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Integral Equations and Operator Theory

Bergman Space Structure, Commutative Algebras of Toeplitz Operators and Hyperbolic Geometry

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Abstract. We exhibit a surprising but natural connection among the Bergman space structure, commutative algebras of Toeplitz operators and pencils of hyperbolic straight lines.

The commutative C^*-algebras of Toeplitz operators on the unit disk can be classified as follows. Each pencil of hyperbolic straight lines determines the set of symbols consisting of functions which are constant on corresponding cycles, the orthogonal trajectories to lines forming a pencil. It turns out that the C^*-algebra generated by Toeplitz operators with this class of symbols is commutative.

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1. Introduction

The aim of this paper is to show that the three following things, which seems to be quite different: Bergman space structure, commutative algebras of Toeplitz operators and pencils of hyperbolic straight lines, are connected in a very natural way. Let us comment on them first.

Bergman space structure. The Bergman space $\mathcal{A}^2(D)$ on a simply connected domain D in the complex plane $\mathbb{C}$ is the subspace of $L_2(D)$ consisting of functions analytic in D. In order to understand better the structure of the Bergman space one would like to find a unitary operator, defined on $L_2(D)$, which maps the Bergman space onto a very simple space transparently lying inside an ambient space. Then one may want to find an isometric isomorphism between the Bergman space and one of the classical Hilbert spaces. Of course, being a separable Hilbert space, the Bergman space can be mapped to one of its standard representatives. But the point

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here is to find an isomorphism which is compatible with and reflects the structure of a certain algebra of (Toeplitz) operators, acting on the initial Bergman space.

Commutative algebras of Toeplitz operators. In a sense these are quite strange objects, as one even may not expect their existence. For example, there are no commutative C^*-algebras of Toeplitz operators acting on the Hardy space, except the trivial case of scalar operators. Nevertheless, it turns out that such algebras do exist in the context of the Bergman space. Moreover, being *commutative* they possess the *non compact semicommutator* property; i.e., the operators $[T_a, T_b) = T_a \cdot T_b - T_{a \cdot b}$ are not compact in general. Note that having a commutative C^* operator algebra and the spectral theorem mapping (provided by an operator from the *Bergman space structure*) one has immediately all corresponding paraphernalia, such as invariant subspaces, spectra, eigenvalues and eigenspaces, etc.

Pencils of hyperbolic straight lines. Contrary to the case of the Euclidean geometry, the hyperbolic one has three different types of pencils: straight lines passing through a point, parallel lines and lines orthogonal to a given one. Each pencil defines a set of *cycles*, orthogonal trajectories to lines forming a pencil.

We describe now briefly how the above three ingredients are connected. Each pencil of straight lines together with corresponding cycles generate a sort of orthogonal coordinate system on the domain D which leads to the tensor product decomposition of $L_2(D)$ into two factors. Further, there exist a unitary operator U, defined on $L_2(D)$ and compatible with this tensor structure, such that the image of the Bergman space under this mapping looks like a tensor product of a closed subspace of the first factor with a one-dimensional subspace of the second, for the two first types of pencils, and in a slightly different form for the third type of pencil. That is, after the unitary transform U both the Bergman space and the Bergman projection possess a very simple and transparent structure. Further eliminating this one-dimensional multiple one obtains an isometric isomorphism R of the Bergman space onto some classical Hilbert space L, which gives an independent description of the Bergman space in nonanalytic terms. Now any Toeplitz operator whose symbol is constant on the cycles of initial pencil is unitary equivalent, by means of the operator R, to a multiplication operator acting on L. Thus the symbols constant on cycles produce a commutative C^*-algebra of Toeplitz operators. Moreover, the operator R provides the spectral theorem mapping for this algebra of Toeplitz operators.

In [9] and [2] certain C^*-algebras of Toeplitz operators, acting on the Bergman spaces on the upper half-plane and on the unit disk respectively, were considered. These algebras appear to be commutative, and the special characteristic of symbols, which made this phenomenon possible, were *symbols depending only on the imaginary part of a variable* for Toeplitz operators on the upper half-plane, and *radial symbols* for Toeplitz operators on the unit disk. Moreover, in each of these cases we have constructed the unitary operator, defined on all of L_2, which captures appropriately the special nature of the situation under consideration, and provides the spectral theorem mapping of the Toeplitz operator algebra onto an algebra of bounded continuous functions, in the first case, and of bounded sequences, in the second.

At this stage natural questions appear: *whether there exist other classes of symbols which generate commutative Toeplitz operator C^*-algebras, whether in each of these cases there exists a corresponding description of the Bergman space, and how these cases can be classified*.

The invariance of the Bergman kernel under biholomorphic automorphisms of a domain suggests immediately other classes of symbols, obtained from the initial ones by means of biholomorphic automorphisms. But, surprisingly, the key to understanding the nature of such classes of symbols, and classifying them lies in the hyperbolic geometry of the unit disk, endowed with the Poincaré metric.

It turns out that the two cases, studied in [9] and [2] respectively can be perfectly treated as the classes of symbols constant on cycles which correspond to the two types of pencils of hyperbolic straight lines: pencils of parallel lines and pencils of lines passing through a given point. But hyperbolic geometry has a third type of pencil, the pencil of straight lines orthogonal to a given line. This suggests considering a new algebra of Toeplitz operators whose symbols are constant on cycles of this type of pencil. As we will see in Section 3 such a class of symbols again produces a commutative C^*-algebra of Toeplitz operators, and again there is an appropriate unitary operator which helps to understand the situation.

Thus the sets of symbols which generate commutative C^*-algebras of Toeplitz operators can be classified as follows: each pencil of hyperbolic straight lines, of the first, second, or third type, defines the set of symbols consisting of functions which are constant on the corresponding cycles. We conjecture that these are the only possible commutative C^*-algebras of Toeplitz operators on the unit disk.

2. Two known model cases

We recall here necessary results from [9, 10] for the first case, and from [2, 8] for the second case.

We begin with the case of the upper half-plane Π in $\mathbb{C}$. Introduce the space $L_2(\Pi)$ with the usual Lebesgue plane measure $d\mu(w) = dudv$, $w = u + iv$, and its Bergman subspace $\mathcal{A}^2(\Pi)$, consisting of all functions analytic in Π. It is well known that the Bergman orthogonal projection B_Π of $L_2(\Pi)$ onto $\mathcal{A}^2(\Pi)$ is given by

$$(B_\Pi\varphi)(w) = -\frac{1}{\pi}\int_\Pi \frac{\varphi(\omega)}{(w-\overline{\omega})^2}\, d\mu(\omega).$$

Introduce the unitary operators

$$U_1 = F \otimes I : \, L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+),$$

where $F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is the Fourier transform

$$(Ff)(u) = \frac{1}{\sqrt{2\pi}}\int_\mathbb{R} e^{-iu\xi} f(\xi)\, d\xi,$$

and

$$U_2 : L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

which is defined by the rule

$$U_2 : \varphi(u,v) \longmapsto \frac{1}{\sqrt{2|x|}}\, \varphi(x, \frac{y}{2|x|}).$$

The inverse operator $U_2^{-1} = U_2^* : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ is given by

$$U_2^{-1} : \varphi(x,y) \longmapsto \sqrt{2|u|}\, \varphi(u, 2|u| \cdot v).$$

Leting $\ell_0(y) = e^{-y/2}$, we have $\ell_0(y) \in L_2(\mathbb{R}_+)$ and $\|\ell_0(y)\| = 1$. Denote by L_0 the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_0(y)$.

Theorem 2.1. *The unitary operator $U = U_2 U_1$ is an isometric isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ under which the Bergman space $\mathcal{A}^2(\Pi)$ is mapped onto $L_2(\mathbb{R}_+) \otimes L_0$,*

$$U : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+) \otimes L_0,$$

where L_0 is the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_0 = e^{-y/2}$.

Introduce the isometric imbedding

$$R_0 : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

by the rule

$$(R_0 f)(x,y) = \chi_+(x)\, f(x)\, \ell_0(y),$$

where $\chi_+(x)$ is the characteristic function of the positive half-line. The adjoint operator $R_0^* : L_2(\Pi) \to L_2(\mathbb{R}_+)$ is given by

$$(R_0^*\varphi)(x) = \chi_+(x) \int_{\mathbb{R}_+} \varphi(x,\eta)\, \ell_0(\eta)\, d\eta,$$

and

$$\begin{aligned} R_0^* R_0 = I \quad &: \quad L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+), \\ R_0 R_0^* = \chi_+ I \otimes P_0 \quad &: \quad L_2(\Pi) \longrightarrow L_2(\mathbb{R}_+) \otimes L_0. \end{aligned}$$

Now the operator $R = R_0^* U$ maps the space $L_2(\Pi)$ onto $L_2(\mathbb{R}_+)$, and the restriction

$$R|_{\mathcal{A}^2(\Pi)} : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+)$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : L_2(\mathbb{R}_+) \longrightarrow \mathcal{A}^2(\Pi) \subset L_2(\Pi)$$

is an isometric isomorphism of $L_2(\mathbb{R}_+)$ onto the subspace $\mathcal{A}^2(\Pi)$ of the space $L_2(\Pi)$.

Remark 2.2. We have

$$\begin{aligned} R\, R^* = I \quad &: \quad L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+), \\ R^* R = B_\Pi \quad &: \quad L_2(\Pi) \longrightarrow \mathcal{A}^2(\Pi). \end{aligned}$$

Introduce the C^*-algebra of bounded measurable symbols which depend only on v (the imaginary part of a variable $w = u + iv$) $\mathcal{A}_\infty = \mathbb{C} \otimes L_\infty(\mathbb{R}_+)$ ($\subset L_\infty(\mathbb{R} \times \mathbb{R}_+)$), and consider the Toeplitz operator algebra $\mathcal{T}(\mathcal{A}_\infty)$ generated by all operators of the form

$$T_a : \varphi \in \mathcal{A}^2(\Pi) \longmapsto B_\Pi a\varphi \in \mathcal{A}^2(\Pi),$$

where $a = a(v) \in \mathcal{A}_\infty$.

Theorem 2.3. *Let $a = a(v)$ be a measurable function on $\mathbb{R}_+$. Then the Toeplitz operator T_a acting on $\mathcal{A}^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_a I = R T_a R^*$, acting on $L_2(\mathbb{R}_+)$. The function $\gamma_a(x)$ is given by*

$$\gamma_a(x) = \int_{\mathbb{R}_+} a(\frac{y}{2x})\, e^{-y}\, dy, \quad x \in \mathbb{R}_+. \tag{2.1}$$

Corollary 2.4. *The algebra $\mathcal{T}(\mathcal{A}_\infty)$ is commutative. The isomorphic imbedding*

$$\tau_\infty : \mathcal{T}(\mathcal{A}_\infty) \longrightarrow C_b(\mathbb{R}_+)$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}_\infty)$,

$$\tau_\infty : T_a \longmapsto \gamma_a(x),$$

where $a = a(v) \in \mathcal{A}_\infty$.

Consider now the unit disk $\mathbb{D}$ in $\mathbb{C}$. Introduce the space $L_2(\mathbb{D})$ with the usual Lebesgue plane measure $d\mu(z) = dxdy$, $z = x + iy$, and its Bergman subspace $\mathcal{A}^2(\mathbb{D})$, consisting of all functions analytic in $\mathbb{D}$. The orthogonal Bergman projection $B_\mathbb{D}$ of $L_2(\mathbb{D})$ onto $\mathcal{A}^2(\mathbb{D})$ is given by

$$(B_\mathbb{D}\varphi)(z) = \frac{1}{\pi} \int_\mathbb{D} \frac{\varphi(\zeta)\, d\mu(\zeta)}{(1 - z\overline{\zeta})^2}.$$

Passing to the polar coordinates we have

$$\begin{aligned} L_2(\mathbb{D}) = L_2(\mathbb{D}, d\mu(z)) &= L_2([0,1), rdr) \otimes L_2([0,2\pi), d\theta) \\ &= L_2([0,1), rdr) \otimes L_2(S^1, \frac{dt}{it}) = L_2([0,1), rdr) \otimes L_2(S^1), \end{aligned}$$

where S^1 is the unit circle, and

$$\frac{dt}{it} = |dt| = d\theta$$

is the element of length.

Introduce the unitary operator

$$U_1 = I \otimes \mathcal{F} :\ L_2([0,1), rdr) \otimes L_2(S^1) \longrightarrow L_2([0,1), rdr) \otimes l_2 = l_2(L_2([0,1), rdr)),$$

where the discrete Fourier transform $\mathcal{F} : L_2(S^1) \to l_2$ is given by

$$\mathcal{F} :\ f \longmapsto c_n = \frac{1}{\sqrt{2\pi}} \int_{S^1} f(t)\, t^{-n}\, \frac{dt}{it}, \quad n \in \mathbb{Z},$$

and its inverse $\mathcal{F}^{-1} = \mathcal{F}^* : l_2 \to L_2(S^1)$ is given by

$$\mathcal{F}^{-1} : \{c_n\}_{n\in\mathbb{Z}} \longmapsto f = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n\, t^n.$$

For each $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ introduce the unitary operator

$$u_n : L_2([0,1), rdr) \longrightarrow L_2([0,1), rdr)$$

by the rule

$$(u_n f)(r) = \frac{1}{\sqrt{n+1}}\, r^{-\frac{n}{n+1}}\, f(r^{\frac{1}{n+1}}),$$

then the inverse operator $u_n^{-1} = u_n^* : L_2([0,1), rdr) \longrightarrow L_2([0,1), rdr)$ is given by

$$(u_n^{-1} f)(r) = \sqrt{n+1}\, r^n\, f(r^{n+1}).$$

Finally, define the unitary operator

$$U_2 : l_2(L_2([0,1), rdr)) \longrightarrow l_2(L_2([0,1), rdr)) = L_2([0,1), rdr) \otimes l_2$$

as follows,

$$U_2 : \{c_n(r)\}_{n\in\mathbb{Z}} \ \longmapsto\ \{(u_{|n|} c_n)(r)\}_{n\in\mathbb{Z}}.$$

Let $\ell_0(r) = \sqrt{2}$; we have $\ell_0(r) \in L_2([0,1), rdr)$ and $\|\ell_0(r)\| = 1$. Denote by L_0 the one-dimensional subspace of $L_2([0,1), rdr)$ generated by $\ell_0(r)$. Denote by l_2^+ the subspace of (two-sided) l_2, consisting of all sequences $\{c_n\}_{n\in\mathbb{Z}}$, such that $c_n = 0$ for all $n \in \mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{Z}_+$, and introduce the sequence $\chi_+ = \{\chi_+(n)\}_{n\in\mathbb{Z}} \in l_\infty$, where $\chi_+(n) = 1$ for $n \in \mathbb{Z}_+$, and $\chi_+(n) = 0$ for $n \in \mathbb{Z}_-$.

Theorem 2.5. *The unitary operator $U = U_2 U_1$ is an isometric isomorphism of the space $L_2(\mathbb{D})$ onto $L_2([0,1), rdr) \otimes l_2$ under which the Bergman space $\mathcal{A}^2(\mathbb{D})$ is mapped onto $L_0 \otimes l_2^+$,*

$$U : \mathcal{A}^2(\mathbb{D}) \longrightarrow L_0 \otimes l_2^+,$$

where L_0 is the one-dimensional subspace of $L_2([0,1), rdr)$ generated by $\ell_0(r) = \sqrt{2}$.

Introduce the isometric imbedding

$$R_0 : l_2^+ \longrightarrow L_2([0,1), rdr) \otimes l_2$$

by the rule

$$R_0 :\ \{c_n\}_{n\in\mathbb{Z}_+} \ \longmapsto\ \ell_0(r)\{\chi_+(n) c_n\}_{n\in\mathbb{Z}}.$$

The adjoint operator $R_0^* : L_2([0,1), rdr) \otimes l_2 \to l_2^+$ is given by

$$R_0^* :\ \{c_n(r)\}_{n\in\mathbb{Z}} \ \longmapsto\ \left\{\chi_+(n) \int_0^1 c_n(\rho)\sqrt{2}\,\rho\, d\rho\right\}_{n\in\mathbb{Z}_+}.$$

Now the operator $R = R_0^* U$ maps the space $L_2(\mathbb{D})$ onto l_2^+, and the restriction

$$R|_{\mathcal{A}^2(\mathbb{D})} : \mathcal{A}^2(\mathbb{D}) \longrightarrow l_2^+$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : l_2^+ \longrightarrow \mathcal{A}^2(\mathbb{D}) \subset L_2(\mathbb{D})$$

is an isometric isomorphism of l_2^+ onto the subspace $\mathcal{A}^2(\mathbb{D})$ of the space $L_2(\mathbb{D})$.

Remark 2.6. We have

$$\begin{aligned} R\,R^* = I &\quad : \quad l_2^+ \longrightarrow l_2^+, \\ R^* R = B_{\mathbb{D}} &\quad : \quad L_2(\mathbb{D}) \longrightarrow \mathcal{A}^2(\mathbb{D}). \end{aligned}$$

Theorem 2.7. *Let $a = a(r)$ be a measurable function on the segment $[0,1]$. Then the Toeplitz operator T_a acting on $\mathcal{A}^2(\mathbb{D})$ is unitary equivalent to the multiplication operator $\gamma_a I = R\,T_a R^*$, acting on l_2^+. The sequence $\gamma_a = \{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$ is given by*

$$\gamma_a(n) = \int_0^1 a(r^{\frac{1}{2(n+1)}})\,dr = (n+1)\int_0^1 a(\sqrt{r})\,r^n\,dr, \quad n \in \mathbb{Z}_+. \tag{2.2}$$

We note that any sequence of the form (2.2) necessarily has some special properties, for example, as it stated in [2], $\lim_{n\to\infty}(\gamma_a(n+1) - \gamma_a(n)) = 0$. That is the set of all sequences (2.2) forms a special subclass in l_∞, which is not the case for Toeplitz operators with radial symbols acting on the Fock space [4], where corresponding set of sequences coincides with l_∞. Thus, contrary to the Fock space case, the product of two bounded Toeplitz operators with radial symbols is not, in general, a Toeplitz operator itself. Some sufficient conditions when this product is a Toeplitz operator are given in [3], but the complete answer to this question remains as an open problem.

3. The third model case

Return to the upper half-plane Π, the space $L_2(\Pi)$, and its Bergman subspace $\mathcal{A}^2(\Pi)$. Passing to polar coordinates we have

$$L_2(\Pi) = L_2(\mathbb{R}_+, rdr) \otimes L_2([0,\pi], d\theta),$$

and

$$\frac{\partial}{\partial\overline{z}} = \frac{\cos\theta + i\sin\theta}{2}\left(\frac{\partial}{\partial r} + i\frac{1}{r}\frac{\partial}{\partial\theta}\right) = \frac{\cos\theta + i\sin\theta}{2r}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial\theta}\right).$$

The Bergman space $\mathcal{A}^2(\Pi)$ can be described alternatively as the (closed) subspace of $L_2(\Pi)$ which consists of all functions satisfying the equation

$$\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial\theta}\right)\varphi(r,\theta) = 0.$$

Introduce the unitary operator

$$U_1 = M \otimes I : L_2(\mathbb{R}_+, rdr) \otimes L_2([0,\pi], d\theta) \longrightarrow L_2(\mathbb{R}) \otimes L_2([0,\pi], d\theta),$$

where the Mellin transform $M : L_2(\mathbb{R}_+, rdr) \longrightarrow L_2(\mathbb{R})$ is given by

$$(M\psi)(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_+} r^{-i\lambda}\,\psi(r)\,dr.$$

The inverse Mellin transform $M^{-1} : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}_+, rdr)$ has the form

$$(M^{-1}\psi)(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} r^{i\lambda-1}\, \psi(\lambda)\, d\lambda.$$

Then, as it is easy to see,

$$(M \otimes I) \left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta} \right) (M^{-1} \otimes I) = (i\lambda - 1) + i\frac{\partial}{\partial \theta} = i\left((\lambda + i) + \frac{\partial}{\partial \theta} \right).$$

Thus the image of the Bergman space $\mathcal{A}_1^2 = U_1(\mathcal{A}^2(\Pi))$ can be described as the (closed) subspace of $L_2(\mathbb{R}) \otimes L_2([0,\pi], d\theta)$ which consists os all functions $\psi(\lambda,\theta)$ satisfying the equation

$$\left((\lambda + i) + \frac{\partial}{\partial \theta} \right) \psi(\lambda,\theta) = 0.$$

This equation easy to solve and its general L_2 solution has the form

$$\psi(\lambda,\theta) = f(\lambda) \cdot \sqrt{\frac{2\lambda}{1 - e^{-2\pi\lambda}}}\; e^{-(\lambda+i)\theta},$$

where $f(\lambda)$ is an arbitrary $L_2(\mathbb{R})$ function, and furthermore

$$\|\psi(\lambda,\theta)\|_{L_2(\mathbb{R}\times[0,\pi])} = \|f(\lambda)\|_{L_2(\mathbb{R})}.$$

Thus we arrive to the following.

Lemma 3.1. *The unitary operator U_1 is an isometric isomorphism of the space $L_2(\Pi)$ onto $L_2(\mathbb{R} \times [0,\pi])$ under which the Bergman space $\mathcal{A}^2(\Pi)$ is mapped onto*

$$\mathcal{A}_1^2 = \left\{ f(\lambda) \cdot \sqrt{\frac{2\lambda}{1 - e^{-2\pi\lambda}}}\; e^{-(\lambda+i)\theta} :\; f(\lambda) \in L_2(\mathbb{R}) \right\}.$$

Introduce the isometric imbedding $R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}_1^2 \subset L_2(\mathbb{R} \times [0,\pi])$ by the rule

$$(R_0 f)(\lambda,\theta) = f(\lambda) \cdot \sqrt{\frac{2\lambda}{1 - e^{-2\pi\lambda}}}\; e^{-(\lambda+i)\theta}.$$

The adjoint operator $R_0^* : L_2(\mathbb{R} \times [0,\pi]) \longrightarrow L_2(\mathbb{R})$ is given by

$$(R_0^*\psi)(\lambda) = \sqrt{\frac{2\lambda}{1 - e^{-2\pi\lambda}}} \int_0^\pi \psi(\lambda,\theta)\, e^{-(\lambda-i)\theta}\, d\theta,$$

and

$$\begin{aligned} R_0^* R_0 = I \quad &: \quad L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}), \\ R_0 R_0^* = B_1 \quad &: \quad L_2(\mathbb{R} \times [0,\pi]) \longrightarrow \mathcal{A}_1^2, \end{aligned}$$

where B_1 is the orthogonal projection of $L_2(\mathbb{R} \times [0,\pi])$ onto $\mathcal{A}_1^2$.

Now the operator $R = R_0^* U_1$ maps the space $L_2(\Pi)$ onto $L_2(\mathbb{R})$, and its restriction

$$R|_{\mathcal{A}^2(\Pi)} : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R})$$

is an isometric isomorphism. The adjoint operator

$$R^* = U_1^* R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}^2(\Pi) \subset L_2(\Pi)$$

is an isometric isomorphism of $L_2(\mathbb{R})$ onto the Bergman subspace $\mathcal{A}^2(\Pi)$ of the space $L_2(\Pi)$.

Remark 3.2. We have

$$\begin{array}{rcl} RR^* = I & : & L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}), \\ R^*R = B_\Pi & : & L_2(\Pi) \longrightarrow \mathcal{A}^2(\Pi). \end{array}$$

Denote by $\mathcal{A}'_\infty$ the C^*-algebra of bounded measurable homogeneous functions on Π of order zero, or functions depending only on the polar coordinate θ. Also consider the Toeplitz operator algebra $\mathcal{T}(\mathcal{A}'_\infty)$ which is generated by all the operators of the form

$$T_a : \varphi \in \mathcal{A}^2(\Pi) \longmapsto B_\Pi a\varphi \in \mathcal{A}^2(\Pi),$$

where $a = a(\theta) \in \mathcal{A}'_\infty$.

Theorem 3.3. *Let $a = a(\theta) \in \mathcal{A}'_\infty$. Then the Toeplitz operator T_a acting on $\mathcal{A}^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma'_a I = R T_a R^*$, acting on $L_2(\mathbb{R})$. The function $\gamma'_a(\lambda)$ is given by*

$$\gamma'_a(\lambda) = \frac{2\lambda}{1 - e^{-2\pi\lambda}} \int_0^\pi a(\theta)\, e^{-2\lambda\theta}\, d\theta, \qquad \lambda \in \mathbb{R}.$$

Proof. Given $a = a(\theta) \in \mathcal{A}'_\infty$, calculate

$$\begin{array}{rcl} R\,T_a\,R^* & = & R\,B_\Pi a B_\Pi R^* = R(R^*R)a(R^*R)R^* \\ & = & (RR^*)RaR^*(RR^*) = RaR^* \\ & = & R_0^*(M \otimes I)a(\theta)(M^{-1} \otimes I)R_0 \\ & = & R_0^* a(\theta) R_0. \end{array}$$

Finally

$$(R_0^* a(\theta) R_0 f)(\lambda) = \frac{2\lambda}{1 - e^{-2\pi\lambda}} \int_0^\pi a(\theta)\, f(\lambda)\, e^{-2\lambda\theta}\, d\theta = \gamma'_a(\lambda) \cdot f(\lambda),$$

where

$$\gamma'_a(\lambda) = \frac{2\lambda}{1 - e^{-2\pi\lambda}} \int_0^\pi a(\theta)\, e^{-2\lambda\theta}\, d\theta, \qquad \lambda \in \mathbb{R}.$$

□

Corollary 3.4. *The algebra $\mathcal{T}(\mathcal{A}'_\infty)$ is commutative. The isomorphic imbedding*

$$\tau'_\infty : \mathcal{T}(\mathcal{A}'_\infty) \longrightarrow C_b(\mathbb{R})$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}'_\infty)$

$$\tau'_\infty : T_a \longmapsto \gamma'_a(\lambda),$$

where $a = a(\theta) \in \mathcal{A}'_\infty$.

Let us mention the differences which this case presents compared with the two cases of the previous section. First of all, here there is no second unitary operator (like U_2 in the previous section) which permits a complete separation of variables in the Bergman space representation, and thus unlike to Theorems 2.1 and 2.5 we have to stop here with Lemma 3.1.

Further, the coordinate systems used (and the systems of cycles later on) in the first two cases contain as a coordinate line the boundary of the domain, and is not compatible with the boundary in the third case. As a corollary, in the first two cases there exist the following transparent connections between the Bergman and Hardy spaces.

Theorem 3.5. *The unitary operator $W = (F^{-1} \otimes I)U$, where U is the unitary operator of Theorem 2.1, is an isometric isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ under which the Bergman space $\mathcal{A}^2(\Pi)$ and the Hardy space $H^2(\Pi)$ are connected by the formula*

$$W(\mathcal{A}^2(\Pi)) = H^2(\Pi) \otimes L_0,$$

where L_0 is the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_0 = e^{-y/_2}$.

Theorem 3.6. *The unitary operator $W = (I \otimes \mathcal{F}^{-1})U$, where U is the unitary operator of Theorem 2.5, is an isometric isomorphism of the space $L_2(\mathbb{D}) = L_2([0,1), rdr) \otimes L_2(S^1, \frac{dt}{it})$ under which the Bergman space $\mathcal{A}^2(\mathbb{D})$ and the Hardy space $H^2(\mathbb{D})$ are connected by the formula*

$$W(\mathcal{A}^2(\mathbb{D})) = L_0 \otimes H^2(\mathbb{D}),$$

where L_0 is the one-dimensional subspace of $L_2([0,1), rdr)$ generated by $\ell_0(r) = \sqrt{2}$.

However, there is no natural connection between the Bergman and Hardy spaces compatible with the third type of description of the Bergman space.

4. Bergman metric

We recall some necessary facts about the Bergman metric; for further details see, for example [1, 7, 11].

Let D be a (bounded or unbounded) simply connected domain with smooth boundary on the complex plane $\mathbb{C}$. The Bergman metric of the domain D is defined by

$$ds^2 = \frac{1}{2} \frac{\partial^2 \ln K_D(z, \overline{z})}{\partial z \, \partial \overline{z}} \, dv(z),$$

where $K_D(z, \overline{\zeta})$ is the Bergman kernel function of the domain D, and $dv(z)$ is the Euclidean volume element. Denote by $\beta_D(z_1, z_2)$ the distance in the Bergman metric between two points z_1 and z_2 in D. An important feature of the Bergman metric is its invariance under the biholomorphic mappings: *if the function $\omega = \alpha(z)$ gives a biholomorphic mapping of the domain D onto a domain G, then*

$$\beta_D(z_1, z_2) = \beta_G(\alpha(z_1), \alpha(z_2)).$$

Denote by $\Delta_D(z_0, r)$ the (open) Bergman metric disk with the center $z_0 \in D$ and radius $r > 0$, i.e.,

$$\Delta_D(z_0, r) = \{z \in D : \beta_D(z_0, z) < r\}.$$

The boundary

$$\mathcal{S}_D(z_0, r) = \{z \in D : \beta_D(z_0, z) = r\}$$

of the disk $\Delta_D(z_0, r)$ is called the Bergman circle with the center z_0 and radius r.

For the case of the unit disk $\mathbb{D}$ the Bergman metric is defined by the form

$$ds^2 = \frac{1}{(1 - |z|^2)^2}\, dv(z),$$

and coincides with the hyperbolic or the Poincaré metric, realizing the non-Euclidean (Lobachevsky) geometry on the disk $\mathbb{D}$. The Bergman distance between two points z_1 and z_2 in $\mathbb{D}$ is given by

$$\beta_{\mathbb{D}}(z_1, z_2) = \frac{1}{2} \ln \frac{|1 - z_1 \overline{z}_2| + |z_1 - z_2|}{|1 - z_1 \overline{z}_2| - |z_1 - z_2|},$$

and in particular, for $z \in \mathbb{D}$,

$$\beta_{\mathbb{D}}(0, z) = \frac{1}{2}\, \ln \frac{1 + |z|}{1 - |z|}.$$

Note, that the Bergman disk $\Delta_{\mathbb{D}}(z_0, r)$ is actually an Euclidean disk whose (Euclidean) center and radius are as follows,

$$C = \frac{1 - s^2}{1 - s^2|z_0|^2}\, z_0, \qquad\qquad R = \frac{1 - |z_0|^2}{1 - s^2|z_0|^2}\, s,$$

where

$$s = \frac{e^r - e^{-r}}{e^r + e^{-r}} = \tanh r \in (0, 1).$$

Note also that the Bergman and the Euclidean centers of $\Delta_{\mathbb{D}}(z_0, r)$ coincide only for disks centered at the origin.

5. Commutative algebras of the Toeplitz operators on the unit disk

We recall first some basic facts from hyperbolic geometry on the unit disk (see, for example [5, 6]). A straight (geodesic or hyperbolic) line here is the part of an Euclidean circle orthogonal to the boundary $\partial\mathbb{D}$. There are three types of pencils of lines in the hyperbolic geometry:

1. All lines passing through a given point $z_0 \in \mathbb{D}$.
2. All lines parallel to a given line (and tending to the same point $z_0 \in \partial\mathbb{D}$).
3. All lines orthogonal to a given line (with endpoints $z_1, z_2 \in \partial\mathbb{D}$).

The orthogonal trajectories to elements of a pencil are called *cycles*, and have the following characterization for each of the above cases:

1. Each cycle is a Bergman circle centered at the point z_0.

2. Each cycle, called in this case *horocycle*, is an Euclidean circle tangent to $\partial\mathbb{D}$ at the point z_0.
3. Each cycle, called in this case *hypercycle*, is the arc of an Euclidean circle connecting the points z_1 and z_2.

In Figures 1, 2, and 3, illustrating the above cases, the cycles are drawn in bold lines. Note that in each case the set of all cycles for a fixed pencil can be characterized as the set of all trajectories equidistant in the Bergman metric to a given cycle.

It turns out that there is a deep connection between the structure of the commutative algebras of Toeplitz operators and the geometric properties of pencils of hyperbolic lines.

With each pencil of lines $\mathcal{P}$ we associate the class $\mathcal{A}_{\mathcal{P}}$ of measurable functions (symbols) $a = a(z)$ which are constant on cycles, and for which the corresponding Toeplitz operators T_a are bounded on the Bergman space $\mathcal{A}^2(\mathbb{D})$. In particular, the class $\mathcal{A}_{\mathcal{P}}$ contains all $L_\infty(\mathbb{D})$-functions which are constant on cycles. For simplicity we restrict ourselves to the case of bounded symbols. That is, in what follows by $\mathcal{A}_{\mathcal{P}}$ we mean the algebra of all $L_\infty(\mathbb{D})$-functions which are constant on cycles of the pencil $\mathcal{P}$.

We will show that each Toeplitz operator C^*-algebra $\mathcal{T}(\mathcal{A}_{\mathcal{P}})$, which is generated by all operators T_a with $a \in \mathcal{A}_{\mathcal{P}}$, is commutative, and we will describe its structure. Moreover, we conjecture that these are the only commutative Toeplitz operator algebras on the unit disk.

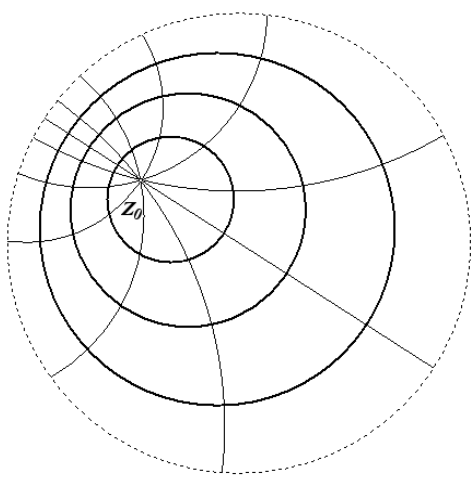

FIGURE 1. Pencil of the first type and corresponding cycles.

Start with a pencil $\mathcal{P} = \mathcal{P}(z_0)$ of the first type, which consists of the all (hyperbolic straight) lines passing through a point $z_0 \in \mathbb{D}$. The functions $a = a(z)$ from $\mathcal{A}(z_0) = \mathcal{A}_{\mathcal{P}(z_0)}$ now are constant on each Bergman circle $\mathcal{S}(z_0, r)$, $r > 0$.

The simplest case of the above situation is the case of $z_0 = 0$. The pencil $\mathcal{P}(0)$ consists of all (Euclidean) diameters of the unit circle $\mathbb{D}$, the set of cycles coincides with the set of all (Euclidean) circles having common center $z_0 = 0$, and the class of symbols $\mathcal{A}(0)$ is the class of radial symbols, mentioned in Section 2 and studied in detail in [2].

Passing to the general case of an arbitrary point $z_0 \in \mathbb{D}$, introduce the Moebius transformation

$$\alpha_{z_0}(z) = \frac{z_0 - z}{1 - z\overline{z}_0}$$

of $\mathbb{D}$ onto itself, which maps the point z_0 to the point 0. Note that this mapping is self-inverse: $\alpha_{z_0}^{-1}(z) = \alpha_{z_0}(z)$. The biholomorphic invariance of the Bergman metric implies that $\mathcal{P}(0) = \alpha_{z_0}(\mathcal{P}(z_0))$ and

$$\mathcal{A}(0) = \{a(\alpha_{z_0}(z)) :\ a(z) \in \mathcal{A}(z_0)\}.$$

Define the unitary operator $U_{z_0} = U_{z_0}^{-1} : L_2(\mathbb{D}) \longrightarrow L_2(\mathbb{D})$ as follows

$$(U_{z_0}\varphi)(z) = \alpha'_{z_0}(z)\varphi[\alpha_{z_0}(z)].$$

It is easy to see that for each $a(z) \in \mathcal{A}(z_0)$ we have

$$T_{a(z)} = U_{z_0} T_{a(\alpha_{z_0}(z))} U_{z_0}.$$

Thus the properties of Toeplitz operators with $\mathcal{A}(z_0)$ symbols follow directly from the results of [2]. Let us formulate here just one theorem.

Theorem 5.1. *The C^*-algebra $\mathcal{T}(\mathcal{A}(z_0))$ is commutative and isomorphically embedded into the algebra l_∞. This embedding*

$$\nu_{z_0} : \mathcal{T}(\mathcal{A}(z_0)) \longrightarrow l_\infty$$

is generated by the following mapping,

$$\nu_{z_0} : T_a \longmapsto \gamma_a^{(z_0)},$$

where $a(z) \in \mathcal{A}(z_0)$, and the sequence $\gamma_a^{(z_0)}$ is given by formula

$$\gamma_a^{(z_0)}(n) = (n+1) \int_0^1 a(\sqrt{\alpha_{z_0}(r)})\, r^n dr, \qquad n \in \mathbb{Z}_+.$$

Each Toeplitz operator T_a with $a(z) \in \mathcal{A}(z_0)$ admits the following spectral decomposition

$$T_a = \sum_{n=0}^{\infty} \gamma_a^{(z_0)}(n) P_n^{(z_0)},$$

where $P_n^{(z_0)}$ is the one-dimensional projection of $\mathcal{A}^2(\mathbb{D})$ onto the one-dimensional space generated by the function

$$\ell_n^{(z_0)}(z) = \sqrt{\frac{n+1}{\pi}} \left(\frac{z_0 - z}{1 - z\overline{z}_0}\right)^n \frac{|z_0|^2 - 1}{(1 - z\overline{z}_0)^2}.$$

Consider now a pencil $\mathcal{P} = \mathcal{P}(z_0)$ of the second type, which consists of all (straight) lines parallel to a given line, and tending to a same point $z_0 \in \partial\mathbb{D}$. The functions $a = a(z)$ from $\mathcal{A}(z_0) = \mathcal{A}_{\mathcal{P}(z_0)}$ now are constant on each horocycle, i.e., on each Euclidean circle tangent to $\partial\mathbb{D}$ at the point z_0.

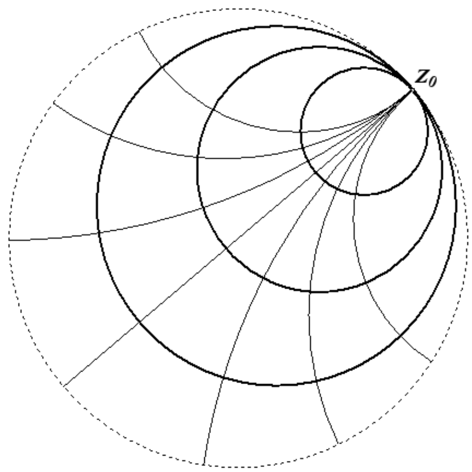

FIGURE 2. Pencil of the second type and corresponding horocycles.

Introduce the Moebius transformation

$$w = \alpha_{z_0}(z) = i\frac{z_0 + z}{z_0 - z} \tag{5.1}$$

of the unit disk $\mathbb{D}$ onto the upper half-plane Π, which maps the point $z_0 \in \partial\mathbb{D}$ to the point $\infty \in \Pi$. Then the pencil $\mathcal{P}(\infty) = \alpha_{z_0}(\mathcal{P}(z_0))$ on the upper half-plane Π consists of all semi-lines which are parallel in Euclidean sense to positive semi-axis $\{w = 0 + iv : v \in \mathbb{R}_+\}$, the set of all horocycles coincides with the set of all Euclidean straight lines parallel to the real axis being the boundary $\mathbb{R} = \partial\Pi$ of the upper half-plain, and the set

$$\mathcal{A}_\infty = \{a(\alpha_{z_0}^{-1}(w)) :\, a(z) \in \mathcal{A}(z_0)\}.$$

consists of functions depending only on v, the imaginary part of $w = u + iv \in \Pi$. Thus we are in the situation mentioned in Section 2 and studied in detail in [9].

The unitary operator $V_{z_0} : L_2(\Pi) \longrightarrow L_2(\mathbb{D})$, where

$$(V_{z_0}\varphi)(z) = i\frac{2z_0}{(z_0 - z)^2}\,\varphi\left(i\frac{z_0 + z}{z_0 - z}\right),$$

provides the following unitary equivalence of the Toeplitz operators,

$$T_{a(z)} = V_{z_0}^{-1} T_{a(\alpha_{z_0}^{-1}(w))} V_{z_0}$$

for each $a(z) \in \mathcal{A}(z_0)$, and where $\alpha_{z_0}^{-1}(w)$ is the mapping inverse to (5.1). Thus as a direct corollary of Theorem 2.3 we have

Theorem 5.2. *The algebra $\mathcal{T}(\mathcal{A}(z_0))$ is commutative. The isomorphic imbedding*

$$\tau_{z_0} : \mathcal{T}(\mathcal{A}(z_0)) \longrightarrow C_b(\mathbb{R}_+)$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}(z_0))$

$$\tau_{z_0} : T_a \longmapsto \gamma_a^{(z_0)}(x) = \int_{\mathbb{R}_+} a(\alpha_{z_0}^{-1}(\frac{y}{2x}))\, e^{-y}\, dy, \quad x \in \mathbb{R}_+,$$

where $a = a(z) \in \mathcal{A}(z_0)$, and $\alpha_{z_0}^{-1}(w)$ is the mapping inverse to (5.1).

Consider finally a pencil $\mathcal{P} = \mathcal{P}(z_1, z_2)$ of the third type, which consists of all (hyperbolic straight) lines orthogonal to a given line with endpoint z_1, $z_2 \in \partial\mathbb{D}$. The functions $a = a(z)$ from $\mathcal{A}(z_1, z_2) = \mathcal{A}_{\mathcal{P}(z_1,z_2)}$ now are constant on each hypercycle, i.e., on each Euclidean arc connecting the points z_1 and z_2.

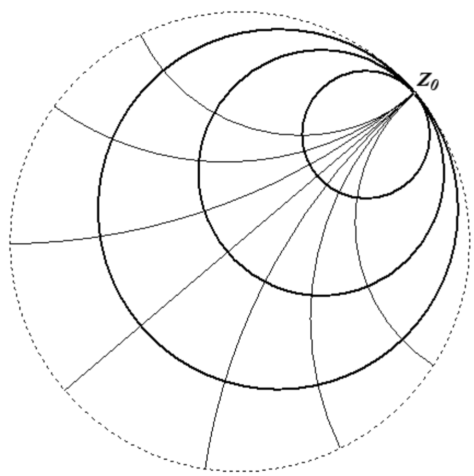

FIGURE 3. Pencil of the third type and corresponding hypercycles.

Introduce the following Moebius transformation of the unit disk $\mathbb{D}$ onto the upper half-plane Π:

$$w = \alpha_{z_1,z_2}(z) = \sqrt{\frac{z_2}{z_1}}\,\frac{z_1 - z}{z - z_2}, \tag{5.2}$$

where the value of the square root is selected to be in the upper half-plane. This transformation maps the points z_1 and z_2 on $\partial\mathbb{D}$ to the points 0 and ∞ on Π, respectively. Then the pencil $\mathcal{P}'(\infty) = \alpha_{z_1,z_2}(\mathcal{P}(z_1, z_2))$ on the upper half-plane Π consists of all Euclidean semi-circles centered at the origin, the set of all hypercycles coincides with the set of all Euclidean rays outgoing from the origin, and the set

$$\mathcal{A}'_\infty = \{a(\alpha_{z_1,z_2}^{-1}(w)) : a(z) \in \mathcal{A}(z_1, z_2)\},$$

where α_{z_1,z_2}^{-1} is the mapping inverse to (5.2), consists of homogeneous of zero order functions.

Introduce the unitary operator $W_{z_1,z_2} : L_2(\Pi) \longrightarrow L_2(\mathbb{D})$ as follows

$$(W_{z_1,z_2}\varphi)(z) = \sqrt{\frac{z_2}{z_1}}\,\frac{z_2 - z_1}{(z - z_2)^2}\,\varphi\left(\sqrt{\frac{z_2}{z_1}}\,\frac{z_1 - z}{z - z_2}\right),$$

then the Toeplitz operator algebra $\mathcal{T}(\mathcal{A}(z_1, z_2))$, we are interested in, is unitary equivalent to the algebra

$$\mathcal{T}(\mathcal{A}'_\infty) = W^{-1}_{z_1,z_2}\mathcal{T}(\mathcal{A}(z_1, z_2))W_{z_1,z_2}$$

studied in Section 3. Thus by Corollary 3.4 we have

Theorem 5.3. *The algebra $\mathcal{T}(\mathcal{A}(z_1, z_2))$ is commutative. The isomorphic imbedding*

$$\tau_{(z_1,z_2)} : \mathcal{T}(\mathcal{A}(z_1, z_2)) \longrightarrow C_b(\mathbb{R})$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}(z_1, z_2))$,

$$\tau_{(z_1,z_2)} : T_a \longmapsto \gamma_a^{(z_1,z_2)}(\lambda) = \frac{2\lambda}{1 - e^{-2\lambda\pi}} \int_0^\pi a(\alpha^{-1}_{z_1,z_2}(e^{i\theta}))\, e^{-2\lambda\theta}\, d\theta, \quad \lambda \in \mathbb{R},$$

where $a(z) \in \mathcal{A}(z_1, z_2)$, and $\alpha^{-1}_{z_1,z_2}(w)$ is the mapping inverse to (5.2).

The pencils of the first and the second types are uniquely defined by a point z_0. This point belongs to $\mathbb{D}$ and is a common point of all lines, for the first case, and belongs to $\partial\mathbb{D}$ and is a point (at infinity) to with tend all parallel lines, for the second case. The pencil of the third type is uniquely defined by two points z_1, $z_2 \in \partial\mathbb{D}$. This point determine the straight line to which all lines of a pencil are orthogonal. Thus, reformulating the results of this section, we can say each point $z_0 \in \overline{\mathbb{D}}$ or each pair z_1, $z_2 \in \partial\mathbb{D}$ determine uniquely a class of symbols for which the corresponding Toeplitz operator C^*-algebra $\mathcal{T}$ is commutative.

The structure of this algebra $\mathcal{T}$ depends on where these points are situated. For the interior points $z_0 \in \mathbb{D}$ corresponding algebra $\mathcal{T}$ is isomorphic to the algebra of *sequences*, while for the points on the boundary $\partial\mathbb{D}$ corresponding algebra $\mathcal{T}$ is isomorphic to the algebra of *functions*. Thus, for example, there exist non-zero compact operators in the first case, and no such operators in the second case; each operator has eigenvalues and one-dimensional eigenspaces in the first case, and only continuous spectra in the second case.

Note finally, that using the biholomorphic mappings the results of this section can be easily extended to the case of an arbitrary simply connected domain in $\mathbb{C}$ with a smooth boundary.

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