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Bergman Space on Tube Domains and Commuting Toeplitz Operators*

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Abstract

Let V be a pointed convex cone in $\mathbb{R}^n$ having vertex at the origin. Denote by $T^V = \mathbb{R}^n + iV$ the tube domain over V, and denote by $\mathcal{A}^2(T^V)$ the Bergman space on T^V, i.e., the subspace of $L_2(T^V)$ consisting of all functions analytic in T^V. Studying the structure of the Bergman space, we obtain several connections between the Bergman and Hardy spaces, as well as between the corresponding Bergman and Szegö projections.

Toeplitz operators with symbols depending only on the cone components are considered. It turns out that the C^*-algebra generated by these operators is commutative.

1 Introduction

Let V be a pointed convex cone in $\mathbb{R}^n$ having vertex at the origin. Denote by $T^V = \mathbb{R}^n + iV$ the tube domain over V, and introduce the space $L_2(T^V)$ with the usual Lebesgue volume measure $d\mu(z) = dxdy$, $z = x + iy \in \mathbb{C}^n$. Denote by $\mathcal{A}^2(T^V)$ the Bergman space on T^V, i.e., the subspace of $L_2(T^V)$ consisting of all functions analytic in T^V, and denote by B_{T^V} the orthogonal Bergman projection of $L_2(T^V)$ onto $\mathcal{A}^2(T^V)$.

Studying the structure of the Bergman space, we obtain several connections between the Bergman and the Hardy spaces, as well as between the corresponding Bergman and Szegö projections. In particular, we construct a partial isometry $\widetilde{R} : L_2(T^V) \to L_2(\mathbb{R}^n)$ for which we have the following isometric isomorphisms between the Bergman $\mathcal{A}^2(T^V)$ and the Hardy $H^2(T^V)$ spaces:

$$\widetilde{R}|_{\mathcal{A}^2(T^V)} : \mathcal{A}^2(T^V) \longrightarrow H^2(T^V), \qquad \widetilde{R}^*|_{H^2(T^V)} : H^2(T^V) \longrightarrow \mathcal{A}^2(T^V).$$

Furthermore, the operators $\widetilde{R}$ and $\widetilde{R}^*$ provide the following decomposition of the Bergman B_{T^V} and the Szegö P_{T^V} projections:

$$\widetilde{R}^*\widetilde{R} = B_{T^V} : L_2(T^V) \longrightarrow \mathcal{A}^2(T^V), \qquad \widetilde{R}\,\widetilde{R}^* = P_{T^V} : L_2(\mathbb{R}^n) \longrightarrow H^2(T^V).$$

For the case of a homogeneous [7] cone V we construct a unitary operator U giving an isometric isomorphism of the space $L_2(T^V) = L_2(\mathbb{R}^n) \otimes L_2(V)$, under which the Bergman space $\mathcal{A}^2(T^V)$ is mapped onto $L_2(V^*) \otimes L_{x_0}$

$$U : \mathcal{A}^2(T^V) \longrightarrow L_2(V^*) \otimes L_{x_0},$$

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where L_{x_0} is the one-dimensional subspace of $L_2(V)$, generated by $\ell_{x_0}(y) = k_V(x_0)^{-1/2}e^{-(x_0,y)}$, where the function $k_V(x)$ is given by (1.1), and x_0 is a fixed point of V^*.

This result shows "how much room the Bergman space $\mathcal{A}^2(T^V)$ occupies inside $L_2(T^V)$", and is the exact Bergman space analog of the well known Hardy space result by Bochner [1, 8]: the Fourier integral transform F gives an isometric isomorphism of the space $L_2(\mathbb{R}^n)$, under which

$$F : H^2(T^V) \longrightarrow L_2(V^*).$$

The tools used in the study of the Bergman space properties serve perfectly for studying the Toeplitz operators with symbols depending only on the cone variable. The more unpredictable result here is that such operators can be diagonalized. More exactly, we prove that each Toeplitz operator T_a with measurable symbol $a = a(y)$, acting on $\mathcal{A}^2(T^V)$, is unitary equivalent to the multiplication operator $\gamma_a I$, acting on $L_2(V^*)$. The function $\gamma_a = \gamma_a(x)$, $x \in V^*$, is given by

$$\gamma_a(x) = \frac{1}{k_V(x)} \int_V a(y)\, e^{-2(x,y)}\, dy,$$

where

$$k_V(x) = \frac{(n-1)!}{2^n} \int_{\mathrm{pr\,V}} \frac{d\sigma}{(x,\sigma)^n} = \frac{i^n}{2^n} \mathcal{K}_{V^*}(ix), \tag{1.1}$$

and $\mathcal{K}_{V^*}(z)$ is the Cauchy kernel [8] of the tube domain T^{V^*}.

It is worth mentioning that the Toeplitz operators with *functional* symbols under consideration, acting on the Bergman space $\mathcal{A}^2(T^V)$, are unitary equivalent to Toeplitz operators with *convolution operator* symbols, acting on the Hardy space $H^2(T^V)$.

An interesting feature of our symbols is that the corresponding Toeplitz operators form a *commutative* algebra. Note, that the commutativity of a Toeplitz operator algebra is quite unusual, and especially for the case of a non strictly pseudoconvex domain. At the same time we prove that the algebra $\mathcal{R}$ generated by the Bergman projection and the multiplication operators aI does non contain any non-zero compact operator, and has *infinite dimensional* irreducible representations.

2 Bergman space and Bergman projection

Let V be a pointed convex cone in $\mathbb{R}^n$ having vertex at the origin. Denote by $T^V = \mathbb{R}^n + iV$ the tube domain over V. Introduce the space $L_2(T^V)$ with the usual Lebesgue volume measure $d\mu(z) = dxdy$, $z = x + iy \in \mathbb{C}^n$. Denote by $\mathcal{A}^2(T^V)$ the Bergman space on T^V, i.e., the subspace of $L_2(T^V)$ consisting of all functions analytic in T^V, and denote by B_{T^V} the orthogonal Bergman projection of $L_2(T^V)$ onto $\mathcal{A}^2(T^V)$.

The first step was already known to Bochner [1]. Introduce the unitary operator

$$U_1 = F \otimes I : L_2(T^V) = L_2(\mathbb{R}^n) \otimes L_2(V) \longrightarrow L_2(\mathbb{R}^n) \otimes L_2(V),$$

where the Fourier transform $F : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ is given by

$$(Ff)(u) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i(x,\xi)} f(\xi)\, d\xi.$$

Then the image of the Bergman space $\mathcal{A}_1^2 = U_1(\mathcal{A}^2(T^V))$ can be described as the (closed) subspace of $L_2(T^V)$, which consists of all functions satisfying the equations

$$(F \otimes I) 2 \frac{\partial}{\partial \overline{z}_k} (F^{-1} \otimes I)\, \varphi = i(x_k + \frac{\partial}{\partial y_k})\, \varphi = 0, \tag{2.1}$$

where $k = \overline{1, n}$.

The equations (2.1) are easy to solve, and their general solution has the form

$$\varphi(x, y) = \psi(x) \cdot e^{-(x,y)}.$$

But the function $\varphi(x, y)$ has to be in $L_2(T^V)$. Calculate

$$\int_{T^V} |\psi(x)|^2 e^{-2(x,y)} dx dy = \int_{\mathbb{R}^n} |\psi(x)|^2 dx \int_V e^{-2(x,y)} dy,$$

thus it is necessary that

$$\operatorname{supp} \psi(x) \subset V^* = \{u \in \mathbb{R}^n : (u, y) > 0, \quad \text{for all} \quad y \in \overline{V}\}.$$

Now, for $x \in V^*$

$$\int_V e^{-2(x,y)} dy = \int_{\mathrm{pr\,V}} d\sigma \int_V e^{-2r(x,\sigma)} r^{n-1} dr = \frac{(n-1)!}{2^n} \int_{\mathrm{pr\,V}} \frac{d\sigma}{(x, \sigma)^n},$$

where $\mathrm{pr\,V} = V \cap S^{n-1}$.

Recall (see, for example [8]), that the function

$$\mathcal{K}_{V^*}(z) = \int_V e^{i(z,\xi)} d\xi = i^n (n-1)! \int_{\mathrm{pr\,V}} \frac{d\sigma}{(z, \sigma)^n}, \quad z \in T^{V^*}$$

is called the Cauchy kernel of a tube domain T^{V^*}, and is analytic in T^{V^*}.

Introduce the function

$$k_V(x) = \frac{(n-1)!}{2^n} \int_{\mathrm{pr\,V}} \frac{d\sigma}{(x, \sigma)^n} = \frac{i^n}{2^n} \mathcal{K}_{V^*}(ix), \quad x \in V^*.$$

Then the space $\mathcal{A}_1^2$ (the set of all L_2-solutions of the equations (2.1)) consists of all the functions

$$\varphi_0(x, y) = \chi_{V^*}(x) \, f(x) \frac{e^{-(x,y)}}{\sqrt{k_V(x)}}, \tag{2.2}$$

where $f(x) \in L_2(\mathbb{R}^n)$, $\chi_{V^*}(x)$ is the characteristic function of the dual cone V^*, and, in addition

$$\|\varphi_0(x, y)\|_{\mathcal{A}_1^2} = \|f(x)\|_{L_2(V^*)}.$$

Lemma 2.1 *The orthogonal projection*

$$B_1 = (F \otimes I) B_{T^V} (F^{-1} \otimes I) : L_2(T^V) \longrightarrow \mathcal{A}_1^2$$

has the form

$$(B_1 \varphi)(x, y) = \chi_{V^*}(x) \frac{1}{k_V(x)} e^{-(x,y)} \int_V \varphi(x, \eta) \cdot e^{-(x,\eta)} d\eta.$$

PROOF. First, for each function $\varphi \in L_2(T^V)$ its image $(B_1\varphi)(x,y)$ has the form (2.2) with

$$f(x) = \chi_{V^*}(x)\frac{1}{\sqrt{k_V(x)}}\int_V \varphi(x,\eta)\cdot e^{-(x,\eta)}\,d\eta.$$

Furthermore, for each function φ_0 of the form (2.2) we have $(B_1\varphi_0)(x,y) = \varphi_0(x,y)$.
In fact,

$$\begin{aligned}(B_1\varphi_0)(x,y) &= \chi_{V^*}(x)\frac{1}{k_V(x)}\,e^{-(x,y)}\int_V \varphi_0(x,\eta)\cdot e^{-(x,\eta)}\,d\eta \\ &= \chi_{V^*}(x)\frac{1}{(k_V(x))^{3/2}}\,e^{-(x,y)}f(x)\int_V e^{-2(x,\eta)}\,d\eta \\ &= \chi_{V^*}(x)\,f(x)\frac{e^{-(x,y)}}{\sqrt{k_V(x)}} = \varphi_0(x,y).\end{aligned}$$

Thus $\operatorname{Im} B_1 = \mathcal{A}_1^2$, $B_1^2 = B_1$, and B_1 is obviously self-adjoint. □

Introduce the isometric imbedding $R_0 : L_2(V^*) \longrightarrow \mathcal{A}_1^2$ by the rule

$$(R_0 f)(x,y) = \chi_{V^*}(x)\,f(x)\,\frac{e^{-(x,y)}}{\sqrt{k_V(x)}}.$$

The adjoint operator $R_0^* : L_2(T^V) \to L_2(V^*)$ is given by

$$(R_0^*\varphi)(x) = \chi_{V^*}(x)\frac{1}{\sqrt{k_V(x)}}\int_V \varphi(x,\eta)\,e^{-(x,\eta)}\,d\eta,$$

and

$$R_0^*R_0 = I : L_2(V^*) \longrightarrow L_2(V^*), \qquad R_0R_0^* = B_1 : L_2(T^V) \longrightarrow \mathcal{A}_1^2.$$

Now the operator $R = R_0^*U_1$ maps the space $L_2(T^V)$ onto $L_2(V^*)$, and the restriction

$$R|_{\mathcal{A}^2(T^V)} : \mathcal{A}^2(T^V) \longrightarrow L_2(V^*)$$

is an isometric isomorphism. The adjoint operator

$$R^* = U_1^*R_0 : L_2(V^*) \longrightarrow \mathcal{A}^2(T^V) \subset L_2(T^V)$$

is an isometric isomorphism of $L_2(V^*)$ onto the subspace $\mathcal{A}^2(T^V)$ of the space $L_2(T^V)$. And furthermore

$$R\,R^* = I : L_2(V^*) \longrightarrow L_2(V^*), \qquad R^*R = B_{T^V} : L_2(T^V) \longrightarrow \mathcal{A}^2(T^V).$$

Theorem 2.2 *The isometric isomorphism $R^* = U_1^*R_0 : L_2(V^*) \longrightarrow \mathcal{A}^2(T^V)$ is given by*

$$(R^*f)(z) = (2\pi)^{-\frac{n}{2}}\int_{V^*}\frac{1}{\sqrt{k_V(\xi)}}\,f(\xi)\,e^{i(\xi,z)}\,d\xi.$$

PROOF. Calculate

$$\begin{aligned}(R^*f)(z) &= (F^{-1}\otimes I)(\chi_{V^*}(\xi)\, f(\xi)\, \frac{1}{\sqrt{k_V(\xi)}}\, e^{-(\xi,y)}) \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \chi_{V^*}(\xi)\, f(\xi)\, \frac{1}{\sqrt{k_V(\xi)}}\, e^{-(\xi,y)}\, e^{i(\xi,x)}\, d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{V^*} \frac{1}{\sqrt{k_V(\xi)}}\, f(\xi)\, e^{i(\xi,x+iy)}\, d\xi.\end{aligned}$$

□

Corollary 2.3 *The inverse isomorphism $R : \mathcal{A}^2(T^V) \longrightarrow L_2(V^*)$ is given by*

$$(R\varphi)(x) = \frac{1}{\sqrt{k_V(x)}}\, (2\pi)^{-\frac{n}{2}} \int_{T^V} \varphi(w)\, e^{-i(x,\overline{w})}\, d\mu(w).$$

The analogous results for symmetric cones (in the context of real Jordan algebras) were given, for example, in [6].

Denote by $H^2(T^V)$ the Hardy space [4, 6] in a tube domain T^V, and denote by P_{T^V} the orthogonal Szegö projection of $L_2(\mathbb{R}^n)$ onto $H^2(T^V)$.

Introduce the operator $\widetilde{R} : L_2(T^V) \to L_2(\mathbb{R}^n)$ as follows

$$\widetilde{R} = F^{-1}R_0^*(F\otimes I).$$

Corollary 2.4 *We have the following isometric isomorphisms between the Bergman $\mathcal{A}^2(T^V)$ and Hardy $H^2(T^V)$ spaces:*

$$\widetilde{R}|_{\mathcal{A}^2(T^V)} : \mathcal{A}^2(T^V) \longrightarrow H^2(T^V), \qquad \widetilde{R}^*|_{H^2(T^V)} : H^2(T^V) \longrightarrow \mathcal{A}^2(T^V).$$

The operators $\widetilde{R}$ and $\widetilde{R}^*$ provide the following decomposition of the Bergman B_{T^V} and Szegö P_{T^V} projections:

$$\widetilde{R}^*\widetilde{R} = B_{T^V} : L_2(T^V) \longrightarrow \mathcal{A}^2(T^V), \qquad \widetilde{R}\,\widetilde{R}^* = P_{T^V} : L_2(\mathbb{R}^n) \longrightarrow H^2(T^V).$$

3 Homogeneous cones.

In this section we consider the case when the cone V ($\subset \mathbb{R}^n$) is homogeneous. Then the dual cone V^* is homogeneous as well [7], and [3, 7] there exists a connected group $G(V^*)$ ($\subset GL_n(\mathbb{R})$) of linear transformations of V^*, such that

i) there is a one-to-one correspondence between the elements of $G(V^*)$ and the points of the cone V^*,

ii) the group $G(V^*)$ acts transitively on V^*.

For each element $g \in G(V^*)$ we have obviously that $\det g > 0$. Fix now a point $x_0 \in V^*$, and let $g_{x_0}(x)$, $x \in V^*$, be the (unique) element of $G(V^*)$, such that $g_{x_0}(x) : x \mapsto x_0$.

Note [7] that the group $G^t(V^*) = \{g^t :\ g \in G(V^*)\}$, where g^t is the transpose of g, preserves the cone V and acts transitively on it.

The operator $U_2 : L_2(T^V) \longrightarrow L_2(T^V)$ defined by

$$(U_2\varphi)(x,y) = \begin{cases} (\det g_{x_0}(x))^{1/2}\varphi(x, g_{x_0}^t(x)y), & \text{for } (x,y) \in V^*\times V \\ \varphi(x,y), & \text{for } (x,y) \in (\mathbb{R}^n\setminus V^*)\times V \end{cases},$$

is obviously unitary, and the operator

$$(U_2^{-1}\varphi)(x,y) = \begin{cases} (\det g_{x_0}^{-1}(x))^{1/2}\varphi(x,(g_{x_0}^t(x))^{-1}y), & \text{for } (x,y)\in V^*\times V \\ \varphi(x,y), & \text{for } (x,y)\in(\mathbb{R}^n\setminus V^*)\times V \end{cases}$$

is its inverse.

On the functions of the form (2.2) we have

$$\begin{aligned}(U_2\varphi_0)(x,y) &= \chi_{V^*}(x)f(x)(\det g_{x_0}(x))^{1/2}\frac{1}{\sqrt{k_v(x)}}\,e^{-(x,g_{x_0}^t(x)y)} \\ &= \chi_{V^*}f(x)\left(\frac{\det g_{x_0}(x)}{k_V(x)}\right)^{1/2} e^{-(x_0,y)}.\end{aligned}$$

Now

$$\begin{aligned}\int_{V^*}|f(x)|^2dx &= \|f\|^2_{L_2(V^*)} = \|\varphi_0\|^2 \\ &= \|U_2\varphi_0\|^2 = \int_{V^*}|f(x)|^2\frac{\det g_{x_0}(x)}{k_V(x)}\,dx\int_V e^{-2(x_0,y)}dy \\ &= \int_{V^*}|f(x)|^2\frac{\det g_{x_0}(x)}{k_V(x)}\,k_V(x_0)dx.\end{aligned}$$

Thus we have the following description of $\det g_{x_0}(x)$ in terms of the Cauchy kernel of the tube domain T^{V^*}:

$$\det g_{x_0}(x) = \frac{k_V(x)}{k_V(x_0)} = \frac{\mathcal{K}_{V^*}(ix)}{\mathcal{K}_{V^*}(ix_0)}.$$

Denote $\mathcal{A}_2^2 = U_2(\mathcal{A}_1^2)$; the operator $B_2 = U_2\,B_1\,U_2^{-1}$ is obviously the orthogonal projection of $L_2(T^V)$ onto $\mathcal{A}_2^2$. Calculate

$$\begin{aligned}(B_2\varphi)(x,y) &= U_2(\chi_{V^*}(x)\,\frac{1}{k_V(x)}\,e^{-(x,y)}\int_V e^{-(x,\eta)}\left(\frac{k_V(x_0)}{k_V(x)}\right)^{1/2}\varphi(x,(g_{x_0}^t(x))^{-1}\eta)\,d\eta) \\ &= U_2(\chi_{V^*}(x)\left(\frac{k_V(x_0)}{k_V(x)}\right)^{1/2}\frac{1}{k_V(x_0)}\,e^{-(x,y)}\int_V\varphi(u,\nu)\,e^{-(x_0,\nu)}\,d\nu) \\ &= \chi_{V^*}(x)\,\frac{1}{k_V(x_0)}\,e^{-(x_0,y)}\int_V\varphi(x,\nu)\,e^{-(x_0,\nu)}\,d\nu.\end{aligned}$$

Introduce $\ell_{x_0}(y) = \frac{1}{\sqrt{k_V(x_0)}}\,e^{-(x_0,y)}$; we have $\ell_{x_0}(y)\in L_2(V)$ and $\|\ell_{x_0}(y)\| = 1$. Denote by L_{x_0} the one-dimensional subspace of $L_2(V)$ generated by $\ell_{x_0}(y)$, then the one-dimensional projection P_{x_0} of $L_2(V)$ onto L_{x_0} has the form

$$(P_{x_0}\psi)(y) = \langle\psi,\ell_{x_0}\rangle\cdot\ell_{x_0} = \frac{1}{\sqrt{k_V(x_0)}}\,e^{-(x_0,y)}\int_V\psi(\nu)\,\frac{1}{\sqrt{k_V(x_0)}}\,e^{-(x_0,\nu)}\,d\nu. \tag{3.1}$$

Thus $B_2 = \chi_{V^*}(x)I\otimes P_{x_0}$, and we have the following theorem.

Theorem 3.1 *The unitary operator $U = U_2U_1$ gives an isometric isomorphism of the space $L_2(T^V) = L_2(\mathbb{R}^n)\otimes L_2(V)$ under which*

1. *the Bergman space $\mathcal{A}^2(T^V)$ is mapped onto $L_2(V^*) \otimes L_{x_0}$*

$$U : \mathcal{A}^2(T^V) \longrightarrow L_2(V^*) \otimes L_{x_0},$$

where L_{x_0} is the one-dimensional subspace of $L_2(V)$ generated by $\ell_{x_0} = \frac{1}{\sqrt{k_V(x_0)}}\, e^{-(x_0,y)}$;

2. *the Bergman projection B_{T^V} is unitary equivalent to the following one,*

$$U\, B_{T^V}\, U^{-1} = \chi_{V^*} I \otimes P_{x_0},$$

where P_{x_0} is the one-dimensional projection (3.1) of $L_2(V)$ onto L_{x_0}.

Remark 3.2 This theorem shows "how much room the Bergman space $\mathcal{A}^2(T^V)$ occupies inside $L_2(T^V)$", and is the exact Bergman space analog of the well known Hardy space result by Bochner [1, 8]: *the Fourier integral transform F gives an isometric isomorphism of the space $L_2(\mathbb{R}^n)$, under which*

$$F : H^2(T^V) \longrightarrow L_2(V^*).$$

Corollary 3.3 *The unitary operator $W = (F^{-1} \otimes I)U_2(F \otimes I)$ gives an isometric isomorphism of the space $L_2(T^V) = L_2(\mathbb{R}^n) \otimes L_2(V)$ under which*

1. *the Bergman $\mathcal{A}^2(T^V)$ and the Hardy $H^2(T^V)$ spaces are connected by the formula*

$$W(\mathcal{A}^2(T^V)) = H^2(T^V) \otimes L_{x_0},$$

2. *the Bergman B_{T^V} and the Szegö P_{T^V} projections are connected by the formula*

$$W\, B_{T^V}\, W^{-1} = P_{T^V} \otimes P_{x_0},$$

where P_{x_0} is the one-dimensional projection (3.1) of $L_2(V)$ onto one-dimensional space L_{x_0} generated by $\ell_{x_0}(y) = \frac{1}{\sqrt{k_V(x_0)}}\, e^{-(x_0,y)} \in L_2(V)$.

4 Toeplitz operators with cone component symbols

Return now to the general case. Consider a pointed convex cone V ($\subset \mathbb{R}^n$), not necessarily homogeneous, the corresponding tube domain T^V, and the Bergman projection B_{T^V} acting on $L_2(T^V)$.

Theorem 4.1 *Let $a = a(y)$, $y \in V$, be a measurable function. Then the Toeplitz operator T_a acting on $\mathcal{A}^2(T^V)$ is unitary equivalent to the multiplication operator $\gamma_a(x)I$ acting on $L_2(V^*)$. The function $\gamma_a(x)$ is given by*

$$\gamma_a(x) = \frac{1}{k_V(x)} \int_V a(y)\, e^{-2(x,y)}\, dy = \int_V a(y)\, \ell_x^2(y)\, dy, \quad x \in V^*, \tag{4.1}$$

where

$$\ell_x(y) = \frac{1}{\sqrt{k_V(x)}}\, e^{-(x,y)} \in L_2(V). \tag{4.2}$$

PROOF. The operator T_a is obviously unitary equivalent to the operator

$$\begin{aligned} R\,T_a R^* &= R\,B_{T^V} a B_{T^V} R^* = R(R^*R)\,a(R^*R)R^* = (RR^*)R\,aR^*(RR^*) \\ &= R\,aR^* = R_0^*(F\otimes I\,)\,a(y)(F^{-1}\otimes I)R_0 = R_0^*\,a(y)R_0. \end{aligned}$$

Calculate now

$$\begin{aligned} (R_0^*\,a(y)R_0 f)(x) &= \int_V \chi_{V^*}(x)\,\frac{1}{\sqrt{k_V(x)}}\,e^{-(x,y)}\chi_{V^*}(x)\,\frac{1}{\sqrt{k_V(x)}}\,a(y)\,f(x)\,e^{-(x,y)}\,d\eta \\ &= \gamma_a(x)\cdot f(x), \end{aligned}$$

where

$$\gamma_a(x) = \frac{1}{k_V(x)}\int_V a(y)\,e^{-2(x,y)}\,dy = \int_V a(y)\,\ell_x^2(y)\,dy, \quad x\in V^*,$$

with $\ell_x(y)$ given by (4.2). $\square$

Corollary 4.2 *Toeplitz operator T_a with measurable symbol $a=a(y)$ is bounded on $L_2(T^V)$ if and only if the function $\gamma_a(x)$ is bounded, and*

$$\|T_a\| = \sup_{x\in V^*}|\gamma_a(x)|.$$

The spectrum of bounded Toeplitz operators coincides with its essential spectrum, and is given by

$$\mathrm{sp}\,T_a = \text{ess-sp}\,T_a = \overline{\{\gamma_a(x) : x\in V^*\}}.$$

Let $\mathcal{A}_\infty = \mathbb{C}\otimes L_\infty(V)\subset L_\infty(T^V)$. For a function $a\in\mathcal{A}_\infty$ the corresponding Toeplitz operator T_a is bounded, and the function $\gamma_a(x)$ is obviously continuous. Denote by $\mathcal{T}(\mathcal{A}_\infty)$ the C^*-algebra generated by all Toeplitz operators T_a with $a\in\mathcal{A}_\infty$.

It is worth mentioning, that the Toeplitz operators with *functional* symbols $a\in\mathcal{A}_\infty$ acting on the Bergman space $\mathcal{A}^2(T^V)$ are unitary equivalent to Toeplitz operators with *convolution operator* symbols acting on the Hardy space $H^2(T^V)$. Really, the Toeplitz operator T_a with $a\in\mathcal{A}_\infty$ is unitary equivalent to the following operator (acting on $H^2(T^V)$) :

$$\begin{aligned} \widetilde{R}\,T_a\widetilde{R}^*|_{H^2(T^V)} &= \widetilde{R}\,B_{T^V} a B_{T^V}\widetilde{R}^*|_{H^2(T^V)} = \widetilde{R}\widetilde{R}^*\widetilde{R}\,a\widetilde{R}^*\widetilde{R}\widetilde{R}^*|_{H^2(T^V)} \\ &= P_{T^V}\widetilde{R}\,a\widetilde{R}^*P_{T^V}|_{H^2(T^V)} = P_{T^V}F^{-1}R_0^*\,a(y)R_0FP_{T^V}|_{H^2(T^V)} \\ &= P_{T^V}F^{-1}\chi_{V^*}(\xi)\,\gamma_a(\xi)FP_{T^V}|_{H^2(T^V)} = T_{\widetilde{\gamma}_a(D)}, \end{aligned}$$

where $\widetilde{\gamma}_a(D) = F^{-1}\widetilde{\gamma}_a(\xi)F$, and $\widetilde{\gamma}_a\in L_\infty(\mathbb{R}^n)$ is a function coinciding with γ_a in the dual cone V^*. An easy calculation shows that

$$\gamma_{a(y)}(\rho x) = \gamma_{a(\frac{y}{\rho})}(x), \quad \text{for each} \quad \rho>0.$$

In particular, this means that if a function $a\in\mathcal{A}_\infty$ is homogeneous of order zero, then the function $\gamma_a(x)$ is, and in this case (extending γ_a to $\widetilde{\gamma}_a$ as a homogeneous of zero order function) the operator $\widetilde{\gamma}_a(D) = F^{-1}\widetilde{\gamma}_a(\xi)F$ is a classical Mikhlin-Calderon-Zygmund type multidimensional singular integral operator (= zero order pseudodifferential operator).

Theorem 4.3 *The algebra $\mathcal{T}(\mathcal{A}_\infty)$ is commutative. The isometric imbedding*

$$\tau : \mathcal{T}(\mathcal{A}_\infty) \longrightarrow C_b(V^*)$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}_\infty)$

$$\tau : T_a \longmapsto \gamma_a(x),$$

where the function $\gamma_a(x)$ is given by (4.1).

PROOF. Follows from Theorem 4.1. □

Note that the commutativity of a Toeplitz operator algebra is quite unusual, and especially for the case of a non-strictly pseudoconvex domain. Normally the studied Toeplitz operator algebras for this type of domains are irreducible, contain the ideal of the compact operators, and even their quotient algebras modulo compact operators are non commutative. The irreducibility of a Toeplitz operator algebra implies, in particular, that this algebra can not have invariant subspaces. At the same time our algebra $\mathcal{T}(\mathcal{A}_\infty)$ has a sufficiently rich structure of invariant subspaces.

Theorem 4.4 *Let Λ be a measurable subset of the dual cone V^* having positive measure. Then the subspace*

$$\mathcal{A}_\Lambda = \left\{ \varphi(z) = (2\pi)^{-\frac{n}{2}} \int_\Lambda f(\xi) \frac{1}{\sqrt{k_V(\xi)}} e^{i(\xi,z)} d\xi \,:\, f(\xi) \in L_2(\Lambda) \right\}$$

is invariant for the algebra $\mathcal{T}(\mathcal{A}_\infty)$.

PROOF. Follows directly from Theorems 4.1 and 2.2. □

Besides the Toeplitz operator algebra $\mathcal{T}(\mathcal{A}_\infty)$, consider the algebra $\mathcal{R}(\mathcal{A}_\infty, B_{T^V})$ generated by all operators of the form

$$A = aI + bB_{T^V},$$

where $a = a(y)$, $b = b(y) \in \mathcal{A}_\infty$, acting on $L_2(T^V)$.

The algebra $\mathcal{R}(\mathcal{A}_\infty, B_{T^V})$ is naturally isomorphic to the algebra

$$\mathcal{R}_0 = (F \otimes I)\mathcal{R}(\mathcal{A}_\infty, B_{T^V})(F^{-1} \otimes I)$$

of operators acting on the space $L_2(T^V) = L_2(\mathbb{R}^n, L_2(V))$. Under this identification the above isomorphism is generated by the following mapping of the generators of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_{T^V})$:

$$B_{T^V} \longmapsto \int_{\mathbb{R}^n}^{\oplus} \chi_{V^*}(x) P_x \, dx, \qquad aI \longmapsto \int_{\mathbb{R}^n}^{\oplus} a(y) I \, dx,$$

where P_x is the one-dimensional projection (3.1) of $L_2(V)$ onto the one-dimensional space L_x generated by the function $\ell_x(y)$ of (4.2). Thus the algebra $\mathcal{R}_0$ splits onto the direct integral (see, for example, [5], Chapter IV) of the algebras $\mathcal{R}_0(x)$, $x \in \mathbb{R}^n$,

$$\mathcal{R}_0 = \int_{\mathbb{R}^n}^{\oplus} \mathcal{R}_0(x) \, dx,$$

where for $x \in V^*$ the algebra $\mathcal{R}_0(x) = \mathcal{R}(L_\infty(V), P_x)$ is generated by one-dimensional projection P_x of the form (3.1) and by the multiplication operators $a(y)I$, $a(y) \in L_\infty(V)$; for $x \in \mathbb{R}^n \setminus V^*$ the algebra $\mathcal{R}_0(x) \cong L_\infty(V)$ is generated only by the multiplication operators $a(y)I$, $a(y) \in L_\infty(V)$.

The proof of the following lemma is trivial.

Lemma 4.5 *For $x \in \mathbb{R}^n \setminus V^*$ all the algebras $\mathcal{R}_0(x)$ are isomorphic to $L_\infty(V)$. The homomorphism*

$$\nu_x : \mathcal{R}(\mathcal{A}_\infty, B_{T^V}) \cong \mathcal{R}_0 \longrightarrow \mathcal{R}_0(x) \cong L_\infty(V) \tag{4.3}$$

is generated by the following mapping of generators of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_{T^V})$

$$\begin{aligned} \nu_x \quad &: \quad aI \longmapsto a(y), \\ \nu_x \quad &: \quad B_{T^V} \longmapsto 0. \end{aligned}$$

Lemma 4.6 *For each $x \in V^*$ the algebra $\mathcal{R}_0(x)$ is irreducible, contains the ideal $\mathcal{K}$ of all compact in $L_2(V)$ operators, and its Calkin algebra $\widehat{\mathcal{R}}_0(x) = \mathcal{R}_0(x)/\mathcal{K}$ is isomorphic to $L_\infty(V)$.*

Proof. To prove the first statement, we show that the algebra $\mathcal{R}_0(x)$ does not have any non-trivial invariant subspace. First, observe that all the invariant subspaces of the subalgebra $\{a(y)I : a \in L_\infty(V)\} \cong L_\infty(V)$ of the algebra $\mathcal{R}_0(x)$ have the form

$$X_M = \{\psi_M(y) = \chi_M(y)\psi(y) : \psi(y) \in L_2(V)\},$$

where M is a measurable subset of V having positive measure. Thus the algebra $\mathcal{R}_0(x)$ has an invariant subspace if and only if there exists a measurable set M with positive measure, such that for each $\psi_M(y) \in X_M$ we have $P_x\psi_M(y) \in X_M$. Further, $P_x\psi_M = \langle \chi_M\psi, \ell_x\rangle \cdot \ell_x$. The function ℓ_x is always nonzero on V, thus $\langle \chi_M\psi, \ell_x\rangle$ has to be zero on M, for all $\psi \in L_2(V)$, but for $\psi = \ell_x$ we have

$$\langle \chi_M\ell_x, \ell_x\rangle = \langle \chi_M\ell_x, \chi_M\ell_x\rangle = \|\chi_M\ell_x\|^2.$$

Thus χ_M has to be zero a.e. on V, and M must have measure zero. Therefore the algebra $\mathcal{R}_0(x)$ does not have any non-trivial invariant subspace, and thus is irreducible. Now the algebra $\mathcal{R}_0(x)$ is obviously a GCR-algebra, contains the one-dimensional operator P_0, and thus [2] contains the whole ideal $\mathcal{K}$ of compact operators. The last statement of the lemma is obvious. □

Lemma 4.7 *For each $x \in V^*$ the homomorphism*

$$\nu_x : \mathcal{R}(\mathcal{A}_\infty, B_{T^V}) \cong \mathcal{R}_0 \longrightarrow \mathcal{R}_0(x) \cong \mathcal{R}(L_\infty(V), P_x)$$

gives the (infinite dimensional) irreducible representation of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_{T^V})$, and for different values of $x \in V^*$ the representations ν_x are not unitary equivalent.

Proof. The first statement follows directly from Lemma 4.6. Let now x_1 and x_2 be two different points of V^*, and let $a(y)$ be a function from $\mathcal{A}_\infty$, such that $\gamma_a(x_1) \neq \gamma_a(x_2)$, where $\gamma_a(x)$ is given by (4.1). Then for the operator $A = \gamma_a(x_1)B_{T^V} - B_{T^V}aB_{T^V} \in \mathcal{R}(\mathcal{A}_\infty, B_{T^V})$ we have

$$\begin{aligned} v_{x_1} \quad &: \quad A \longmapsto (\gamma_a(x_1) - \gamma_a(x_1))P_{x_1} = 0, \\ v_{x_2} \quad &: \quad A \longmapsto (\gamma_a(x_1) - \gamma_a(x_2))P_{x_2} \neq 0. \end{aligned}$$

Thus the representations v_{x_1} and v_{x_2} are not unitary equivalent. □

Theorem 4.8 *The algebra $\mathcal{R}(\mathcal{A}_\infty, B_{T^V})$ does not contain any non-zero compact operator. All its infinite dimensional representations are parameterized by the points $x \in V^*$ and are of the form*

$$\nu_x : \mathcal{R}(\mathcal{A}_\infty, B_{T^V}) \cong \mathcal{R}_0 \longrightarrow \mathcal{R}(L_\infty(V), P_x),$$

where

$$\begin{aligned} v_x &: a(y)I \in \mathcal{R}(\mathcal{A}_\infty, B_{T^V}) \longmapsto a(y)I \in \mathcal{R}(L_\infty(V), P_x), \\ v_x &: B_{T^V} \in \mathcal{R}(\mathcal{A}_\infty, B_{T^V}) \longmapsto P_x \in \mathcal{R}(L_\infty(V), P_x). \end{aligned}$$

All finite dimensional irreducible representations of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_{T^V})$ are one-dimensional, and are the composition of the homomorphism (4.3) of Lemma 4.5 with one-dimensional representations of $L_\infty(V)$.

Proof. Follows directly from Lemmas 4.5, 4.6 and 4.7. ☐

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