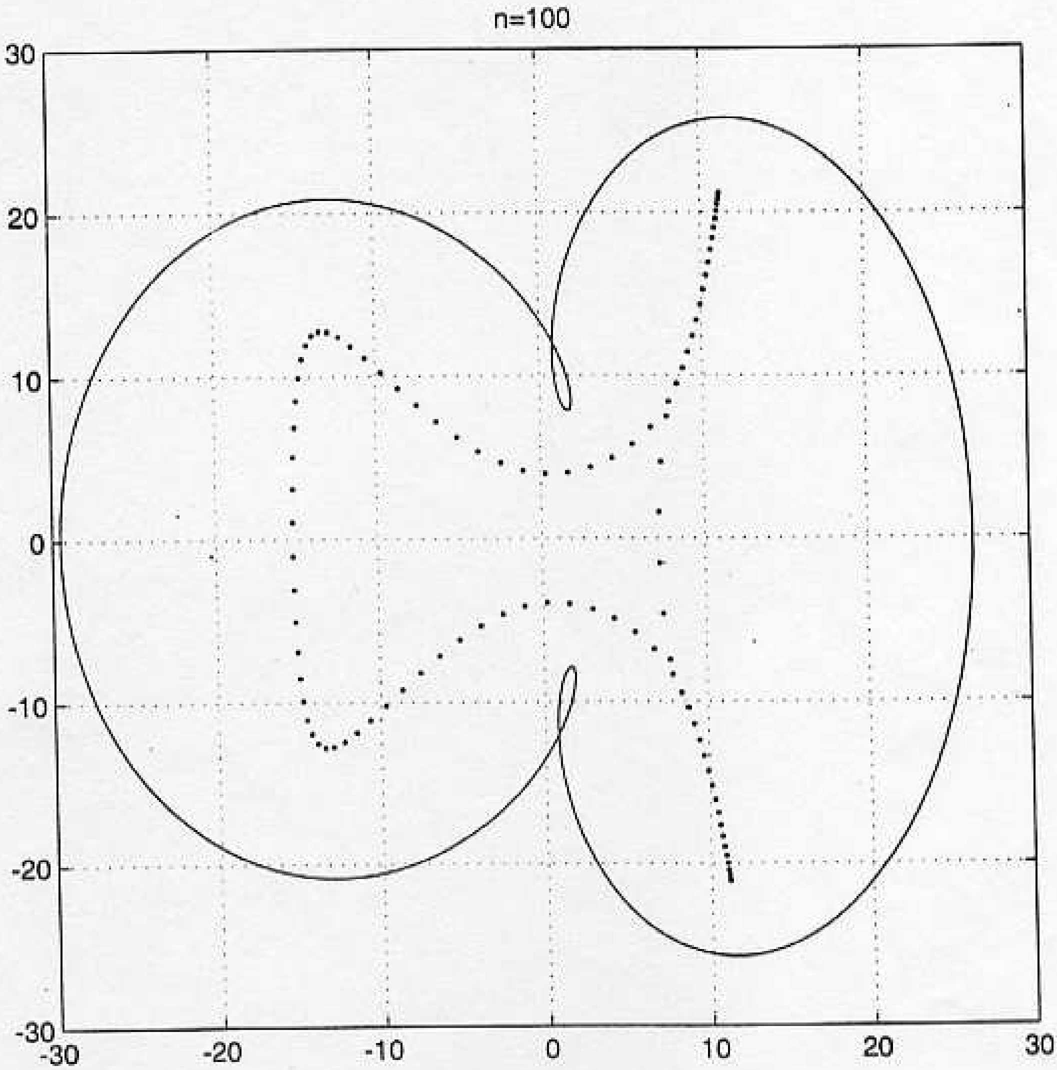
sp $T(a)$



$$sp(T_n(a)) = \bigcup_{k=0}^{n} \{\lambda_k\}$$

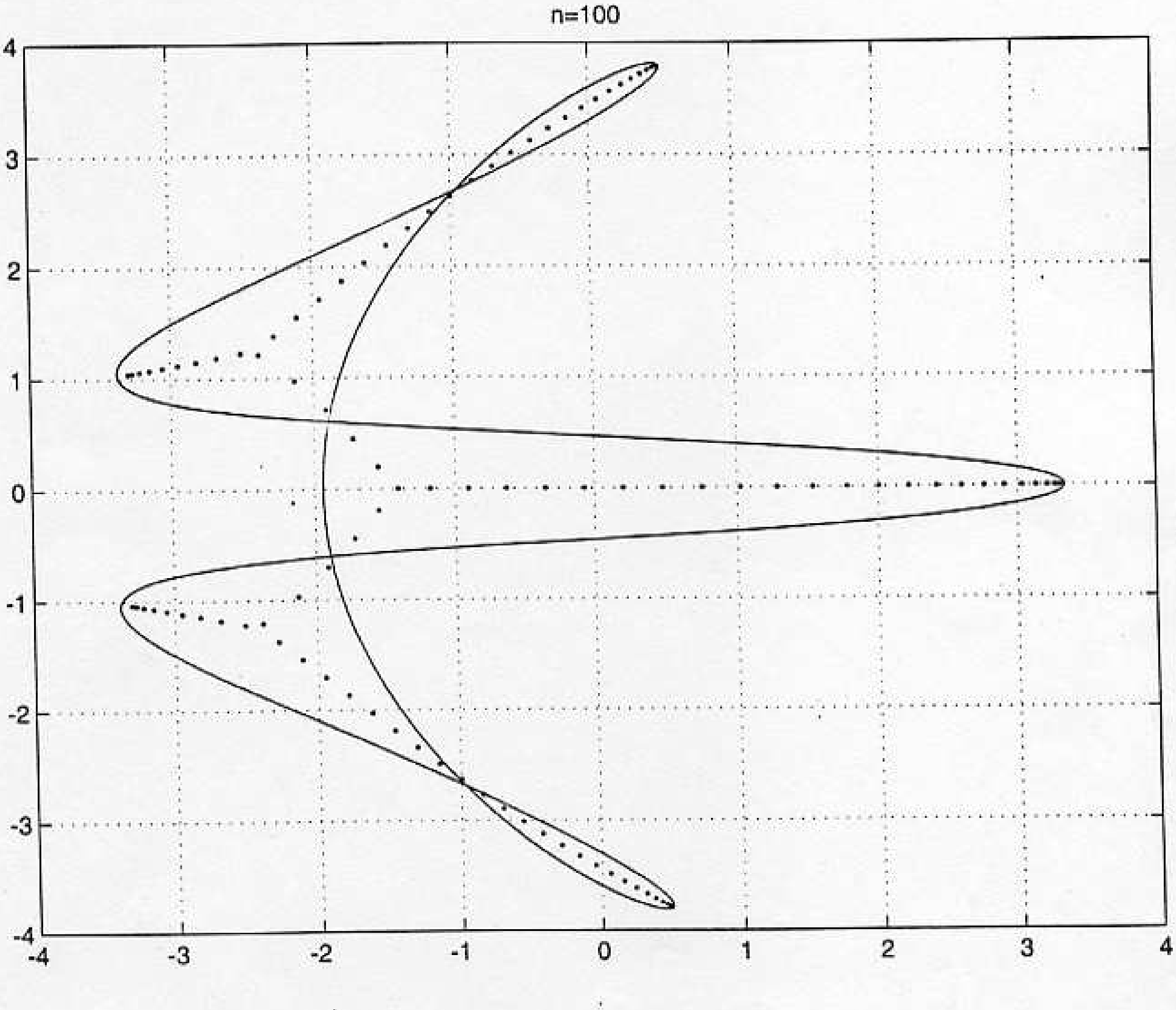
$$\Lambda(a) = \limsup_{n \to \infty} spT_n(a)$$

$$a(t) = -2t^{-3} - 2t^{-2} + \frac{i}{100}t^{-1} + 20t + \frac{1}{2}t^2 + 10t^3$$



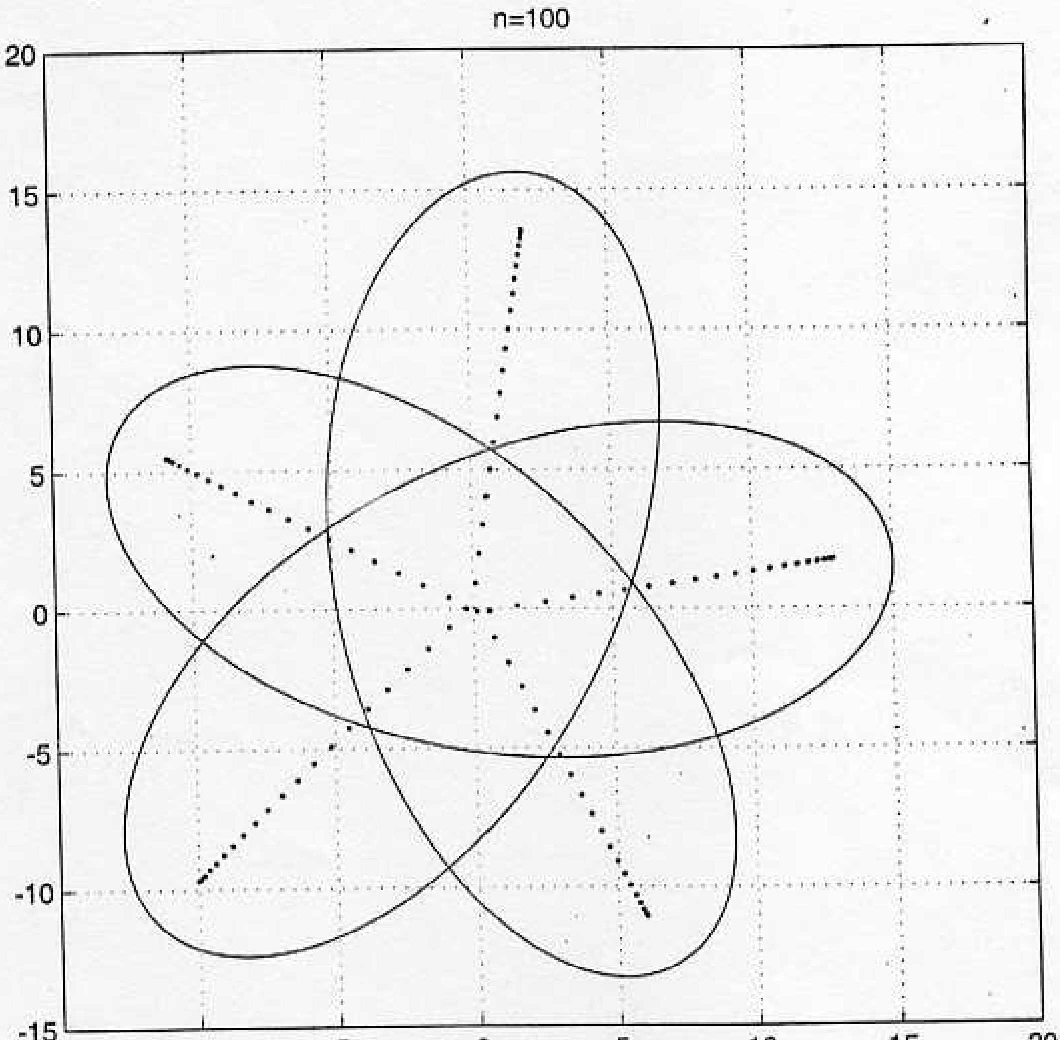
I

$$a(t) = -\frac{13}{10}t^{-3} - \frac{1}{10}t^{-2} - \frac{11}{7}t^{-1} - \frac{7}{10} + t + 1,5t^2 - 0.77t^3$$

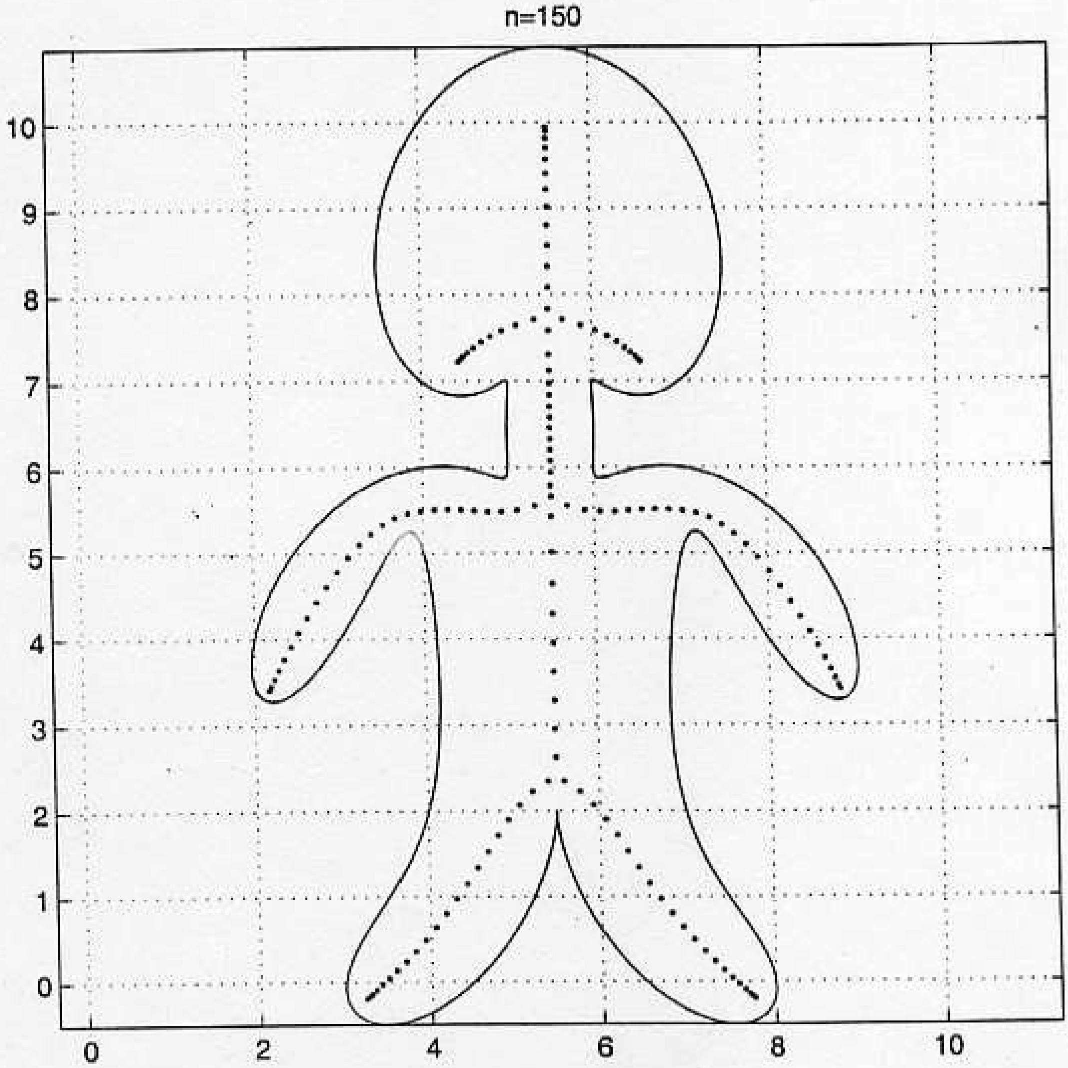


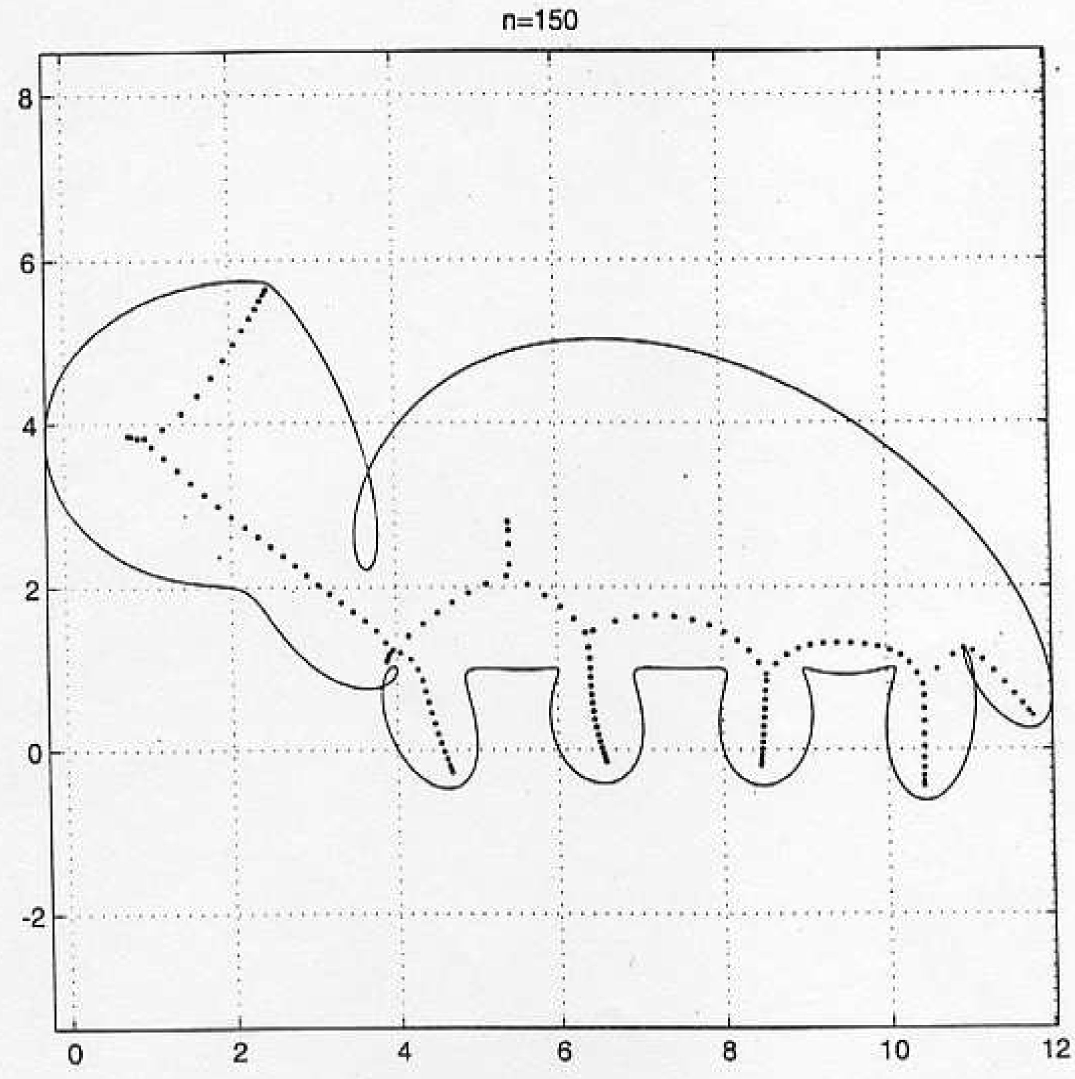
II

$$a(t) = t^{-4} - \frac{i}{10i} t^{-3} + (i+5)t^{-2} - \frac{i}{2} t^{-1} + \frac{i}{2} t^{2} + (10+i)t^{3} - \cancel{\frac{it^{4}}{?}}$$



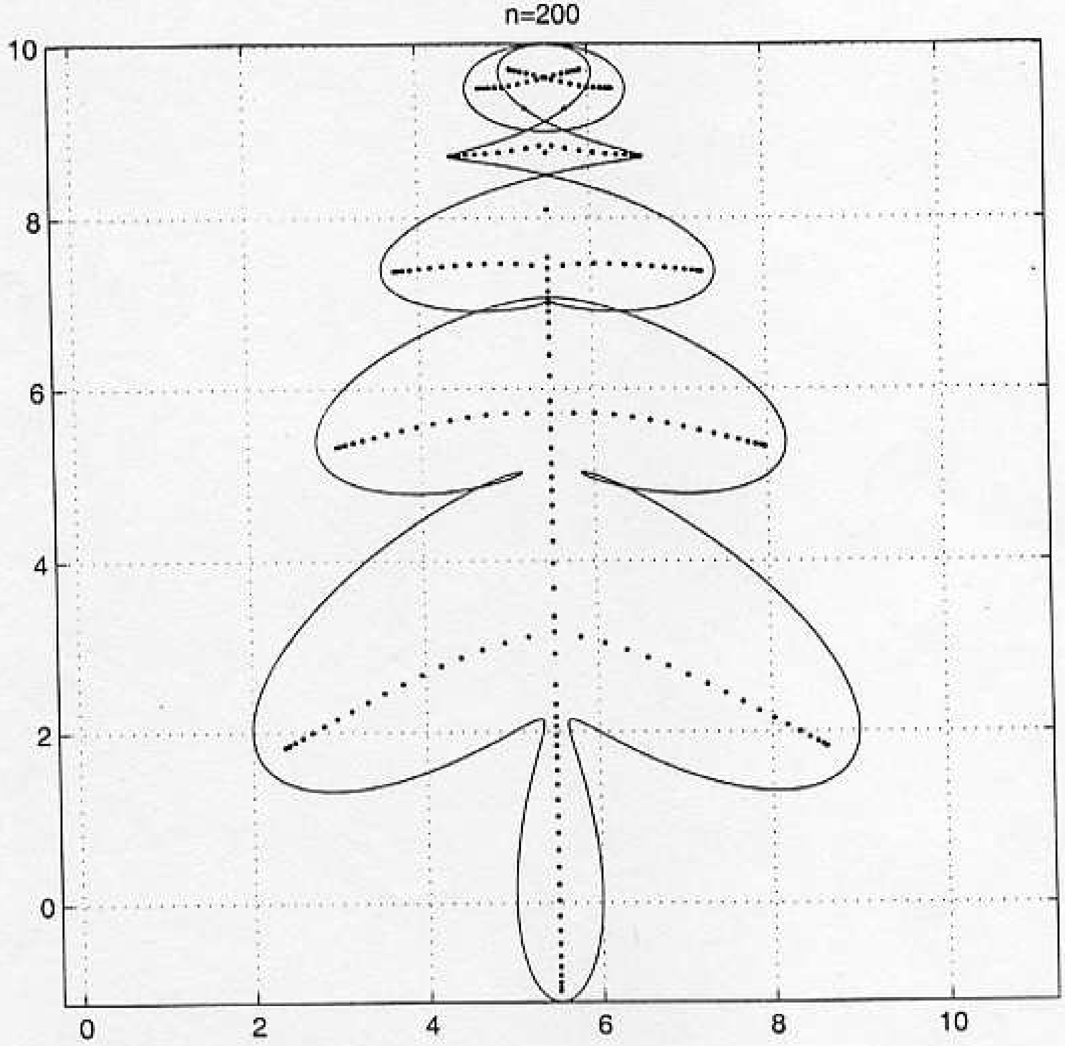
III

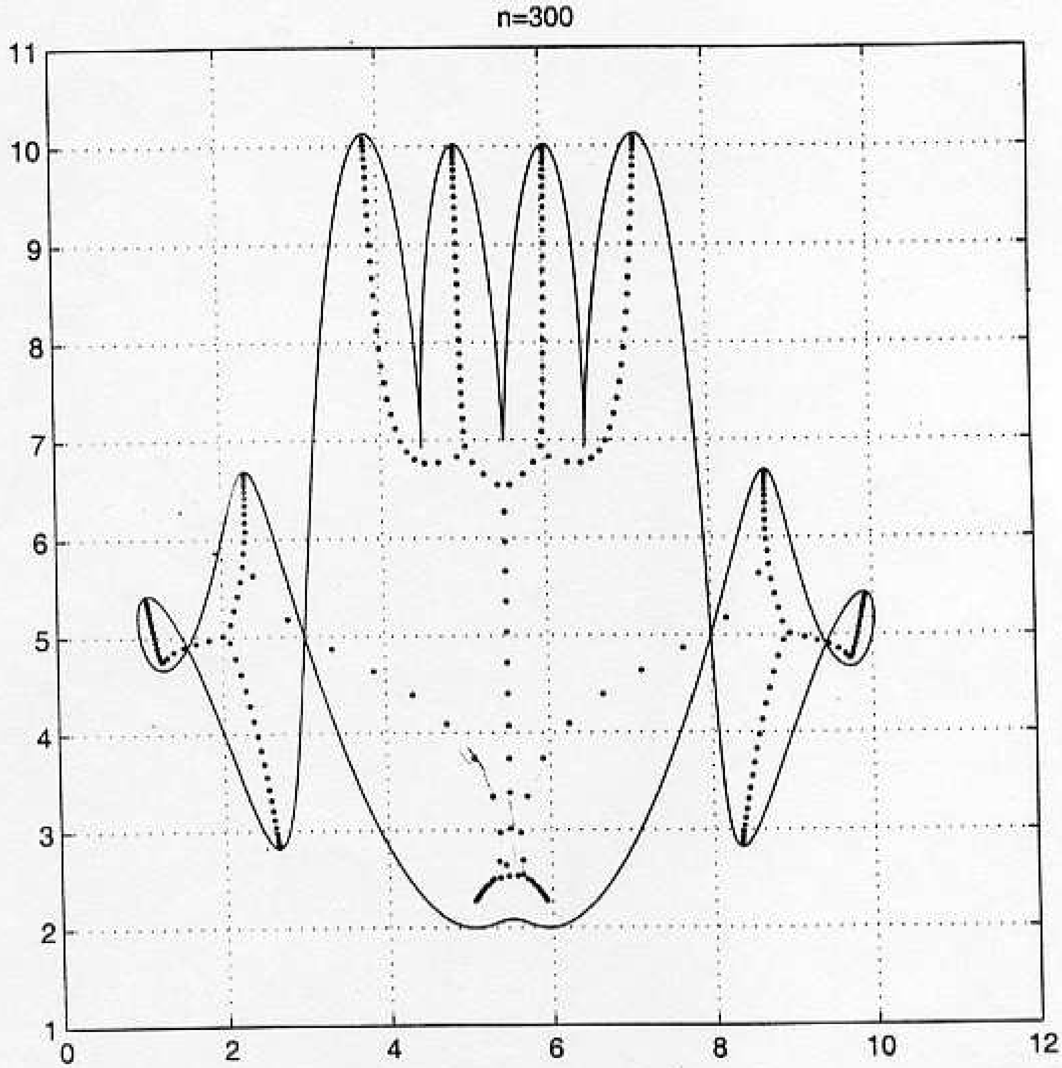

n=150
10
9
8
7
6
5
4
3
2
1
0
0
2
4
6
8
10



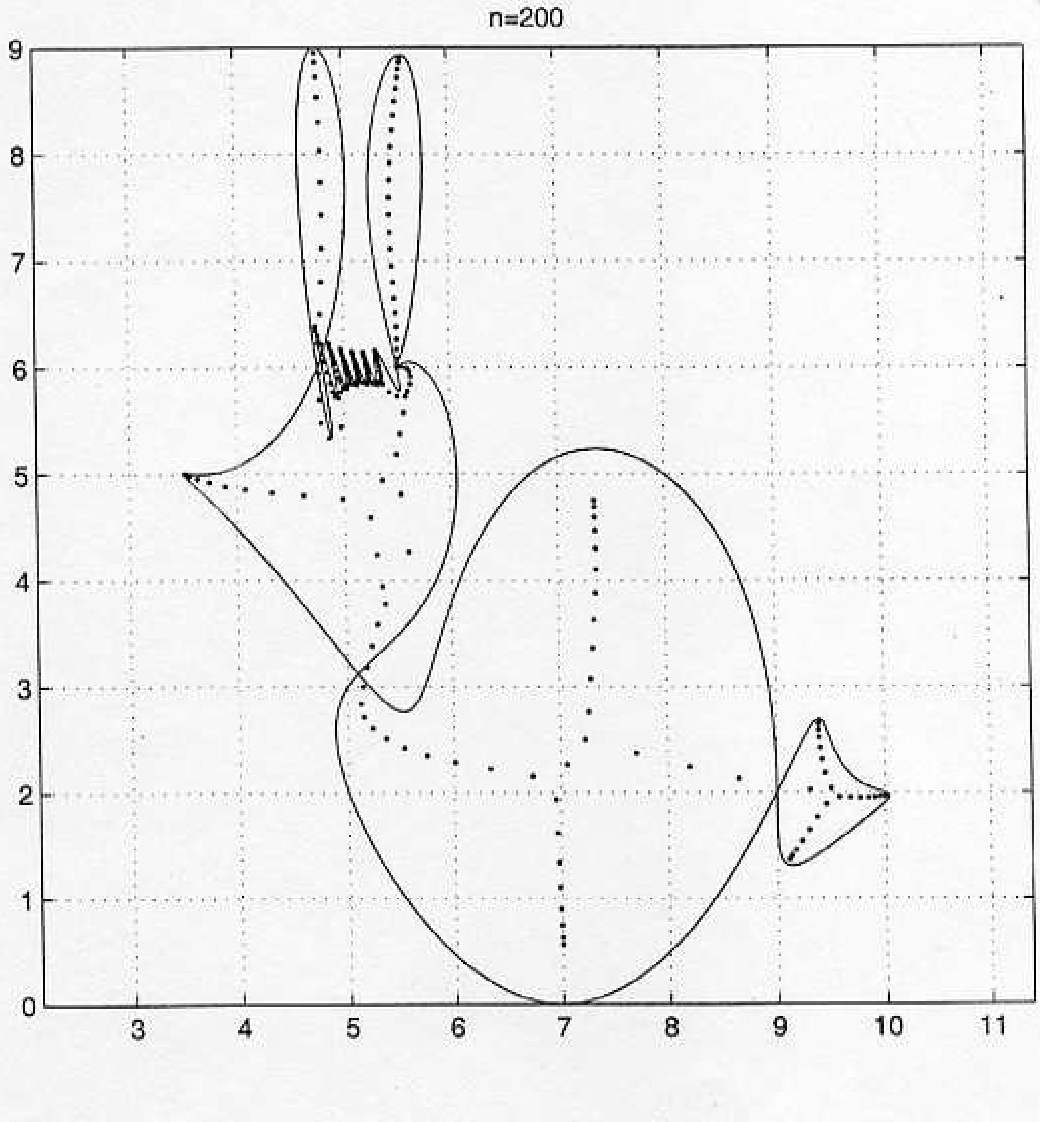
V

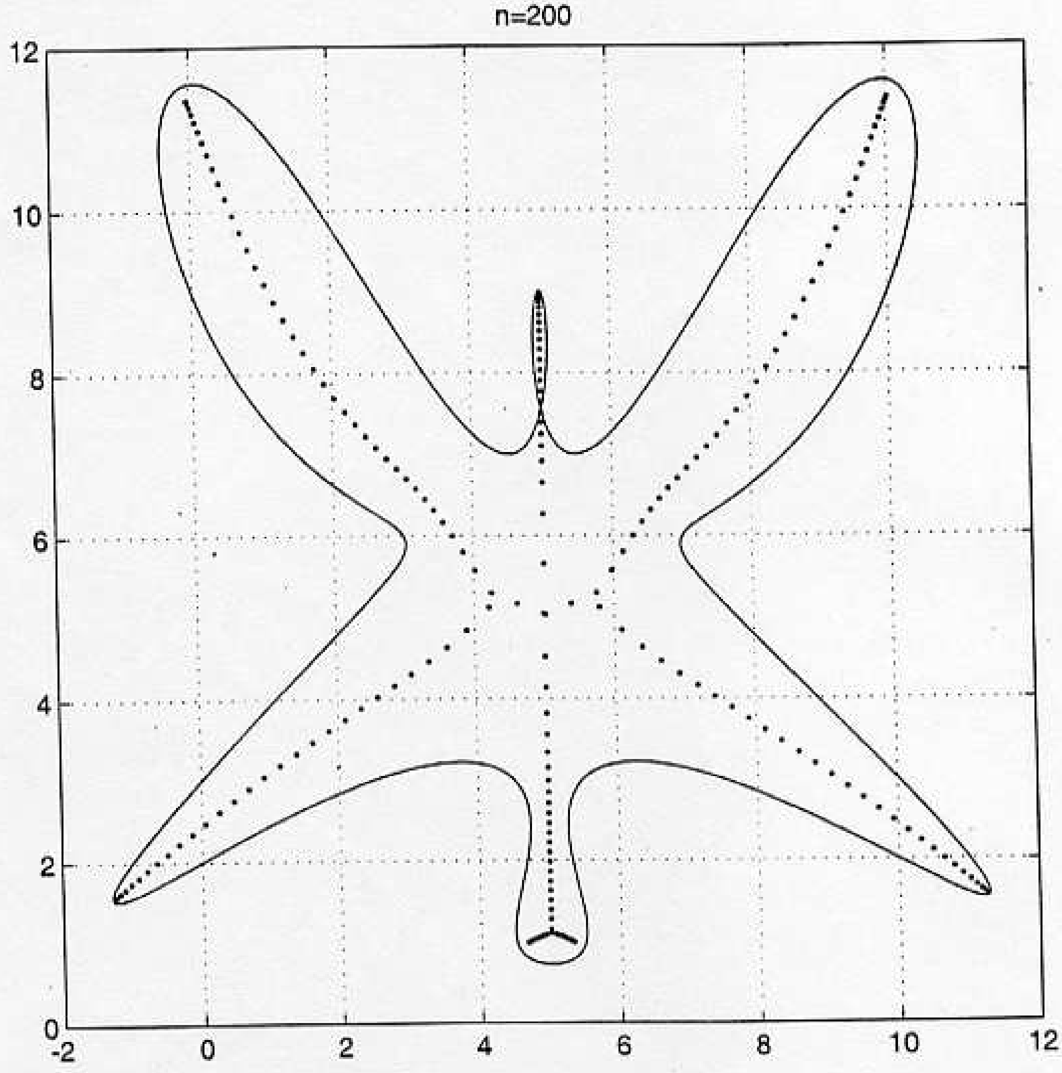
$b(12) = 5.5 + 9i$





VII


n=200
9
8
7
6
5
4
3
2
1
0
3
4
5
6
7
8
9
10
11



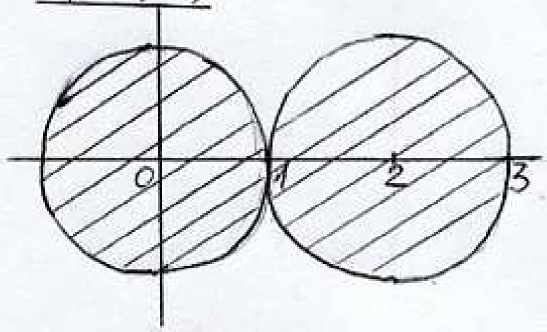
IX

Theorem 5 (Bö, Gru, Sp) *There exist* $\{c_n\}$ *and* c *in* L^∞ *such that* $\|c_n - c\|_{L^\infty} \to 0$ *and*

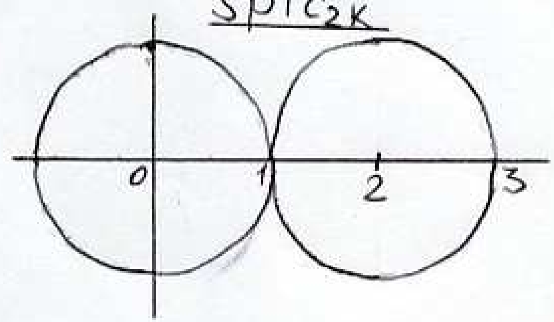
$$\operatorname{sp} T(c) = \operatorname{sp}_{ess} T(c) = \overline{\mathbb{D}} \cup (2 + \overline{\mathbb{D}}),$$

$$\operatorname{sp} T(c_n) = \operatorname{sp}_{ess} T(c_n) = \begin{cases} \mathbb{T} \cup (2+\mathbb{T}) & \text{if } n \text{ is odd,} \\ \mathbb{T} \cup (2+\overline{\mathbb{D}}) & \text{if } n \text{ is even.} \end{cases}$$

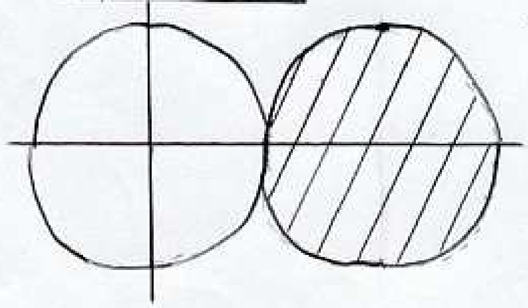
spT(c)



spTc2k



spT(c2k+1)



$$\liminf_{n\to\infty} \ \operatorname{sp}\ T(c_n) = \mathbb{T} \cup (2+\mathbb{T})$$
$$\limsup_{n\to\infty} \ \operatorname{sp}\ T(c_n) = \mathbb{T} \cup (2+\mathbb{D})$$
$$\operatorname{sp}\ T(c_n) = \overline{\mathbb{D}} \cup (2+\overline{\mathbb{D}})$$

Theorem 6 (Schmidt and Spitzer, 1961) *Let* $a(t) = \sum_{k=-r}^{s} a_k t^k (r \geq 1,\ s \geq 1,\ a_s \neq 0,\ a_{-r} \neq 0)$ *be a Laurent polynomial, let* $z_1(\lambda), z_2(\lambda), \ldots, z_{r+s}(\lambda)$ *be the zeros of the polynomial* $z^r(a(t) - \lambda)$*, labelled so that*

$$|z_1(\lambda)| \leq |z_2(\lambda)| \leq \ldots \leq |z_{r+s}(\lambda)|.$$

Then

$$\Lambda(a) = \{\lambda \in \mathbb{C} \mid |z_r(\lambda)| = |z_{r+1}(\lambda)|\}.$$

THE MAIN QUESTION:

IS THE SET $\Lambda(a)$ STABLE?

The answer is yes for polynomial of bounded degree.

$\boxed{\textbf{The answer is NO in general case.}}$

Theorem 7 (Böttcher, Grudsky) *There exist $a^n \in C(\mathbb{T})$ such that $||a^n - a||_{L_\infty} \to 0$, but*

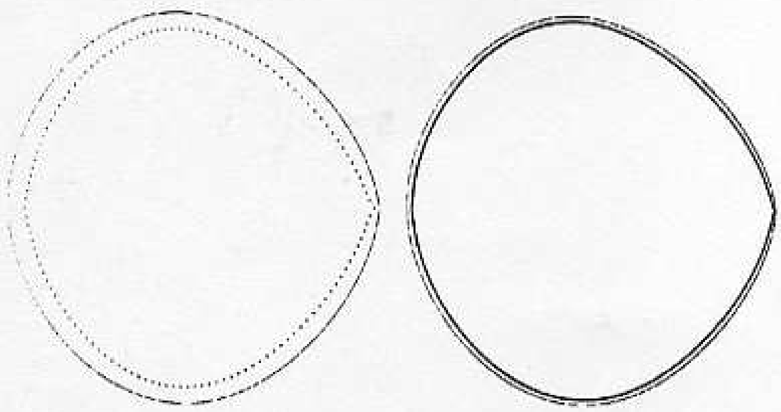
$$\limsup_{n\to\infty} \Lambda(a^n) \neq \Lambda(a)$$

$$a(t) = t^{-1}(33 - (t + t^2)(1 - t^2)^{3/4})$$

$$\Lambda(a) = \{\lambda \in \mathbb{C} \mid \exists t \in \mathbb{T}, a(t) = \lambda\}$$

$$0 \notin \Lambda(a)$$

$$\boxed{\begin{gathered} a^n = t^{-1}(33 - (t + t^2)\Sigma_{j=0}^{n} \binom{\frac{3}{4}}{j} t^{2j}) \\ \|a^n - a\|_{L_\infty} \to 0 \quad \textit{and} \\ 0 \in \Lambda(a^n), \quad \textit{for all } n \geq 2 \end{gathered}}$$



(a) (b)

Figure 1. The range $a(\mathbf{T})$ and the eigenvalues of $T_{128}(a)$ (a) and $T_{512}(a)$ (b).

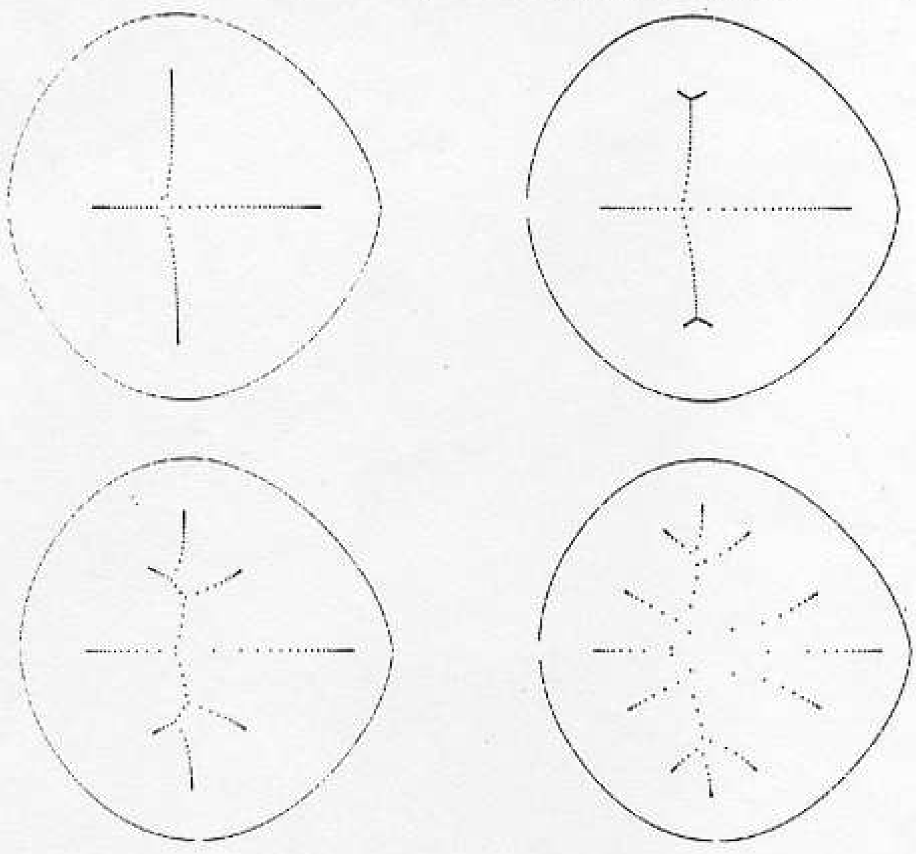


Figure 2. The range $a(\mathbf{T})$ and the eigenvalues of $T_{128}(P_n a)$ for $n = 4, 6, 8, 12$.

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