Contemporary Mathematics

Toeplitz Operators on the Bergman Spaces: Inside-the-Domain Effects

Nikolai L. Vasilevski

Contents

1. Introduction

Let D be either the unit disk $\mathbb{D}$ or the upper half-plane Π in the complex plane $\mathbb{C}$, i.e., one of the two canonical representatives of a simply connected domain in $\mathbb{C}$ with a smooth boundary. There are two principally different concepts of the Hilbert space of analytic functions associated with D: the Hardy space $H^2(D)$ – the set of all $L_2(\partial D)$-functions which admit analytic extension to D, and the Bergman space $\mathcal{A}^2(D)$: the set of all $L_2(D)$-functions which are analytic in D. In both cases there are orthogonal projections: the Szegö projection $P_D : L_2(\partial D) \longrightarrow H^2(D)$ and the Bergman projection $B_D : L_2(D) \longrightarrow \mathcal{A}^2(D)$. And in both cases one of the principal operators under study is the so-called Toeplitz operator which, for given $a \in L_\infty(\partial D)$, is defined on the Hardy space as

$$T_a : \varphi \in H^2(D) \longmapsto P_D a\varphi \in H^2(D),$$

and, for given $a \in L_\infty(D)$, defined on the Bergman space as

$$T_a : \varphi \in \mathcal{A}^2(D) \longmapsto B_D a\varphi \in \mathcal{A}^2(D).$$

In both cases the function a is called the *symbol* of the Toeplitz operator T_a.

2000 *Mathematics Subject Classification.* 30C40, 46E22, 47B35, 47L15.

This work was partially supported by CONACYT Project 27934-E, México.

In spite of many similar properties these cases are quite different. One of the key reasons for this is that for the Hardy space setting there is only one direction to move – "along the boundary", while for the Bergman case there is an extra direction – "inside the domain".

The aim of these notes is to explore the impact of this extra direction on the properties of the Bergman space and Toeplitz operators acting on it.

The notes are based on lectures given at the Second Summer School on Analysis, Cuernavaca, Mexico, June, 2000. I would like to thank Salvador Pérez Esteva and Carlos Villegas for inviting me and for the beautiful organization of the School.

2. Preliminaries

2.1. Bergman kernel function and Bergman projection. Let D be a bounded simply connected domain in the complex plain $\mathbb{C}$ with smooth boundary γ. Consider the space $L_2(D)$ and its (closed) subspace $\mathcal{A}^2(D)$ consisting of analytic in D functions, the so-called *Bergman space over the domain* D.

Important properties of the Bergman space are given in the following statement (see, for example, [**22**]).

STATEMENT 2.1. *Given a compact set $K \Subset D$, there is a constant C_K, depending on K, such that*

$$\sup_{z\in K} |f(z)| \leq C_K \|f\|_{\mathcal{A}^2(D)},$$

for all $f \in \mathcal{A}^2(D)$.

From the statement it follows that for any fixed point $z \in D$ the functional $\varphi_z : f \longmapsto f(z)$ is linear and bounded. Thus by the Riesz representation theorem there exists a unique element $k_z \in \mathcal{A}^2(D)$ such that $\varphi_z = \langle \cdot, k_z \rangle$; that is,

$$f(z) = \int_D f(\zeta)\, \overline{k_z(\zeta)}\, dv(\zeta),$$

where $dv(\zeta)$ is the usual plane measure in D. The function $K_D(z,\overline{\zeta}) = \overline{k_z(\zeta)}$ is called the *Bergman kernel function* of the domain D, and has the following reproducing property

$$f(z) = \int_D K_D(z,\overline{\zeta})\, f(\zeta)\, dv(\zeta), \tag{2.1}$$

for all $f(z) \in \mathcal{A}^2(D)$.

LEMMA 2.2. *The Bergman kernel function is hermitian symmetric,*

$$K_D(z,\overline{\zeta}) = \overline{K_D(\zeta,\overline{z})}.$$

Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be any orthonormal base in $\mathcal{A}^2(D)$, then

$$K_D(z,\overline{\zeta}) = \sum_{k=1}^{\infty} \varphi_k(z)\overline{\varphi_k(\zeta)}. \tag{2.2}$$

PROOF. The function $k_\zeta(z) = \overline{K_D(\zeta,\overline{z})}$ belongs to $\mathcal{A}^2(D)$, thus by (2.1) we have

$$\begin{aligned}\overline{K_D(\zeta,\overline{z})} &= \int_D K_D(z,\overline{\tau})\,\overline{K_D(\zeta,\overline{\tau})}\,dv(\tau)\\ &= \overline{\int_D K_D(\zeta,\overline{\tau})\,\overline{K_D(z,\overline{\tau})}\,dv(\tau)}\\ &= \overline{\overline{K_D(z,\overline{\zeta})}} = K_D(z,\overline{\zeta}).\end{aligned}$$

Let now $\{\varphi_k\}$ be an orthonormal basis in $\mathcal{A}^2(D)$, then the analytic function $k_\zeta(z) = \overline{K_D(\zeta,\overline{z})} = K_D(z,\overline{\zeta})$ admits the following expansion

$$K_D(z,\overline{\zeta}) = \sum_{k=1}^{\infty} c_k(\zeta)\,\varphi_k(z),$$

where the coefficients $c_k(\zeta)$ are given by

$$\begin{aligned}c_k(\zeta) &= \langle k_\zeta(z),\varphi_k(z)\rangle = \int_D K_D(z,\overline{\zeta})\,\overline{\varphi_k(z)}\,dv(z)\\ &= \overline{\int_D K_D(\zeta,\overline{z})\,\varphi_k(z)\,dv(z)} = \overline{\varphi_k(\zeta)}.\end{aligned}$$

That is,

$$K_D(z,\overline{\zeta}) = \sum_{k=1}^{\infty} \varphi_k(z)\overline{\varphi_k(\zeta)}.$$

□

EXAMPLE 2.3. Kernel function for the unit disk.
It is well known thet the system of functions $\varphi_k(z) = \sqrt{\frac{k}{\pi}}\,z^{k-1}$, $k \in \mathbb{N}$, forms an orthonormal base in $\mathcal{A}^2(\mathbb{D})$, where $\mathbb{D}$ is the unit disk. Therefore

$$K_{\mathbb{D}}(z,\overline{\zeta}) = \sum_{k=1}^{\infty} \frac{k}{\pi}\,(z\overline{\zeta})^{k-1} = \frac{1}{\pi(1-z\overline{\zeta})^2}.$$

The explicit formula for the Bergman kernel function is known for very few domains. Apart from formula (2.2) the next lemma gives additional information about the Bergman kernel function.

LEMMA 2.4. *Let $G(z,\zeta)$ be the Green function of a domain D, then*

$$K_D(z,\overline{\zeta}) = -\frac{2}{\pi}\cdot\frac{\partial^2 G(z,\zeta)}{\partial z\,\partial\overline{\zeta}}.$$

The Bergman kernel function admits the following integral representation

$$K_D(z,\overline{\zeta}) = \frac{1}{2\pi^2 i}\int_\gamma \frac{dt}{(t-z)^2(\overline{t}-\overline{\zeta})} + l(z,\overline{\zeta}), \tag{2.3}$$

where the function $l(z,\overline{\zeta})$ is analytic on z and $\overline{\zeta}$ in $D\times D$, and is continuous on $\overline{D}\times\overline{D}$.

Let the function $\omega = \alpha(z)$ be a biholomorphic mapping of the domain D onto a domain G. Then

$$K_D(z,\overline{\zeta}) = K_G(w,\overline{\omega})\cdot\alpha'(z)\cdot\overline{\alpha'(\zeta)}. \tag{2.4}$$

PROOF. Proofs of the first two statements can be found, for example, in [**10, 15**]. Proving the third statement, we assume that the system $\{f_k(w)\}_{k\in\mathbb{N}}$ is an orthonormal base in the space $\mathcal{A}^2(G)$. Then the equality

$$\delta_{j,k} = \int_G f_j(w)\,\overline{f_k(w)}\,dv(w) = \int_D f_j(\alpha(z))\,\overline{f_k(\alpha(z))}\,\alpha'(z)\,\overline{\alpha'(z)}\,dv(z)$$

shows that the system $\varphi_k(z) = f_k(\alpha(z))\,\alpha'(z)$, $k\in\mathbb{N}$, is an orthonormal base in $\mathcal{A}^2(D)$. Thus

$$\begin{aligned} K_D(z,\overline{\zeta}) &= \sum_{k=1}^{\infty}\varphi_k(z)\,\overline{\varphi_k(\zeta)} = \sum_{k=1}^{\infty} f_k(\alpha(z))\alpha'(z)\,\overline{f_k(\alpha(\zeta))\alpha'(\zeta)} \\ &= \sum_{k=1}^{\infty} f_k(\alpha(z))\,\overline{f_k(\alpha(\zeta))}\cdot\alpha'(z)\cdot\overline{\alpha'(\zeta)} = K_G(w,\overline{\omega})\cdot\alpha'(z)\cdot\overline{\alpha'(\zeta)}. \end{aligned}$$

□

EXAMPLE 2.5. Kernel function for the upper half-plane.
The Moebius transformation

$$z = \frac{w-i}{1-iw}$$

maps the upper half-plane Π onto the unit disk $\mathbb{D}$. Thus the Bergman kernel function for the upper half-plane has the form

$$\begin{aligned} K_\Pi(w,\overline{\omega}) &= K_{\mathbb{D}}(z,\overline{\zeta})\cdot\alpha'(w)\cdot\overline{\alpha'(\omega)} \\ &= \frac{1}{\pi\left(1-\frac{w-i}{1-iw}\,\frac{\overline{\omega}+i}{1+i\overline{\omega}}\right)^2}\cdot\frac{2}{(1-iw)^2}\cdot\frac{2}{(1+i\overline{\omega})^2} \\ &= -\frac{1}{\pi\,(w-\overline{\omega})^2}. \end{aligned}$$

The projection of $L_2(D)$ onto $\mathcal{A}^2(D)$ is called the *Bergman projection*, and has the following integral representation

$$(B_D\varphi)(z) = \int_D K_D(z,\overline{\zeta})\varphi(\zeta)dv(\zeta).$$

In the sequel we will use the following properties of the operator B_D.

STATEMENT 2.6. *1. The operator B_D is the orthogonal projection of $L_2(D)$ onto $\mathcal{A}^2(D)$, i.e., bounded on $L_2(D)$, selfadjoint, and $B_D^2 = B_D$.*
2. The following equality holds

$$B_D = I - S_D S_D^* + L, \tag{2.5}$$

where

$$(S_D\varphi)(z) = -\frac{1}{\pi}\int_D\frac{\varphi(\zeta)\,dv(\zeta)}{(\zeta-z)^2}, \quad (S_D^*\varphi)(z) = -\frac{1}{\pi}\int_D\frac{\varphi(\zeta)\,dv(\zeta)}{(\overline{\zeta}-\overline{z})^2},$$

and L is a compact operator. In the case of the unit disk $\mathbb{D}$ the last (compact) summand in (2.6) is equal to zero.

PROOF. The first part follows from Theorem 2.19, Remark 2.20 and the form of a system of coherent states in $\mathcal{A}^2(D)$. The proof of the second part is given, for example, in [**15**]. □

REMARK 2.7. The formula (2.5) shows that the Bergman projection B_D belongs to the algebra generated by two-dimensional singular integral (zero-order pseudo-differential) operators on the domain D. This permits us to use corresponding results and tools studying the properties of the Bergman projection.

LEMMA 2.8. *For each function $a(z) \in C(\overline{D})$ the commutator $[B_D, a(z)I] = B_D a(z)I - a(z)B_D$ is compact.*

PROOF. Follows immediately from the representation (2.5) and the results of Mikhlin about two–dimensional singular integral operators (see, for example, [**26**]). $\square$

THEOREM 2.9. *1. Let $a(z) \in L_\infty(D)$ be continuous at all points of $\gamma = \partial D$ with $a|_\gamma \equiv 0$. Then the operators $a(z)B_D$ and $B_D a(z)I$ are compact.*
2. Let $a(z) \in L_\infty(D)$ be continuous at all points of $\gamma = \partial D$. Then the commutator $[B_D, a(z)I] = B_D a(z)I - a(z)B_D$ is compact.

PROOF. Because of $B_D a(z)I = (\overline{a(z)}B_D)^*$ it is sufficient to prove the first statement for the operator $a(z)B_D$. Given $a(z)$ and $\varepsilon > 0$, introduce

$$a_\varepsilon(z) = \begin{cases} 0, & \text{if } \inf_{t\in\gamma} |z-t| < \varepsilon, \\ a(z), & \text{otherwise .} \end{cases}$$

The integral operator $a_\varepsilon(z)B_D$ has a bounded kernel, and thus is compact on $L_2(D)$. Finally, the operator $a(z)B_D$ is compact as a uniform limit (as $\varepsilon \to 0$) of the compact operators $a_\varepsilon(z)B_D$.

To prove the second statement for given $a(z)$, we introduce a function $b(z)$ continuous on $\overline{D}$ and such that $a|_\gamma = b|_\gamma$. Then the operator $[B_D, a(z)I] = [B_D, b(z)I] - (a-b)B_D + B_D(a-b)$ is compact as a sum of compact operators. $\square$

Let again the function $\omega = \alpha(z)$ be a biholomorphic mapping of the domain D onto a domain G, and let $z = \beta(\omega)$ be the inverse mapping. Introduce the operator $W : L_2(G) \to L_2(D)$ by the rule

$$(W\varphi)(z) = \alpha'(z)\varphi[\alpha(z)],$$

then the inverse (and adjoint) operator $W^{-1} = W^* : L_2(D) \to L_2(G)$ is given by

$$(W^{-1}\varphi)(\omega) = \beta'(\omega)\varphi[\beta(\omega)].$$

LEMMA 2.10. *We have $B_D = WB_G W^{-1}$.*

PROOF. Calculate, using the equality $\beta'(\alpha(\zeta))\,\alpha'(\zeta) = 1$ and the property (2.4) of the Bergman kernel function,

$$\begin{aligned}
(WB_G W^{-1}\varphi)(z) &= \alpha'(z)\int_G K_G(\alpha(z),\overline{\omega})\,\beta'(\omega)\varphi(\beta(\omega))\,dv(\omega) \\
&= \int_D K_G(\alpha(z),\overline{\alpha(\zeta)})\,\alpha'(z)\,\beta'(\alpha(\zeta))\,\varphi(\zeta)\,\alpha'(\zeta))\,\overline{\alpha'(\zeta))}\,dv(\zeta) \\
&= \int_D K_G(\alpha(z),\overline{\alpha(\zeta)})\,\alpha'(z)\,\overline{\alpha'(\zeta))}\,\varphi(\zeta)\,dv(\zeta) \\
&= \int_D K_D(z,\overline{\zeta})\,\varphi(\zeta)\,dv(\zeta) = (B_D\varphi)(z).
\end{aligned}$$

$\square$

2.2. "Continuous" theory and local properties of the Bergman projection. Let $\mathcal{A}$ be a C^*–algebra of functions on $\overline{D}$ with the properties

(1) $C(\overline{D}) \subset \mathcal{A} \subset L_\infty(D)$,
(2) each function in $\mathcal{A}$ is continuous at all points of $\gamma = \partial D$.

We will study the C^*–algebra $\mathcal{R} = \mathcal{R}(\mathcal{A}; B_D)$ generated by all operators of the form

$$A = a(z)I + b(z)B_D + K,$$

where $a, b \in \mathcal{A}$ and K is a compact operator, acting on $L_2(D)$.

The operators B_D and $I - B_D$ are orthogonal and complementary projections, and have compact commutators with multiplication operators aI, $a \in \mathcal{A}$, by Theorem 2.9. Thus for the operators

$$A_n = a_n(z)I + b_n(z)B_D + K_n = a_n(I - B_D) + (a_n + b_n)B_D + K_n,$$

where $a_n, b_n \in \mathcal{A}$, K_n are compact operators, $n = 1, 2$, we have

$$\begin{aligned} A_1 + A_2 &= (a_1 + a_2)(I - B_D) + [(a_1 + b_1) + (a_2 + b_2)]B_D + K', \\ A_1 \cdot A_2 &= a_1 \cdot a_2(I - B_D) + (a_1 + b_1) \cdot (a_2 + b_2)B_D + K'', \end{aligned}$$

where K' and K'' are compact operators.

A good approach to the final result gives the following lemma.

LEMMA 2.11. *Let $a, b \in \mathcal{A}$, and assume that the function a is invertible in $\mathcal{A}$ and the function $(a + b)|_\gamma$ is invertible in $C(\gamma)$. Then the operator*

$$A = a(z)I + b(z)B_D + K,$$

where K is a compact operator, is Fredholmian on the space $L_2(D)$.

PROOF. Let a^{-1} be the inverse to a, and $(a+b)^{(-1)}$ be a function from $\mathcal{A}$ such that $(a + b)^{(-1)}|_\gamma$ is the inverse to $(a + b)|_\gamma$ in $C(\gamma)$. Then for the operator

$$R = a^{-1}(I - B_D) + (a + b)^{(-1)}B_D$$

we have

$$\begin{aligned} AR &= (I - B_D) + (a + b)(a + b)^{(-1)}B_D + K' \\ &= I + ((a + b)(a + b)^{(-1)} - 1)B_D + K', \\ RA &= (I - B_D) + (a + b)(a + b)^{(-1)}B_D + K'' \\ &= I + ((a + b)(a + b)^{(-1)} - 1)B_D + K'', \end{aligned}$$

where K' and K'' are compact.

By Theorem 2.9 the operator $((a + b)(a + b)^{(-1)} - 1)B_D$ is compact, and thus R is two–sided regularizer for A. □

Recall the standard local principle (see, for example, [**12, 36, 39, 41**]) which we will be using later on. Let $\widehat{\mathcal{R}}$ be a unital C^*-algebra and $\mathcal{Z}$ be its central commutative subalgebra having the same unit. Denote by M the compact of the maximal ideals of the algebra $\mathcal{Z}$; then, of course, $\mathcal{Z} \cong C(M)$. Now there is a canonical procedure "to localize with respect to the points of M". For each point $t_0 \in M$ denote by J_{t_0} the maximal ideal in the algebra $\mathcal{Z}$ with corresponds to the point $t_0 \in M$, then $J(t_0) = J_{t_0} \cdot \widehat{\mathcal{R}}$ is the closed two-sided ideal of the algebra $\widehat{\mathcal{R}}$ generated by J_{t_0}. The quotient algebra $\widehat{\mathcal{R}}(t_0) = \widehat{\mathcal{R}}/J(t_0)$ is called the local algebra at the point $t_0 \in M$. Having described all the local algebras, the initial C^*-algebra

$\widehat{\mathcal{R}}$ can be recovered as the C^*-algebra of all continuous sections of the canonical C^*-bundle over the compact space M. These sections are valued in the local algebras $\widehat{\mathcal{R}}(t_0)$ for each point $t_0 \in M$.

We describe now the symbol algebra $\mathrm{Sym}\,\mathcal{R} = \widehat{\mathcal{R}} = \mathcal{R}/\mathcal{K}$ of the algebra $\mathcal{R} = \mathcal{R}(\mathcal{A}, B_D)$. The letter $\mathcal{K}$ will always denote the ideal of all compact operators on the space under consideration. Denote by π the natural projections

$$\pi : \mathcal{R} \longrightarrow \widehat{\mathcal{R}} = \mathcal{R}/\mathcal{K}.$$

It is easy to see that the algebra $\pi(\mathcal{A}) \cong \mathcal{A}$ is a central commutative C^*–subalgebra of the algebra $\widehat{\mathcal{R}}$. Denote by $\widehat{D}$ the compact of maximal ideals of the algebra $\mathcal{A}$; it is obvious that $\gamma \subset \widehat{D}$ and $\pi(\mathcal{A}) \cong C(\widehat{D})$. Let further $\widetilde{a} \in C(\widehat{D})$ be the Gelfand image of $a \in \mathcal{A}$. We will use the above local principle, localizing with respect to the points of $\widehat{D}$. Denote by $\widehat{\mathcal{R}}(t_0)$ the local algebra, corresponding to the point $t_0 \in \widehat{D}$, and by $\pi(t_0)$ the natural projection

$$\pi(t_0) : \widehat{\mathcal{R}} \longrightarrow \widehat{\mathcal{R}}(t_0).$$

Let further

$$\pi_{t_0} = \pi(t_0) \circ \pi : \mathcal{R} \to \widehat{\mathcal{R}}(t_0),$$

and $\pi(B_D) = \widehat{B}_D$, $\pi_{t_0}(B_D) = \pi(t_0)(\widehat{B}_D) = \widehat{B}_D(t_0)$.

There are two different types of the local algebras $\widehat{\mathcal{R}}(t_0)$, corresponding to the cases $t_0 \in \widehat{D} \setminus \gamma$ and $t_0 \in \gamma \subset \widehat{D}$, respectively.

Start with the first case. Let $t_0 \in \widehat{D} \setminus \gamma$. It is easy to see that $\pi(a_1 I) \overset{t_0}{\sim} \pi(a_2 I)$ if and only if $\widetilde{a}_1(t_0) = \widetilde{a}_2(t_0)$, and thus $\pi_{t_0}(aI) = \widetilde{a}(t_0)$. Denote by $e_{t_0}(z)$ a function from $\mathcal{A}$ such that $e_{t_0}|_\gamma \equiv 0$ and $\widetilde{e}_{t_0}(t_0) = 1$. Then the operator $e_{t_0}(z)B_D$ is compact, and we have

$$\widehat{B}_D = 1 \cdot \widehat{B}_D \overset{t_0}{\sim} \pi(e_{t_0} I)\pi(B_D) = \pi(e_{t_0}(z)B_D) = 0.$$

Thus we arrive at the following lemma.

Lemma 2.12. *Let $t_0 \in \widehat{D} \setminus \gamma$. Then the local algebra $\widehat{\mathcal{R}}(t_0)$ is isomorphic to $\mathbb{C}$, and identifying them we have*

$$\pi_{t_0} : A = a(z)I + b(z)B_D + K \longmapsto \widetilde{a}(t_0) \in \mathbb{C}.$$

Let now $t_0 \in \gamma \subset \widehat{D}$. First of all there are not too many possibilities for local algebras $\widehat{\mathcal{R}}(t_0)$ in this case. The operator B_D being an orthogonal projection may have only three possibilities: $\widehat{B}_D(t_0)$ is equal either to zero, to the identity, or to a selfadjoint idempotent in $\widehat{\mathcal{R}}(t_0)$. Knowing that $\pi_{t_0}(aI) = \widetilde{a}(t_0) \in \mathbb{C}$, the algebra $\widehat{\mathcal{R}}(t_0)$ is isomorphic either to $\mathbb{C}$ (in two first cases), or to $\mathbb{C}^2$ (in the third case).

Lemma 2.13. *For each $t_0 \in \gamma \subset \widehat{D}$ the local algebras $\widehat{\mathcal{R}}(t_0)$ are isomorphic among themselves.*

Proof. Having in mind the local algebras we may assume that $\mathcal{A} = C(\overline{D})$. Let $\alpha(z)$ be a conformal mapping from D onto unit disk $\mathbb{D}$ with $1 = \alpha(t_0)$. Then the operator $(W\varphi)(z) = \alpha'(z)\varphi[\alpha(z)]$ generates by Lemma 2.10 the isomorphism $A \mapsto W^*AW$ of the algebras $\mathcal{R}(C(\overline{D}); B_D)$ and $\mathcal{R}(C(\overline{\mathbb{D}}); B_{\mathbb{D}})$, which in turn induces the isomorphism of each local algebra $\widehat{\mathcal{R}}(t_0)$ onto a specific one: $\widehat{\mathcal{C}}^*(C(\overline{\mathbb{D}}); B_{\mathbb{D}})(1)$. □

LEMMA 2.14. *Let $t_0 \in \gamma$, then the local algebra $\widehat{\mathcal{R}}(t_0)$ is isomorphic to $\mathbb{C}^2$, identifying them we have*

$$\pi_{t_0} : A = a(z)I + b(z)B_D + K \longmapsto (a(t_0), a(t_0) + b(t_0)) \in \mathbb{C}^2. \tag{2.6}$$

PROOF. Both the operators B_D and $I - B_D$ are non compact, thus $\pi(B_D) = \widehat{B}_D$ is a selfadjoint idempotent in $\widehat{\mathcal{R}}$. The form of local algebras $\widehat{\mathcal{R}}(t_0)$ shows that the algebra $\widehat{\mathcal{R}}$ is commutative. Its maximal ideal space $\mathfrak{M}$ may be identified with the union of one–dimensional representations of all local algebras, and must be compact. By previous considerations it must be the disjoint union of $\widehat{D}$ and some (closed) subset of γ. By Lemma 2.13 this subset must be either $\emptyset$ or all of γ. But the Gelfand transform of $\widehat{B}_D$ is a characteristic function of some non empty clopen (closed and open) subset of $\mathfrak{M}$, which is constant on $\widehat{D}$. Thus $\mathfrak{M} = \widehat{D} \sqcup \gamma$. Hence for each $t_0 \in \gamma$, $\widehat{B}_D(t_0)$ is an idempotent, and $\widehat{\mathcal{R}}(t_0) \cong \mathbb{C}^2$. Checking (2.6) one needs to use that $\widetilde{a}(t_0) = a(t_0)$ for the points $t_0 \in \gamma$. □

The descriptions of the local algebras $\widehat{\mathcal{R}}(t_0)$ together with the standard arguments lead directly to the following theorem.

THEOREM 2.15. *The algebra $\mathcal{R} = \mathcal{R}(\mathcal{A}, B_D)$ coincides with the set of all operators*

$$A = a(z)I + b(z)B_D + K,$$

where $a, b \in \mathcal{A}$, and K is a compact operator.

The symbol algebra $\operatorname{Sym}\mathcal{R}$ of the algebra $\mathcal{R}$ is isomorphic to the algebra $\mathcal{A} \oplus C(\gamma)$. Identifying them, the symbol homomorphism

$$\operatorname{sym} : \mathcal{R} \longrightarrow \operatorname{Sym}\mathcal{R} = \mathcal{A} \oplus C(\gamma)$$

is given by

$$\operatorname{sym} : A = a(z)I + b(z)B_D + K \longmapsto (a(z), a(t) + b(t)),$$

where $z \in \overline{D}$, $t \in \gamma$.

The operator $A = a(z)I + b(z)B_D + K$ is Fredholmian if and only if its symbol is invertible, i.e., the function a is invertible in $\mathcal{A}$ and the function $(a + b)|_\gamma$ is invertible in $C(\gamma)$.

Passing to the index calculation we start with the following result of Dzhuraev (see, for example, [**13**]). Let $a, b \in C(\overline{D})$, and $a(z) \neq 0$ on $\overline{D}$, $a(t) + b(t) \neq 0$ on γ, then the index of the Fredholm operator

$$A = a(z)I + b(z)B_D + K$$

is calculated by the formula

$$\operatorname{Ind} A = -\frac{1}{2\pi}\{\arg(a(t) + b(t))\}_\gamma.$$

A simple trick shows that this formula contains a hidden summand. In fact,

$$A = a(z)I + b(z)B_D + K = a(z)(I + \frac{b(z)}{a(z)}B_D + K_1) = a(z)A_1,$$

then obviously

$$\begin{aligned}\operatorname{Ind} A &= \operatorname{Ind} A_1 = -\frac{1}{2\pi}\{\arg \frac{a(t)+b(t)}{a(t)}\}_\gamma \\ &= -\frac{1}{2\pi}\{\arg(a(t)+b(t))\}_\gamma + \frac{1}{2\pi}\{\arg a(t)\}_\gamma.\end{aligned}$$

In the case of continuous coefficients a and b the last summand is always equal to zero, but for more general coefficients it needs to be present.

THEOREM 2.16. *Let the operator*

$$A = a(z)I + b(z)B_D + K,$$

where $a, b \in \mathcal{A}$, K is a compact operator, be Fredholmian, then

$$\operatorname{Ind} A = \frac{1}{2\pi}\{\arg a(t)\}_\gamma - \frac{1}{2\pi}\{\arg(a(t)+b(t))\}_\gamma.$$

PROOF. Start with

$$A = a(z)I + b(z)B_D + K = a(z)(I + \frac{b(z)}{a(z)}B_D + K_1) = a(z)A_1,$$

and consider the operator $A_2 = I + c(z)B_D$, where $c(z)$ is a continuous on $\overline{D}$ function whose restriction on γ coincides with $\frac{b}{a}|_\gamma$. Then obviously $\operatorname{Ind} A = \operatorname{Ind} A_1 = \operatorname{Ind} A_2$. For the last one apply the Dzhuraev formula. □

Introduce now the C^*-algebra $\mathcal{T} = \mathcal{T}(\mathcal{A})$ generated by all Toeplitz operators

$$T_a = B_D\, a(z)|_{\mathcal{A}^2(D)} : \mathcal{A}^2(D) \longrightarrow \mathcal{A}^2(D)$$

with symbols $a(z) \in \mathcal{A}$.

COROLLARY 2.17. *The algebra $\mathcal{T}$ coincides with the set of all operators of the form*

$$T = T_a + K,$$

where $a(z) \in \mathcal{A}$ and K is a compact operator. The symbol algebra $\operatorname{Sym}\mathcal{T} = \mathcal{T}/\mathcal{K}$ *of the algebra $\mathcal{T}$ is isomorphic to the algebra $C(\gamma)$. Identifying them, the symbol homomorphism*

$$\operatorname{sym} : \mathcal{T} \longrightarrow \operatorname{Sym}\mathcal{T} = C(\gamma)$$

is given by

$$\operatorname{sym} : T = T_a + K \longmapsto a(t),$$

where $t \in \gamma$.

The operator $T = T_a + K$ is Fredholmian if and only if its symbol is invertible, i.e., the function $a(t) \neq 0$ on γ, and

$$\operatorname{Ind} T = -\frac{1}{2\pi}\{\arg a(t)\}_\gamma.$$

2.3. Wick and anti-Wick symbols of operators. We introduce the necessary ingredients of the Berezin theory (see, for example, [**6, 7, 8, 9**]).

Let H be a Hilbert space, and $\{\varphi_g\}_{g\in G}$ be a subset of elements of H parameterized by elements g of some set G hawing measure $d\mu$.

Then $\{\varphi_g\}_{g\in G}$ is called a *system of coherent states* if for all $\varphi \in H$

$$\|\varphi\|^2 = (\varphi,\varphi) = \int_G |(\varphi,\varphi_g)|^2 d\mu,$$

or, equivalently, if for all $\varphi_1, \varphi_2 \in H$

$$(\varphi_1,\varphi_2) = \int_G (\varphi_1,\varphi_g)\overline{(\varphi_2,\varphi_g)}d\mu. \tag{2.7}$$

Define the isomorphic inclusion

$$V : H \longrightarrow L_2(G)$$

by the rule

$$V : \varphi \in H \longmapsto f = f(g) = (\varphi,\varphi_g) \in L_2(G).$$

By (2.7) we have

$$(\varphi_1,\varphi_2) = \langle f_1, f_2\rangle, \tag{2.8}$$

where $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are the scalar products on H and $L_2(G)$, respectively.

Further

$$f_h(g) = V\varphi_h = (\varphi_h,\varphi_g) = \overline{(\varphi_g,\varphi_h)} = \overline{V\varphi_g} = \overline{f_g(h)}.$$

Let $H_2(G) = V(H) \subset L_2(G)$.

THEOREM 2.18. *The function $f \in L_2(G)$ is an element of $H_2(G)$ if and only if for all $h \in G$*

$$\langle f, f_h\rangle = f(h).$$

PROOF. Let $f(h) = (\varphi,\varphi_h) \in H_2(G)$, then by (2.8)

$$\langle f, f_h\rangle = (\varphi,\varphi_h) = f(h).$$

Conversely, let $f \in L_2(G)$ with

$$\langle f, f_h\rangle = f(h),$$

for all $h \in G$. Consider

$$L_2(G) = H_2(G) \oplus H_2^{\perp}(G)$$

and correspondingly

$$f = f_0 + f^{\perp}.$$

Then we have

$$\langle f^{\perp}, f_h\rangle = \langle f, f_h\rangle - \langle f_0, f_h\rangle = f(h) - f_0(h) = f^{\perp}(h).$$

Now, for all $h \in G$, $f_h(g) = (\varphi_h,\varphi_g) \in H_2(G)$, thus

$$f^{\perp}(h) = \langle f^{\perp}, f_h\rangle = 0,$$

or $f^{\perp} \equiv 0$, or $f = f_0 + 0 \in H_2(G)$. □

THEOREM 2.19. *The operator*

$$(Pf)(g) = \int_G (\varphi_t,\varphi_g) f(t) d\mu(t)$$

is the orthogonal projection of $L_2(G)$ onto $H_2(G)$.

PROOF. We have

$$\begin{aligned}
\langle (Pf)(g), f_h(g)\rangle &= \int_G \left(\int_G f(t)\overline{(\varphi_g,\varphi_t)}d\mu(t)\right)\overline{(\varphi_h,\varphi_g)}d\mu(g)\\
&= \int_G f(t)d\mu(t)\int_G \overline{(\varphi_g,\varphi_t)}\,\overline{(\varphi_h,\varphi_g)}d\mu(g)\\
&= \int_G f(t)d\mu(t)\int_G (\varphi_t,\varphi_g)\,\overline{(\varphi_h,\varphi_g)}d\mu(g)\\
&= \int_G f(t)(\varphi_t,\varphi_h)d\mu(t) = (Pf)(h).
\end{aligned}$$

Thus $\operatorname{Im} P = H_2(G)$. Let $f(g) = (\varphi,\varphi_g) \in H_2(G)$, then

$$(Pf)(g) = \int_G (\varphi,\varphi_t)\,\overline{(\varphi_g,\varphi_t)}d\mu(t) = (\varphi,\varphi_g) = f(g).$$

Thus $P|_{H_2(G)} = I$, or $P^2 = P$. Finally, the operator P is obviously selfadjoint. Being an orthogonal projection the operator P is bounded, and $\|P\| = 1$. $\square$

REMARK 2.20. Let $f_g(t) = V\varphi_g = (\varphi_g,\varphi_t)$, $g \in G$, be the image in $H_2(G)$ of the system of coherent states $\{\varphi_g\}_{g\in G}$, then

$$(Pf)(g) = \langle f, f_g\rangle.$$

The function $a(g)$, $g \in G$, is called the *anti–Wick symbol* (or *contravariant*) of an operator $A : H \to H$ if

$$VAV^{-1}|_{H_2(G)} = Pa(g)P = Pa(g)I|_{H_2(G)} : H_2(G) \longrightarrow H_2(G),$$

or, in other terminology, the operator $VAV^{-1}|_{H_2(G)}$ is the Toeplitz operator

$$T_{a(g)} = Pa(g)I|_{H_2(G)} : H_2(G) \longrightarrow H_2(G),$$

with the symbol $a(g)$.

Given an operator $T : H \to H$, introduce the (Wick) function

$$\widetilde{a}(g,h) = \frac{(T\varphi_h,\varphi_g)}{(\varphi_h,\varphi_g)}, \qquad g,\, h \in G. \tag{2.9}$$

If the operator T has an anti-Wick symbol, that is $VTV^{-1} = T_{a(g)}$ for some function $a = a(g)$, then

$$\widetilde{a}(g,h) = \frac{\langle T_a f_h, f_g\rangle}{\langle f_h, f_g\rangle}, \qquad g,\, h \in G,$$

and

$$\begin{aligned}
(T_a f)(g) &= \int_G a(t)f(t)f_t(g)d\mu(t) = \int_G a(t)f_t(g)d\mu(t)\int_G f(h)f_h(t)d\mu(h)\\
&= \int_G f(h)d\mu(h)\int_G a(t)f_t(g)f_h(t)d\mu(t)\\
&= \int_G f(h)d\mu(h)\ \frac{f_h(g)}{\langle f_h, f_g\rangle}\ \int_G a(t)f_h(t)\overline{f_g(t)}d\mu(t)\\
&= \int_G \widetilde{a}(g,h)f(h)f_h(g)d\mu(h). \qquad (2.10)
\end{aligned}$$

The restriction of the function $\widetilde{a}(g,h)$ onto the diagonal

$$\widetilde{a}(g) = \widetilde{a}(g,g) = \frac{(T\varphi_g, \varphi_g)}{(\varphi_g, \varphi_g)}, \qquad g \in G,$$

is called the *Wick* (or *covariant,* or *Berezin*) symbol of the operator $T : H \to H$, and the formula (2.10) gives the representation of the operator T_a in terms of the Wick symbol.

THEOREM 2.21. *The Wick and anti–Wick symbols of an operator $A : H \to H$ are connected by the Berezin transform*

$$\widetilde{a}(g) = \int_G a(t) \frac{(\varphi_g, \varphi_t)(\varphi_t, \varphi_g)}{(\varphi_g, \varphi_g)} d\mu(t).$$

PROOF. Really

$$\begin{aligned}
\widetilde{a}(g) &= \frac{(A\varphi_g, \varphi_g)}{(\varphi_g, \varphi_g)} = \frac{\langle Paf_g, f_g\rangle}{(\varphi_g, \varphi_g)} = \frac{\langle af_g, Pf_g\rangle}{(\varphi_g, \varphi_g)} \\
&= \frac{\langle af_g, f_g\rangle}{(\varphi_g, \varphi_g)} = \frac{1}{(\varphi_g, \varphi_g)} \int_G a(t) f_g(t) \overline{f_g(t)} d\mu(t) \\
&= \int_G a(t) \frac{(\varphi_g, \varphi_t)(\varphi_t, \varphi_g)}{(\varphi_g, \varphi_g)} d\mu(t).
\end{aligned}$$

$\square$

The above notions fit perfectly in the Bergman space situation. The reproducing property of the Bergman kernel function (2.1) shows that the system of functions $k_\zeta(z) = \overline{k_z(\zeta)} = K_D(z, \overline{\zeta})$, $\zeta \in D$, forms a system of coherent states in the space $\mathcal{A}^2(D)$. That is, in our context, we have $G = D$, $d\mu = dv(\zeta)$, $H = H_2(G) = \mathcal{A}^2(D)$, $L_2(G) = L_2(D)$, $\varphi_g = f_g = k_g$, where $g = \zeta \in D$. In Section 6 we calculate Wick symbols for Toeplitz operators with the radial (anti-Wick) symbols on the unit disk. It turns out that the writing of these operators in the terms of Wick symbol gives the spectral decomposition of such operators. For the use of the Berezin transform in the context of Toeplitz operators for another classes of symbols, see, for example, [**46**].

3. Bergman and Poly-Bergman Spaces

Let $H^2_+(\mathbb{R})$ be the Hardy space on the upper half-plane Π in $\mathbb{C}$, and $P^+_{\mathbb{R}}$ be the (orthogonal) Szegö projection of $L_2(\mathbb{R})$ onto $H^2_+(\mathbb{R})$. Recall one old and well known result: *the Fourier transform F gives an isometrical isomorphism of the space $L_2(\mathbb{R})$, under which*

(1) *the Hardy space $H^2_+(\mathbb{R})$ is mapped onto $L_2(\mathbb{R}_+)$*

$$F : H^2_+(\mathbb{R}) \longrightarrow L_2(\mathbb{R}_+),$$

(2) *the Szegö projection $P^+_{\mathbb{R}} : L_2(\mathbb{R}) \longrightarrow H^2_+(\mathbb{R})$ is unitary equivalent to the following one*

$$F\, P^+_{\mathbb{R}}\, F^{-1} = \chi_+ I.$$

Furthermore, if $H^2_-(\mathbb{R})$ is the Hardy space on the lower half-plane and $P^-_{\mathbb{R}}$ is the (orthogonal) Szegö projection of $L_2(\mathbb{R})$ onto $H^2_-(\mathbb{R})$, then

$$L_2(\mathbb{R}) = H^2_+(\mathbb{R}) \oplus H^2_-(\mathbb{R}),$$

and the *same* Fourier transform gives

(1) $F(H^2_-(\mathbb{R})) = L_2(\mathbb{R}_-)$,

(2) $F\,P^-_{\mathbb{R}}\,F^{-1} = \chi_- I$.

Here we use the obvious notations: $\mathbb{R}_+$ ($\mathbb{R}_-$) is the positive (negative) half-line, and χ_+ (χ_-) is the characteristic function of $\mathbb{R}_+$ ($\mathbb{R}_-$).

Consider now the space $L_2(D)$ and its Bergman subspace $\mathcal{A}^2(D)$. A number of natural questions arises:

- how much room does the Bergman space $\mathcal{A}^2(D)$ occupy incide $L_2(D)$,
- does there exist a complete decomposition of $L_2(D)$ onto analytic and analytic like spaces,
- does there exist an unitary operator on $L_2(D)$, which *simultaneously* reduces all the pieces of decomposition to some simple transparent form,
- are there any connections between these pieces and the Hardy spaces?

In this section we give answers to these questions for the case when D is the upper half-plane Π, and in the next section – for the case of the unit disk $\mathbb{D}$. For more details see [**44, 43**].

3.1. Bergman space and Bergman projection. Let Π be the upper half-plane in $\mathbb{C}$, and consider the space $L_2(\Pi)$, with the usual Lebesgue plane measure $dv(w) = dudv$, $w = u + iv$, and its Bergman subspace $\mathcal{A}^2(\Pi)$. The Bergman projection B_Π of $L_2(\Pi)$ onto $\mathcal{A}^2(\Pi)$ has the form

$$(B_\Pi\varphi)(w) = \int_\Pi K_\Pi(w,\overline{\omega})\varphi(\omega)\,dv(\omega),$$

where the Bergman kernel function $K_\Pi(w,\overline{\omega})$ of the upper half-plane is given by (see Example 2.5)

$$K_\Pi(w,\overline{\omega}) = -\frac{1}{\pi}\cdot\frac{1}{(w-\overline{\omega})^2}.$$

Note, that the Bergman space $\mathcal{A}^2(\Pi)$ can be described alternatively as the (closed) subspace of $L_2(\Pi)$, which consists of all functions satisfying the equation

$$2\frac{\partial}{\partial\overline{w}}\,\varphi = \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\varphi = 0,$$

where $w = u + iv$.

Introduce the unitary operator

$$U_1 = F\otimes I:\ L_2(\Pi) = L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+),$$

where the Fourier transform $F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is given by

$$(Ff)(u) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-iu\xi}f(\xi)\,d\xi.$$

Then the image of the Bergman space $\mathcal{A}^2_1 = U_1(\mathcal{A}^2(\Pi))$ can be described as the (closed) subspace of $L_2(\Pi)$, which consists of all functions satisfying the equation

$$(F\otimes I)2\frac{\partial}{\partial\overline{w}}(F^{-1}\otimes I)\,\varphi = i(u+\frac{\partial}{\partial v})\,\varphi = 0. \tag{3.1}$$

The equation (3.1) is easy to solve, and its general solution has the form

$$\varphi(u,v) = \psi(u)\cdot e^{-uv}.$$

The condition $\varphi \in L_2(\Pi)$ suggests (and we will check this at the moment) that the space $\mathcal{A}_1^2$ consists of all the functions

$$\varphi_0(u,v) = \chi_+(u)\,\sqrt{2u}\,f(u)\cdot e^{-uv}, \tag{3.2}$$

where $f(u) \in L_2(\mathbb{R})$, and $\chi_+(u)$ is the characteristic function of the positive half-line, here, in addition

$$\|\varphi_0(u,v)\|_{\mathcal{A}_1^2} = \|f(u)\|_{L_2(\mathbb{R}_+)}.$$

The orthogonal projection $B_1 : L_2(\Pi) \to \mathcal{A}_1^2$ has obviously the form

$$B_1 = (F\otimes I)B_\Pi(F^{-1}\otimes I).$$

Calculate

$$\begin{aligned}(F^{-1}\otimes I)B_1\varphi &= B_\Pi(F^{-1}\otimes I)\varphi = \langle (F^{-1}\otimes I)\varphi, \overline{K_\Pi(w,\overline{\omega})}\rangle \\ &= \langle \varphi, (F\otimes I)\overline{K_\Pi(w,\overline{\omega})}\rangle = \langle \varphi, (F\otimes I)K_\Pi(\omega,\overline{w})\rangle.\end{aligned}$$

By [**4**] the Fourier transform (with respect to $\xi = \operatorname{Re}\omega$) of the function

$$K_\Pi(\omega,\overline{w}) = -\frac{1}{\pi}\cdot\frac{1}{(\omega-\overline{w})^2} = \frac{1}{\pi}\cdot\frac{1}{(i\overline{w}+\eta-i\xi)^2}$$

is equal to

$$\sqrt{\frac{2}{\pi}}\,\chi_+(\xi)\,\xi\cdot e^{-(\eta+i\overline{w})\xi}.$$

Thus

$$\begin{aligned}(F^{-1}\otimes I)B_1\varphi &= \langle \varphi, (F\otimes I)K_\Pi(\omega,\overline{w})\rangle \\ &= \sqrt{\frac{2}{\pi}}\int_\Pi \varphi(\xi,\eta)\,\chi_+(\xi)\,\xi\cdot e^{-(\eta-iw)\xi}\,d\xi\,d\eta \\ &= \sqrt{\frac{2}{\pi}}\int_\Pi \varphi(\xi,\eta)\,\chi_+(\xi)\,\xi\cdot e^{-(\eta+v)\xi}\cdot e^{iu\xi}\,d\xi\,d\eta \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}_+} 2\xi\,\chi_+(\xi)\,\varphi(\xi,\eta)\cdot e^{-(\eta+v)\xi}\,d\eta\right)e^{iu\xi}\,d\xi \\ &= (F^{-1}\otimes I)\left(2u\,\chi_+(u)\cdot e^{-uv}\int_{\mathbb{R}_+}\varphi(u,\eta)\cdot e^{-\eta u}\,d\eta\right).\end{aligned}$$

Or

$$(B_1\varphi)(u,v) = 2u\,\chi_+(u)\cdot e^{-uv}\int_{\mathbb{R}_+}\varphi(u,\eta)\cdot e^{-\eta u}\,d\eta.$$

Thus for each function $\varphi \in L_2(\Pi)$ its image $(B_1\varphi)(u,v)$ has the form (3.2) with

$$f(u) = \sqrt{2u}\int_{\mathbb{R}_+}\varphi(u,\eta)\cdot e^{-\eta u}\,d\eta,$$

and furthermore, for each function φ_0 of the form (3.2) we have

$$(B_1\varphi_0)(u,v) = \varphi_0(u,v).$$

In fact

$$\begin{aligned}(B_1\varphi_0)(u,v) &= 2u\,\chi_+(u)\cdot e^{-uv}\int_{\mathbb{R}_+}\varphi_0(u,\eta)\cdot e^{-\eta u}\,d\eta\\ &= 2u\,\chi_+(u)\cdot e^{-uv}\sqrt{2u}\,f(u)\int_{\mathbb{R}_+}e^{-2\eta u}\,d\eta\\ &= \chi_+(u)\sqrt{2u}\,f(u)\cdot e^{-uv}\int_{\mathbb{R}_+}2u\,e^{-2\eta u}\,d\eta\\ &= \chi_+(u)\sqrt{2u}\,f(u)\cdot e^{-uv}=\varphi_0(u,v).\end{aligned}$$

Thus $\mathcal{A}_1^2$ coincides with the space of all functions of the form (3.2).

Introduce the unitary operator

$$U_2 : L_2(\Pi) = L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+)$$

by the rule

$$U_2 : \varphi(u,v) \longmapsto \frac{1}{\sqrt{2|x|}}\,\varphi(x,\frac{y}{2|x|}). \tag{3.3}$$

Then the inverse operator $U_2^{-1} = U_2^* : L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+)$ is given by

$$U_2^{-1} : \varphi(x,y) \longmapsto \sqrt{2|u|}\,\varphi(u,2|u|\cdot v).$$

Denote $\mathcal{A}_2^2 = U_2(\mathcal{A}_1^2)$; the operator $B_2 = U_2\,B_1\,U_2^{-1}$ is obviously the orthogonal projection of $L_2(\Pi)$ onto $\mathcal{A}_2^2$. We calculate

$$\begin{aligned}(B_2\varphi)(x,y) &= U_2(2u\,\chi_+(u)\,e^{-uv}\int_{\mathbb{R}_+}e^{-u\eta}\sqrt{2u}\,\varphi(u,2u\eta)\,d\eta)\\ &= U_2(\sqrt{2u}\,\chi_+(u)\,e^{-uv}\int_{\mathbb{R}_+}\varphi(u,\nu)\,e^{-\nu/2}\,d\nu)\\ &= \chi_+(x)\,e^{-y/2}\int_{\mathbb{R}_+}\varphi(x,\nu)\,e^{-\nu/2}\,d\nu.\end{aligned}$$

Introducing $\ell_0(y) = e^{-y/2}$, we have $\ell_0(y)\in L_2(\mathbb{R}_+)$ and $\|\ell_0(y)\| = 1$. Denote by L_0 the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_0(y)$, then the one-dimensional projection P_0 of $L_2(\mathbb{R}_+)$ onto L_0 has the form

$$(P_0\psi)(y) = \langle\psi,\ell_0\rangle\cdot\ell_0 = e^{-y/2}\int_{\mathbb{R}_+}\psi(\nu)\,e^{-\nu/2}\,d\nu. \tag{3.4}$$

Thus

$$B_2 = \chi_+(x)I\otimes P_0.$$

The above work leads to the following theorem.

THEOREM 3.1. *The unitary operator $U = U_2U_1$ gives an isometrical isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+)$ under which*

(1) *the Bergman space $\mathcal{A}^2(\Pi)$ is mapped onto $L_2(\mathbb{R}_+)\otimes L_0$*

$$U : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+)\otimes L_0,$$

where L_0 is the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by $\ell_0 = e^{-y/2}$,

(2) *the Bergman projection* B_Π *is unitary equivalent to the following one*

$$U\, B_\Pi\, U^{-1} = \chi_+ I \otimes P_0,$$

where P_0 *is the one-dimensional projection (3.4) of* $L_2(\mathbb{R}_+)$ *onto* L_0.

REMARK 3.2. The above result describes the structure of the Bergman space inside $L_2(\Pi)$, and is a "Bergman version" of the following "Hardy version" result, mentioned in the introduction to this section: the Fourier transform F gives an isometrical isomorphism of the space $L_2(R)$, under which

(1) the Hardy space $H^2_+(\mathbb{R})$ is mapped onto $L_2(\mathbb{R}_+)$

$$F : H^2_+(\mathbb{R}) \longrightarrow L_2(\mathbb{R}_+),$$

(2) the Szegö projection $P^+_{\mathbb{R}} : L_2(\mathbb{R}) \longrightarrow H^2_+(\mathbb{R})$ is unitary equivalent to the following one

$$F\, P^+_{\mathbb{R}}\, F^{-1} = \chi_+ I.$$

Introduce the isometric imbedding

$$R_0 : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

by the rule

$$(R_0 f)(x,y) = \chi_+(x)\, f(x)\, \ell_0(y),$$

here the function $f(x)$ is extended to an element of $L_2(\mathbb{R})$ by setting $f(x) \equiv 0$, for $x < 0$. The image of R_0 is obviously coincides with the space $\mathcal{A}^2_2$. The adjoint operator $R_0^* : L_2(\Pi) \to L_2(\mathbb{R}_+)$ is given by

$$(R_0^*\varphi)(x) = \chi_+(x) \int_{\mathbb{R}_+} \varphi(x,\eta)\, \ell_0(\eta)\, d\eta,$$

and

$$\begin{aligned} R_0^* R_0 = I \quad &: \quad L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+), \\ R_0 R_0^* = B_2 \quad &: \quad L_2(\Pi) \longrightarrow \mathcal{A}^2_2 = L_2(\mathbb{R}_+) \otimes L_0. \end{aligned}$$

Now the operator $R = R_0^* U$ maps the space $L_2(\Pi)$ onto $L_2(\mathbb{R}_+)$, and the restriction

$$R|_{\mathcal{A}^2(\Pi)} : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+)$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : L_2(\mathbb{R}_+) \longrightarrow \mathcal{A}^2(\Pi) \subset L_2(\Pi)$$

is an isometric isomorphism of $L_2(\mathbb{R}_+)$ onto the subspace $\mathcal{A}^2(\Pi)$ of the space $L_2(\Pi)$.

REMARK 3.3. We have

$$\begin{aligned} R\, R^* = I \quad &: \quad L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+), \\ R^* R = B_\Pi \quad &: \quad L_2(\Pi) \longrightarrow \mathcal{A}^2(\Pi). \end{aligned}$$

THEOREM 3.4. *The isometric isomorphism*

$$R^* = U^* R_0 : L_2(\mathbb{R}_+) \longrightarrow \mathcal{A}^2(\Pi)$$

is given by

$$(R^* f)(z) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} \sqrt{\xi}\, f(\xi)\, e^{iz\cdot\xi}\, d\xi. \tag{3.5}$$

PROOF. Calculate

$$\begin{aligned}(R^* f)(z) &= (U_1^* U_2^* R_0 f)(z) \\ &= (F^{-1} \otimes I)(\chi_+(\xi)\, f(\xi)\, \sqrt{2\xi}\, e^{-xy}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_+(\xi)\, f(\xi)\, \sqrt{2\xi}\, e^{-xy}\, e^{ix\xi}\, d\xi \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} \sqrt{\xi}\, f(\xi)\, e^{i(x+iy)\cdot\xi}\, d\xi.\end{aligned}$$

□

COROLLARY 3.5. *The inverse isomorphism*

$$R : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+)$$

is given by

$$(R\varphi)(x) = \sqrt{x}\, \frac{1}{\sqrt{\pi}} \int_{\Pi} \varphi(w)\, e^{-i\,\overline{w}\cdot x}\, d\mu(w). \tag{3.6}$$

3.2. Connections between Bergman and Hardy spaces. The first immediate result follows directly from Theorem 3.1 and Remark 3.2.

THEOREM 3.6. *The unitary operator $W = (F^{-1} \otimes I)U_2(F \otimes I)$ gives an isometrical isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ under which*

(1) *the Bergman $\mathcal{A}^2(\Pi)$ and the Hardy $H_+^2(\mathbb{R})$ spaces are connected by the formula*

$$W(\mathcal{A}^2(\Pi)) = H_+^2(\mathbb{R}) \otimes L_0,$$

(2) *the Bergman B_Π and the Szegö $P_{\mathbb{R}}^+$ projections are connected by the formula*

$$W\, B_\Pi\, W^{-1} = P_{\mathbb{R}}^+ \otimes P_0,$$

where P_0 is the one-dimensional projection (3.4) of $L_2(\mathbb{R}_+)$ onto the one-dimensional space L_0 generated by $\ell_0(y) = e^{-y/2} \in L_2(\mathbb{R}_+)$.

For a domain D (which will be Π in this section) besides the Bergman space $\mathcal{A}^2(D)$ introduce the space $\widetilde{\mathcal{A}}^2(D)$ as the (closed) subspace of $L_2(D)$ consisting of all anti-analytic functions in D. It is well known (see, for example, [**15**]) that if $K_D(z, \overline{\zeta})$ is the Bergman kernel function of the domain D, then the orthogonal projection $\widetilde{B}_D$ of $L_2(D)$ onto $\widetilde{\mathcal{A}}^2(D)$ is given by

$$(\widetilde{B}_D \varphi)(z) = \int_D K_D(\zeta, \overline{z})\, \varphi(\zeta)\, d\mu(\zeta).$$

THEOREM 3.7. *The unitary operator $U = U_2 U_1$ gives an isometrical isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ under which*

(1) *the space $\widetilde{\mathcal{A}}^2(\Pi)$ is mapped onto $L_2(\mathbb{R}_-) \otimes L_0$*

$$U : \widetilde{\mathcal{A}}^2(\Pi) \longrightarrow L_2(\mathbb{R}_-) \otimes L_0,$$

where L_0 is the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_0 = e^{-y/2}$,

(2) *the projection $\widetilde{B}_\Pi$ is unitary equivalent to the following one*

$$U\,\widetilde{B}_\Pi\,U^{-1} = \chi_- I \otimes P_0,$$

where χ_- is the characteristic function of the negative half-line, and P_0 is the one-dimensional projection (3.4) of $L_2(\mathbb{R}_+)$ onto L_0.

PROOF. One follows all the steps of the proof of Theorem 3.1. Note only that all the $L_2(\Pi)$ – solutions of the equation

$$(u - \frac{\partial}{\partial v})\varphi = 0$$

are of the form

$$\varphi(u,v) = \chi_-(u)\,\sqrt{2|u|}\,f(u)\,e^{u\,v},$$

where $f(u) \in L_2(\mathbb{R})$, and that the Fourier image (with respect to $\xi = \operatorname{Re}\omega$) of the function

$$\overline{K_\Pi(\omega.\overline{w})} = -\frac{1}{\pi}\,\frac{1}{(\overline{\omega} - w)^2}$$

is equal (see, for example, [**4**]) to the following function

$$-\sqrt{\frac{2}{\pi}}\,\xi\,\chi_-(\xi)\,e^{(\eta - iw)\cdot\xi}.$$

The projection $\widetilde{B}_1 = U_1\,\widetilde{B}_\Pi\,U_1^{-1}$ onto the space $\widetilde{\mathcal{A}}_1^2 = U_1(\widetilde{\mathcal{A}}^2(\Pi))$ has the form

$$(\widetilde{B}_1\varphi)(u.v) = 2|u|\,\chi_-(u)\,e^{uv}\int_{\mathbb{R}_+}\varphi(u,\eta)\,e^{u\eta}\,d\eta.$$

□

Introduce now the Hardy space $H^2_-(\mathbb{R})$ as the subspace of $L_2(\mathbb{R})$ consisting of functions having analytic extension to the lower half-plane. It is well known that the operator $P^-_{\mathbb{R}} = F^{-1}\,\chi_-(\xi)\,F$ is the orthogonal (Szegö) projection of $L_2(\mathbb{R})$ onto $H^2_-(\mathbb{R})$.

COROLLARY 3.8. *The unitary operator $W = (F^{-1} \otimes I)U_2(F \otimes I)$ gives an isometrical isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ under which*

(1) *the spaces $\widetilde{\mathcal{A}}^2(\Pi)$ and $H^2_-(\mathbb{R})$ are connected by the formula*

$$W(\widetilde{\mathcal{A}}^2(\Pi)) = H^2_-(\mathbb{R}) \otimes L_0,$$

(2) *the projections $\widetilde{B}_\Pi$ and $P^-_{\mathbb{R}}$ are connected by the formula*

$$W\,\widetilde{B}_\Pi\,W^{-1} = P^-_{\mathbb{R}} \otimes P_0,$$

where P_0 is the one-dimensional projection (3.4) of $L_2(\mathbb{R}_+)$ onto one-dimensional space L_0 generated by $\ell_0(y) = e^{-y/2} \in L_2(\mathbb{R}_+)$.

REMARK 3.9. The direct sum of both Hardy spaces coincides with the whole space

$$L_2(\mathbb{R}) = H^2_+(\mathbb{R}) \oplus H^2_-(\mathbb{R}),$$

while the direct sum $\mathcal{A}^2(\Pi) \oplus \widetilde{\mathcal{A}}^2(\Pi)$ has the (sufficiently large) orthogonal complement

$$\mathcal{A}^\perp(\Pi) = L_2(\Pi) \ominus (\mathcal{A}^2(\Pi) \oplus \widetilde{\mathcal{A}}^2(\Pi)),$$

whose image under the mapping W is equal to

$$W(\mathcal{A}^\perp(\Pi)) = L_2(\mathbb{R}) \otimes L_0^\perp.$$

3.3. Poly-Bergman spaces, decomposition of $L_2(\Pi)$. Analogously to the Bergman spaces $\mathcal{A}^2(\Pi)$ and $\widetilde{\mathcal{A}}^2(\Pi)$, of respectively analytic and anti-analytic functions from $L_2(\Pi)$, introduce the spaces of poly-analytic and poly-anti-analytic functions (see, for example, [**2, 3, 14, 15**]), the poly-Bergman spaces.

Define the space $\mathcal{A}_n^2(\Pi)$ of n-analytic functions as the (closed) subspace of $L_2(\Pi)$ of all functions $\varphi = \varphi(w, \overline{w}) = \varphi(u, v)$, which satisfy the equation

$$\left(\frac{\partial}{\partial \overline{w}}\right)^n \varphi = \frac{1}{2^n} \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)^n \varphi = 0.$$

Similarly, define the space $\widetilde{\mathcal{A}}_n^2(\Pi)$ of n-anti-analytic functions as the (closed) subspace of $L_2(\Pi)$ of all functions $\varphi = \varphi(w, \overline{w}) = \varphi(u, v)$, which satisfy the equation

$$\left(\frac{\partial}{\partial w}\right)^n \varphi = \frac{1}{2^n} \left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)^n \varphi = 0.$$

Of course, we have $\mathcal{A}_1^2(\Pi) = \mathcal{A}^2(\Pi)$ and $\widetilde{\mathcal{A}}_1^2(\Pi) = \widetilde{\mathcal{A}}^2(\Pi)$, for $n = 1$, as well as $\mathcal{A}_n^2(\Pi) \subset \mathcal{A}_{n+1}^2(\Pi)$ and $\widetilde{\mathcal{A}}_n^2(\Pi) \subset \widetilde{\mathcal{A}}_{n+1}^2(\Pi)$, for each $n \in \mathbb{N}$.

Recall (see, for example, [**5**]), that the Laguerre polinomial $L_n(y)$ of degree n, $n = 0, 1, 2, \ldots$, and type 0 is defined by

$$\begin{aligned} L_n(y) &= L_n^0(y) = \frac{e^y}{n!}\,\frac{d^n}{dy^n}(e^{-y}\,y^n) \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\,\frac{(-y)^k}{k!}, \qquad y \in \mathbb{R}_+, \end{aligned} \tag{3.7}$$

and that the system of functions

$$\ell_n(y) = e^{-y/2} L_n(y), \qquad n = 0, 1, 2, \ldots \tag{3.8}$$

form an ortonormal base in the space $L_2(\mathbb{R}_+)$.

Denote by L_n, $n = 0, 1, 2, \ldots$, the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by the function $\ell_n(y)$. Note that for $n = 0$ this definition gives exactly the previously defined space L_0. Further, let

$$L_n^{\oplus} = \bigoplus_{k=0}^{n} L_k$$

be the direct sum of the first $(n+1)$ spaces.

THEOREM 3.10. *The unitary operator $U = U_2 U_1 : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ maps the space $\mathcal{A}_n^2(\Pi)$ of n-analytic functions onto the space $L_2(\mathbb{R}_+) \otimes L_{n-1}^{\oplus}$.*

PROOF. The space $U(\mathcal{A}_n^2(\Pi))$ obviously coincides with the set of all functions from $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ which satisfy the equation

$$\begin{aligned} U\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)^n U^{-1}\varphi &= i^n\, U_2 \left(u + \frac{\partial}{\partial v}\right)^n U_2^{-1}\varphi \\ &= i^n\, |x|^n \left(\mathrm{sign}(x) + 2\frac{\partial}{\partial y}\right)^n \varphi = 0. \end{aligned}$$

It is easy to see that the intersection of the general solution of this equation with the space $L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ coincides with the set of all functions

$$\sum_{k=0}^{n-1} \chi_+(x)\, \psi_k(x)\, y^k e^{-y/2},$$

where $\psi_k(x) \in L_2(\mathbb{R})$, for all $k = \overline{0, n-1}$, or rearranging polynomials on y, with the set of all functions

$$\sum_{k=0}^{n-1} \chi_+(x)\, f_k(x)\, L_k(y) e^{-y/2} = \sum_{k=0}^{n-1} \chi_+(x)\, f_k(x)\, \ell_k(y),$$

where $f_k(x) \in L_2(\mathbb{R})$, for all $k = \overline{0, n-1}$. □

Introduce the space $\mathcal{A}^2_{(n)}$ of true-n-analytic functions by

$$\mathcal{A}^2_{(n)} = \mathcal{A}^2_n \ominus \mathcal{A}^2_{n-1},$$

for $n > 1$, and by $\mathcal{A}^2_{(1)} = \mathcal{A}^2_1$, for $n = 1$; then of course

$$\mathcal{A}^2_n = \bigoplus_{k=1}^{n} \mathcal{A}^2_{(k)}.$$

Corollary 3.11. *The unitary operator $U : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ maps the space $\mathcal{A}^2_{(n)}(\Pi)$ of true-n-analytic functions onto the space $L_2(\mathbb{R}_+) \otimes L_{n-1}$.*

Analogously in the anti-analytic situations we have the following assertions.

Theorem 3.12. *The unitary operator $U : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ maps the space $\widetilde{\mathcal{A}}^2_n(\Pi)$ of n-anti-analytic functions onto the space $L_2(\mathbb{R}_-) \otimes L^{\oplus}_{n-1}$.*

Proof. Follow all the steps of the proof of Theorem 3.10. The space $U(\widetilde{\mathcal{A}}^2_n(\Pi))$ coincides with the set of all functions from $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$, which satisfy the equation

$$\begin{aligned} U\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)^n U^{-1}\varphi &= i^n\, U_2 \left(u - \frac{\partial}{\partial v}\right)^n U_2^{-1}\varphi \\ &= i^n\, |x|^n \left(\operatorname{sign}(x) - 2\frac{\partial}{\partial y}\right)^n \varphi = 0. \end{aligned}$$

Now the intersection of the general solution of this equation with the space $L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ coincides with the set of all functions

$$\sum_{k=0}^{n-1} \chi_-(x)\, \psi_k(x)\, y^k e^{-y/2},$$

where $\psi_k(x) \in L_2(\mathbb{R})$, for all $k = \overline{0, n-1}$, or rearranging polynomials on y, with the set of all functions

$$\sum_{k=0}^{n-1} \chi_-(x)\, f_k(x)\, L_k(y) e^{-y/2} = \sum_{k=0}^{n-1} \chi_-(x)\, f_k(x)\, \ell_k(y),$$

where $f_k(x) \in L_2(\mathbb{R})$, for all $k = \overline{0, n-1}$. □

Symmetrically, introduce the space $\widetilde{\mathcal{A}}^2_{(n)}$ of true-n-anti-analytic functions by

$$\widetilde{\mathcal{A}}^2_{(n)} = \widetilde{\mathcal{A}}^2_n \ominus \widetilde{\mathcal{A}}^2_{n-1},$$

for $n > 1$, and by $\widetilde{\mathcal{A}}^2_{(1)} = \widetilde{\mathcal{A}}^2_1$, for $n = 1$, analogously,

$$\widetilde{\mathcal{A}}^2_n = \bigoplus_{k=1}^{n} \widetilde{\mathcal{A}}^2_{(k)}.$$

COROLLARY 3.13. *The unitary operator $U : L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+) \to L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+)$ maps the space $\widetilde{\mathcal{A}}^2_{(n)}(\Pi)$ of true-n-anti-analytic functions onto the space $L_2(\mathbb{R}_-) \otimes L_{n-1}$.*

The above results lead to the following theorem.

THEOREM 3.14. *We have the following isometrical isomorphisms and decompositions of spaces:*

(1) *Isomorphic images of poly-analytic spaces*

$$\begin{aligned} W &: \mathcal{A}^2_{(n)}(\Pi) \longrightarrow H^2_+(\mathbb{R}) \otimes L_{n-1}, \\ W &: \mathcal{A}^2_n(\Pi) \longrightarrow H^2_+(\mathbb{R}) \otimes \bigoplus_{k=0}^{n-1} L_k, \\ W &: \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)}(\Pi) \longrightarrow H^2_+(\mathbb{R}) \otimes L_2(\mathbb{R}_+). \end{aligned}$$

(2) *Isomorphic images of poly-anti-analytic spaces*

$$\begin{aligned} W &: \widetilde{\mathcal{A}}^2_{(n)}(\Pi) \longrightarrow H^2_-(\mathbb{R}) \otimes L_{n-1}, \\ W &: \widetilde{\mathcal{A}}^2_n(\Pi) \longrightarrow H^2_-(\mathbb{R}) \otimes \bigoplus_{k=0}^{n-1} L_k, \\ W &: \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}^2_{(k)}(\Pi) \longrightarrow H^2_-(\mathbb{R}) \otimes L_2(\mathbb{R}_+). \end{aligned}$$

(3) *Decomposition of the space $L_2(\Pi)$*

$$\begin{aligned} L_2(\Pi) &= \bigoplus_{k=1}^{\infty} (\mathcal{A}^2_{(k)}(\Pi) \oplus \widetilde{\mathcal{A}}^2_{(k)}(\Pi)) \\ &= \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)}(\Pi) \oplus \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}^2_{(k)}(\Pi). \end{aligned}$$

Here L_n is the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by the function $\ell_n(y) = e^{-y/2}L_n(y)$, where the Laguerre polinomial $L_n(y)$ of degree n is given by (3.7).

4. Bergman Type Spaces on the Unit Disk

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Rearranging the basis of $L_2(\mathbb{D})$, L. Peng, R. Rochberg and Z. Wu [**28**] proved that the space $L_2(\mathbb{D})$ can be decomposed into a direct sum of Bergman type spaces

$$L_2(\mathbb{D}) = \bigoplus_{n=1}^{\infty} \mathcal{A}^2_{(n)} \oplus \bigoplus_{n=1}^{\infty} \widetilde{\mathcal{A}}^2_{(n)}, \tag{4.1}$$

where $\mathcal{A}^2_{(n)} = \ker(\overline{z}\,\partial/\partial\overline{z})^n \ominus \ker(\overline{z}\,\partial/\partial\overline{z})^{n-1}$, $\widetilde{\mathcal{A}}^2_{(n)} = \ker(z\,\partial/\partial z)^n \ominus \ker(z\,\partial/\partial z)^{n-1}$. Studying this question we follow the ideas of Section 3, which allows us to obtain more information.

4.1. Bergman space and Bergman projection. Consider the space $L_2(\mathbb{D})$ with usual Lebesgue plane measure $dv(z) = dxdy$, $z = x + iy$, and its Bergman subspace $\mathcal{A}^2(\mathbb{D})$. The Bergman projection $B_{\mathbb{D}}$ of $L_2(\mathbb{D})$ onto $\mathcal{A}^2(\mathbb{D})$ has the form (see Example 2.3)

$$(B_{\mathbb{D}}\varphi)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)\,dv(\zeta)}{(1 - z\overline{\zeta})^2}.$$

The Bergman space $\mathcal{A}^2(\mathbb{D})$ can be described alternatively as the (closed) subspace of $L_2(\mathbb{D})$ which consists of all functions satisfying the equation

$$\frac{\partial}{\partial\overline{z}}\varphi = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\varphi = 0,$$

where $z = x + iy$.

Passing to polar coordinates we have

$$\begin{aligned} L_2(\mathbb{D}) = L_2(\mathbb{D}, d\mu(z)) &= L_2([0,1), rdr) \otimes L_2([0,2\pi), d\alpha) \\ &= L_2([0,1), rdr) \otimes L_2(S^1, \frac{dt}{it}) = L_2([0,1), rdr) \otimes L_2(S^1), \end{aligned}$$

where S^1 is the unit circle, and

$$\frac{dt}{it} = |dt| = d\alpha$$

is the element of length; in addition

$$\frac{\partial}{\partial\overline{z}} = \frac{\cos\alpha + i\sin\alpha}{2}\left(\frac{\partial}{\partial r} + i\frac{1}{r}\frac{\partial}{\partial\alpha}\right) = \frac{t}{2}\left(\frac{\partial}{\partial r} - \frac{t}{r}\frac{\partial}{\partial t}\right).$$

Introduce the unitary operator

$$U_1 = I \otimes \mathcal{F} :\, L_2([0,1), rdr) \otimes L_2(S^1) \longrightarrow L_2([0,1), rdr) \otimes l_2 = l_2(L_2([0,1), rdr)),$$

where the Fourier transform $\mathcal{F} : L_2(S^1) \to l_2$ is given by

$$\mathcal{F} :\, f \longmapsto c_n = \frac{1}{\sqrt{2\pi}}\int_{S^1} f(t)\,t^{-n}\,\frac{dt}{it}, \quad n \in \mathbb{Z},$$

and its inverse $\mathcal{F}^{-1} = \mathcal{F}^* : l_2 \to L_2(S^1)$ is given by

$$\mathcal{F}^{-1} :\, \{c_n\}_{n\in\mathbb{Z}} \longmapsto f = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} c_n\,t^n.$$

Calculate

$$\begin{aligned}(I\otimes\mathcal{F})\frac{t}{2}\left(\frac{\partial}{\partial r}-\frac{t}{r}\frac{\partial}{\partial t}\right)(I\otimes\mathcal{F}^{-1}) \quad &: \quad \{c_n(r)\}_{n\in\mathbb{Z}}\longmapsto\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}c_n(r)\,t^n \\ &\longmapsto \quad \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\frac{t}{2}\left(\frac{\partial}{\partial r}-\frac{n}{r}\right)c_n(r)\,t^n \\ &\longmapsto \quad \left\{\frac{1}{2}\left(\frac{\partial}{\partial r}-\frac{n-1}{r}\right)c_{n-1}(r)\right\}_{n\in\mathbb{Z}},\end{aligned}$$

or

$$(I\otimes\mathcal{F})\frac{t}{2}\left(\frac{\partial}{\partial r}-\frac{t}{r}\frac{\partial}{\partial t}\right)(I\otimes\mathcal{F}^{-1})\{c_n(r)\}_{n\in\mathbb{Z}}=\left\{\frac{1}{2}\left(\frac{\partial}{\partial r}-\frac{n-1}{r}\right)c_{n-1}(r)\right\}_{n\in\mathbb{Z}}.$$

Thus the image of the Bergman space $\mathcal{A}_1^2=U_1(\mathcal{A}^2(\mathbb{D}))$ can be described as the (closed) subspace of $L_2([0,1),rdr)\otimes l_2=l_2(L_2([0,1),rdr))$ which consists of all sequences $\{c_n(r)\}_{n\in\mathbb{Z}}$ satisfying the equations

$$\frac{1}{2}\left(\frac{\partial}{\partial r}-\frac{n}{r}\right)c_n(r)=0,\quad n\in\mathbb{Z}. \tag{4.2}$$

The equations (4.2) are easy to solve, and their general solutions have the form

$$c_n(r)=c_n'\,r^n=\sqrt{2(|n|+1)}\,c_n\,r^n,\quad n\in\mathbb{Z}.$$

But each function $c_n(r)=\sqrt{2(|n|+1)}\,c_n\,r^n$ has to be in $L_2([0,1),rdr)$, which implies that $c_n(r)\equiv 0$, for each $n<0$. Thus the space $\mathcal{A}_1^2$ ($\subset L_2([0,1),rdr)\otimes l_2=l_2(L_2([0,1),rdr))$) coincides with the space of all two-sided sequences $\{c_n(r)\}_{n\in\mathbb{Z}}$ with

$$c_n(r)=\begin{cases}\sqrt{2(n+1)}\,c_n\,r^n, & \text{if } n\in\mathbb{Z}_+\\ 0, & \text{if } n\in\mathbb{Z}_-\end{cases},$$

where $\mathbb{Z}_+=\{0\}\cup\mathbb{N}$, $\mathbb{Z}_-=\mathbb{Z}\setminus\mathbb{Z}_+$, and

$$\|\{c_n(r)\}_{n\in\mathbb{Z}}\|=\left(\sum_{n\in\mathbb{Z}_+}|c_n|^2\right)^{1/2}=\|\{c_n\}_{n\in\mathbb{Z}_+}\|_{l_2}.$$

For each $n\in\mathbb{Z}_+$ introduce the unitary operator

$$u_n: L_2([0,1),rdr)\longrightarrow L_2([0,1),rdr)$$

by the rule

$$(u_nf)(r)=\frac{1}{\sqrt{n+1}}\,r^{-\frac{n}{n+1}}\,f(r^{\frac{1}{n+1}}),$$

then the inverse operator $u_n^{-1}=u_n^*: L_2([0,1),rdr)\longrightarrow L_2([0,1),rdr)$ is given by

$$(u_n^{-1}f)(r)=\sqrt{n+1}\,r^n\,f(r^{n+1}).$$

Finally, define the unitary operator

$$U_2: l_2(L_2([0,1),rdr))\longrightarrow l_2(L_2([0,1),rdr))=L_2([0,1),rdr)\otimes l_2$$

as follows:

$$U_2:\{c_n(r)\}_{n\in\mathbb{Z}}\ \longmapsto\ \{(u_{|n|}c_n)(r)\}_{n\in\mathbb{Z}}.$$

Then the space $\mathcal{A}_2^2=U_2(\mathcal{A}_1^2)$ coincides with the space of all sequences $\{d_n(r)\}_{n\in\mathbb{Z}}$, where

$$d_n=u_n(\sqrt{2(n+1)}\,c_nr^n)=\sqrt{2}\,c_n,$$

for $n \in \mathbb{Z}_+$, and $d_n(r) \equiv 0$, for $n \in \mathbb{Z}_-$.
We introduce some notation. Let $\ell_0(r) = \sqrt{2}$; we have $\ell_0(r) \in L_2([0,1), rdr)$ and $\|\ell_0(r)\| = 1$. Denote by L_0 the one-dimensional subspace of $L_2([0,1), rdr)$ generated by $\ell_0(r)$, then the one-dimensional projection P_0 of $L_2([0,1), rdr)$ onto L_0 has the form

$$(P_0 f)(r) = \langle f, \ell_0 \rangle \cdot \ell_0 = \sqrt{2} \int_0^1 f(\rho)\, \sqrt{2}\rho \, d\rho. \tag{4.3}$$

Denote by l_2^+ (l_2^-) the subspace of (two-sided) l_2 consisting of all sequences $\{c_n\}_{n\in\mathbb{Z}}$, such that $c_n = 0$ for all $n \in \mathbb{Z}_-$ ($n \in \mathbb{Z}_+$). Then $l_2 = l_2^+ \oplus l_2^-$, and denote by p^+ (p^-) the orthogonal projection of l_2 onto l_2^+ (l_2^-). Introduce the sequences $\chi_\pm = \{\chi_\pm(n)\}_{n\in\mathbb{Z}} \in l_\infty$, where $\chi_\pm(n) = 1$ for $n \in \mathbb{Z}_\pm$, and $\chi_\pm(n) = 0$ for $n \in \mathbb{Z}_\mp$. Then obviously $p^\pm = \chi_\pm I$.

Now $\mathcal{A}_2^2 = L_0 \otimes l_2^+$, and the orthogonal projection B_2 of $l_2(L_2([0,1), rdr)) = L_2([0,1), rdr) \otimes l_2$ onto $\mathcal{A}_2^2$ has obviously the form

$$B_2 = P_0 \otimes p^+.$$

The above work leads to the following theorem.

Theorem 4.1. *The unitary operator $U = U_2U_1$ gives an isometric isomorphism of the space $L_2(\mathbb{D})$ onto $L_2([0,1), rdr) \otimes l_2$ under which*

(1) *the Bergman space $\mathcal{A}^2(\mathbb{D})$ is mapped onto $L_0 \otimes l_2^+$*

$$U : \mathcal{A}^2(\mathbb{D}) \longrightarrow L_0 \otimes l_2^+,$$

where L_0 is the one-dimensional subspace of $L_2([0,1), rdr)$, generated by $\ell_0(r) = \sqrt{2}$,

(2) *the Bergman projection $B_{\mathbb{D}}$ is unitary equivalent to the following one*

$$U\, B_{\mathbb{D}}\, U^{-1} = P_0 \otimes p^+,$$

where P_0 is the one-dimensional projection (4.3) of $L_2([0,1), rdr)$ onto L_0.

Introduce the isometric imbedding

$$R_0 : l_2^+ \longrightarrow L_2([0,1), rdr) \otimes l_2$$

by the rule

$$R_0 : \ \{c_n\}_{n\in\mathbb{Z}_+} \ \longmapsto \ \ell_0(r)\{\chi_+(n)c_n\}_{n\in\mathbb{Z}},$$

where we extend the sequence $\{c_n\}_{n\in\mathbb{Z}_+}$ to an element of l_2 setting $c_n = 0$ for negative indices $n < 0$. The image of R_0 is obviously coincides with the space $\mathcal{A}_2^2$. The adjoint operator $R_0^* : L_2([0,1), rdr) \otimes l_2 \to l_2^+$ is given by

$$R_0^* : \ \{c_n(r)\}_{n\in\mathbb{Z}} \ \longmapsto \ \left\{ \int_0^1 c_n(\rho) \sqrt{2}\, \rho \, d\rho \right\}_{n\in\mathbb{Z}_+},$$

and

$$\begin{aligned} R_0^* R_0 = I \quad &: \quad l_2^+ \longrightarrow l_2^+ \\ R_0 R_0^* = B_2 \quad &: \quad L_2([0,1), rdr) \otimes l_2 \longrightarrow \mathcal{A}_2^2 = L_0 \otimes l_2^+. \end{aligned}$$

Now the operator $R = R_0^* U$ maps the space $L_2(\mathbb{D})$ onto l_2^+, and the restriction

$$R|_{\mathcal{A}^2(\mathbb{D})} : \mathcal{A}^2(\mathbb{D}) \longrightarrow l_2^+$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : l_2^+ \longrightarrow \mathcal{A}^2(\mathbb{D}) \subset L_2(\mathbb{D})$$

is an isometric isomorphism of l_2^+ onto the subspace $\mathcal{A}^2(\mathbb{D})$ of the space $L_2(\mathbb{D})$.

REMARK 4.2. We have

$$\begin{aligned} R\,R^* = I &\quad : \quad l_2^+ \longrightarrow l_2^+, \\ R^* R = B_{\mathbb{D}} &\quad : \quad L_2(\mathbb{D}) \longrightarrow \mathcal{A}^2(\mathbb{D}). \end{aligned}$$

THEOREM 4.3. *The isometric isomorphism*

$$R^* = U^* R_0 : l_2^+ \longrightarrow \mathcal{A}^2(\mathbb{D})$$

is given by

$$R^* : \{c_n\}_{n\in\mathbb{Z}_+} \longmapsto \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}_+} \sqrt{2(n+1)}\, c_n\, z^n. \tag{4.4}$$

PROOF. Calculate

$$\begin{aligned} R^* = U_1^* U_2^* R_0 \quad &: \quad \{c_n\}_{n\in\mathbb{Z}_+} \longmapsto U_1^* U_2^* (\{\sqrt{2}\, c_n\}_{n\in\mathbb{Z}_+}) \\ &= \quad U_1^*(\{\sqrt{2}\, c_n\, r^n\}_{n\in\mathbb{Z}_+}) \\ &= \quad \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}_+} \sqrt{2(n+1)}\, c_n (rt)^n = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}_+} \sqrt{2(n+1)}\, c_n\, z^n. \end{aligned}$$

□

COROLLARY 4.4. *The inverse isomorphism*

$$R : \mathcal{A}^2(\mathbb{D}) \longrightarrow l_2^+$$

is given by

$$R : \varphi(z) \longmapsto \left\{ \frac{\sqrt{2(n+1)}}{\sqrt{2\pi}} \int_{\mathbb{D}} \varphi(z)\, \overline{z}^n\, d\mu(z) \right\}_{n\in\mathbb{Z}_+}. \tag{4.5}$$

REMARK 4.5. The above operator R is defined on the whole of $L_2(\mathbb{D})$ and its restriction to $\mathcal{A}^2(\mathbb{D})$ gives an isomorphism onto l_2^+. For the restriction one has an alternative formula

$$R : \varphi(z) \longmapsto \left\{ \frac{\sqrt{2\pi}}{\sqrt{2(n+1)}} \frac{\varphi^{(n)}(0)}{n!} \right\}_{n\in\mathbb{Z}_+}.$$

Introduce the operator $\widetilde{R} : L_2(\mathbb{D}) \to L_2(S^1)$ as follows,

$$\widetilde{R} = \mathcal{F}^{-1} R.$$

COROLLARY 4.6. *We have the following isometric isomorphisms between the Bergman $\mathcal{A}^2(\mathbb{D})$ and the Hardy $H^2(\mathbb{D})$ spaces:*

$$\begin{aligned} \widetilde{R}|_{\mathcal{A}^2(\mathbb{D})} &\quad : \quad \mathcal{A}^2(\mathbb{D}) \longrightarrow H^2(\mathbb{D}), \\ \widetilde{R}^*|_{H^2(\mathbb{D})} &\quad : \quad H^2(\mathbb{D}) \longrightarrow \mathcal{A}^2(\mathbb{D}). \end{aligned}$$

The operators $\widetilde{R}$ and $\widetilde{R}^*$ provide the following decomposition of the Bergman $B_{\mathbb{D}}$ and the Szegö P_{S^1} projections:

$$\begin{aligned} \widetilde{R}^*\widetilde{R} = B_{\mathbb{D}} \quad &: \quad L_2(\mathbb{D}) \longrightarrow \mathcal{A}^2(\mathbb{D}), \\ \widetilde{R}\,\widetilde{R}^* = P_{S^1} \quad &: \quad L_2(S^1) \longrightarrow H^2(\mathbb{D}). \end{aligned}$$

Another connection between the Bergman and the Hardy spaces, and between the corresponding projections is given in the following theorem.

THEOREM 4.7. *The unitary operator $W = (I \otimes \mathcal{F}^{-1})U_2(I \otimes \mathcal{F})$ is an isometric isomorphism of the space $L_2(\mathbb{D}) = L_2([0,1), rdr) \otimes L_2(S^1, \frac{dt}{it})$ under which*

(1) *the Bergman $\mathcal{A}^2(\mathbb{D})$ and the Hardy $H^2(\mathbb{D})$ spaces are connected by the formula*
$$W(\mathcal{A}^2(\mathbb{D})) = L_0 \otimes H^2(\mathbb{D}),$$
(2) *the Bergman $B_{\mathbb{D}}$ and the Szegö $P^+_{S^1}$ projections are connected by the formula*
$$W\,B_{\mathbb{D}}\,W^{-1} = P_0 \otimes P^+_{S^1},$$

where P_0 is the one-dimensional projection (4.3) of $L_2([0,1), rdr)$ onto the one-dimensional space L_0 generated by $\ell_0(r) = \sqrt{2} \in L_2([0,1), rdr)$.

PROOF. Follows directly from Theorem 4.1.

□

In addition to the Bergman space $\mathcal{A}^2(\mathbb{D})$ of functions analytic in $\mathbb{D}$, introduce the space $\widetilde{\mathcal{A}}^2(\mathbb{D})$ as the (closed) subspace of $L_2(\mathbb{D})$ consisting of all functions which are anti-analytic in $\mathbb{D}$ and have the value zero at the point $0 \in \mathbb{D}$ (otherwise the spaces $\mathcal{A}^2(\mathbb{D})$ and $\widetilde{\mathcal{A}}^2(\mathbb{D})$ would intersect in the constants). Denote by $\widetilde{B}_{\mathbb{D}}$ the orthogonal Bergman projection of $L_2(\mathbb{D})$ onto $\widetilde{\mathcal{A}}^2(\mathbb{D})$

THEOREM 4.8. *The unitary operator $U = U_2U_1$ is an isometric isomorphism of the space $L_2(\mathbb{D})$ onto $L_2([0,1), rdr) \otimes l_2$ under which*

(1) *the space $\widetilde{\mathcal{A}}^2(\mathbb{D})$ is mapped onto $L_0 \otimes l_2^-$*
$$U : \widetilde{\mathcal{A}}^2(\mathbb{D}) \longrightarrow L_0 \otimes l_2^-,$$
where L_0 is the one-dimensional subspace of $L_2([0,1), rdr)$ generated by $\ell_0(r) = \sqrt{2}$,

(2) *the projection $\widetilde{B}_{\mathbb{D}}$ is unitary equivalent to the following one*
$$U\,\widetilde{B}_{\mathbb{D}}\,U^{-1} = P_0 \otimes p^-,$$
where $p^- = \chi_- I$ is the orthogonal projection of (two-sided) l_2 onto l_2^-, and P_0 is the one-dimensional projection (4.3) of $L_2([0,1), rdr)$ onto L_0.

PROOF. The space $\widetilde{\mathcal{A}}^2(\mathbb{D})$ can be described alternatively as the (closed) subspace of $L_2(\mathbb{D})$ which consists of all functions satisfying the equation

$$\mathcal{D}\,\varphi = z\,\frac{\partial}{\partial z}\,\varphi = \frac{z}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\varphi = 0,$$

where $z = x + iy$. Passing to polar coordinates we have

$$\mathcal{D} = \frac{r}{2}\left(\frac{\partial}{\partial r} + \frac{t}{r}\,\frac{\partial}{\partial t}\right).$$

Calculate

$$\begin{aligned}(I\otimes\mathcal{F})\mathcal{D}(I\otimes\mathcal{F}^{-1}) \quad &: \quad \{c_n(r)\}_{n\in\mathbb{Z}} \longmapsto \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} c_n(r)\,t^n \\ &\longmapsto \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \frac{r}{2}\left(\frac{\partial}{\partial r}+\frac{n}{r}\right)c_n(r)\,t^n \\ &\longmapsto \left\{\frac{r}{2}\left(\frac{\partial}{\partial r}+\frac{n}{r}\right)c_n(r)\right\}_{n\in\mathbb{Z}},\end{aligned}$$

Now

$$U_2U_1\mathcal{D}U_1^{-1}U_2^{-1} = U_2\left\{\frac{r}{2}\left(\frac{\partial}{\partial r}+\frac{n}{r}\right)c_n(r)\right\}_{n\in\mathbb{Z}} U_2^{-1} = \{\mathcal{D}_n\}_{n\in\mathbb{Z}},$$

where

$$\mathcal{D}_n = \begin{cases} \frac{n+1}{2}\left(\frac{2n}{n+1}+r\,\frac{d}{dr}\right), & \text{if } \ n\in\mathbb{Z}_+ \\ \frac{|n|+1}{2}\,r\,\frac{d}{dr}, & \text{if } \ n\in\mathbb{Z}_- \end{cases}. \tag{4.6}$$

The general solution of the equations $\mathcal{D}_n c_n(r)=0$, $n\in\mathbb{Z}$, has the form

$$\begin{aligned} c_n(r) &= c_n\,r^{-\frac{2n}{n+1}}, \quad \text{for} \quad n\in\mathbb{Z}_+, \\ c_n(r) &= \sqrt{2}\,c_n, \quad \text{for} \quad n\in\mathbb{Z}_-\,. \end{aligned}$$

All functions from $\widetilde{\mathcal{A}}^2(\mathbb{D})$ have the value zero at origin, which implies that $c_0(r)=c_0=0$. Again, each $c_n(r)$, $n\in\mathbb{N}$, has to be in $L_2([0,1),rdr)$, which implies that $c_n=0$, for all $n\in\mathbb{N}$.

Then the space $\widetilde{\mathcal{A}}_2^2 = U(\widetilde{\mathcal{A}}^2(\mathbb{D})$ coincides with the space of all sequences $\{c_n(r)\}_{n\in\mathbb{Z}}$, where

$$c_n(r) = \begin{cases} 0, & \text{if } \ n\in\mathbb{Z}_+ \\ \sqrt{2}\,c_n, & \text{if } \ n\in\mathbb{Z}_- \end{cases},$$

and $\|\{c_n(r)\}\|_{l_2(L_2([0,1),rdr))} = \|\{c_n\}\|_{l_2^-}$.

Thus $\widetilde{\mathcal{A}}_2^2 = L_0\otimes l_2^-$, and the orthogonal projection $\widetilde{B}_2$ of $l_2(L_2([0,1),rdr)) = L_2([0,1),rdr)\otimes l_2$ onto $\widetilde{\mathcal{A}}_2^2$ has the form $\widetilde{B}_2 = P_0\otimes p^-$. □

Introduce now the Hardy space $H^2(\mathbb{D}^-)$ as the subspace of $L_2(S^1)$ consisting of the functions having analytic extension into $\mathbb{C}\setminus\overline{\mathbb{D}}$ and vanishing at infinity. Denote by $P_{S^1}^-$ is the orthogonal (Szegö) projection of $L_2(S^1)$ onto $H^2(\mathbb{D}^-)$.

Corollary 4.9. *The unitary operator $W=(I\otimes\mathcal{F}^{-1})U_2(I\otimes\mathcal{F})$ gives an isometric isomorphism of the space $L_2(\mathbb{D}) = L_2([0,1),rdr)\otimes L_2(S^1,\frac{dt}{it})$ under which*

(1) *the spaces $\widetilde{\mathcal{A}}^2(\mathbb{D})$ and $H^2(\mathbb{D}^-)$ are connected by the formula*

$$W(\widetilde{\mathcal{A}}^2(\mathbb{D})) = L_0\otimes H^2(\mathbb{D}^-),$$

(2) *the projections $\widetilde{B}_{\mathbb{D}}$ and $P_{S^1}^-$ are connected by the formula*

$$W\,\widetilde{B}_{\mathbb{D}}\,W^{-1} = P_0\otimes P_{S^1}^-,$$

where P_0 is the one-dimensional projection (4.3) of $L_2([0,1),rdr)$ onto one-dimensional space L_0 generated by $\ell_0(r)=\sqrt{2}\in L_2([0,1),rdr)$.

4.2. Poly-Bergman type spaces, decomposition of $L_2(\mathbb{D})$. Besides the operator

$$\mathcal{D} = z\,\frac{\partial}{\partial z} = \frac{r}{2}\left(\frac{\partial}{\partial r} + \frac{t}{r}\,\frac{\partial}{\partial t}\right)$$

introduce

$$\overline{\mathcal{D}} = \overline{z}\,\frac{\partial}{\partial \overline{z}} = \frac{r}{2}\left(\frac{\partial}{\partial r} - \frac{t}{r}\,\frac{\partial}{\partial t}\right).$$

Analogously to the Bergman spaces $\mathcal{A}^2(\mathbb{D})$ and $\widetilde{\mathcal{A}}^2(\mathbb{D})$, which can be treated as the L_2 – kernels of $\overline{\mathcal{D}}$ and $\mathcal{D}$ respectively, following [**28**] introduce the spaces of poly-$\overline{\mathcal{D}}$-analytic and poly-$\mathcal{D}$-analytic functions, the poly-Bergman type spaces.

Define the space $\mathcal{A}^2_n(\mathbb{D})$ of n-$\overline{\mathcal{D}}$-analytic functions as the (closed) subspace of $L_2(\mathbb{D})$ of all functions $\varphi = \varphi(z,\overline{z}) = \varphi(x,y)$, which satisfy the equation

$$\overline{\mathcal{D}}^n\varphi = \left(\overline{z}\,\frac{\partial}{\partial \overline{z}}\right)^n \varphi = 0.$$

Similarly, define the space $\widetilde{\mathcal{A}}^2_n(\mathbb{D})$ of n-$\mathcal{D}$-analytic functions as the (closed) subspace of $L_2(\mathbb{D})$ of all functions $\varphi = \varphi(z,\overline{z}) = \varphi(x,y)$, which satisfy the equation

$$\mathcal{D}^n\varphi = \left(z\,\frac{\partial}{\partial z}\right)^n \varphi = 0,$$

and have the value zero at origin.

Of course, we have $\mathcal{A}^2_1(\mathbb{D}) = \mathcal{A}^2(\mathbb{D})$ and $\widetilde{\mathcal{A}}^2_1(\mathbb{D}) = \widetilde{\mathcal{A}}^2(\mathbb{D})$, for $n = 1$, as well as $\mathcal{A}^2_n(\mathbb{D}) \subset \mathcal{A}^2_{n+1}(\mathbb{D})$ and $\widetilde{\mathcal{A}}^2_n(\mathbb{D}) \subset \widetilde{\mathcal{A}}^2_{n+1}(\mathbb{D})$, for each $n \in \mathbb{N}$.

Recall that the system of functions $\{e^{-x/2}L_n(x)\}_{n\in\mathbb{Z}_+}$, where $L_n(x)$, $n \in \mathbb{Z}_+$, are the Laguerre polynomials (3.7), forms an orthonormal base in $L_2(\mathbb{R}_+)$, that is

$$\int_0^\infty L_n(x)\,L_m(x)\,e^{-x}\,dx = \delta_{n,m}. \tag{4.7}$$

Following [**28**] introduce

$$\ell_n(r) = \sqrt{2}\,L_n(\log r^{-2}) = \sqrt{2}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\,\frac{2^k}{k!}\,(\log r)^k, \tag{4.8}$$

where $r \in (0,1]$ and $n \in \mathbb{Z}_+$. Changing variables in (4.7) we have

$$\int_0^1 \ell_n(r)\,\ell_m(r)\,rdr = \delta_{n,m};$$

i.e., the system of functions $\{\ell_n(r)\}_{n\in\mathbb{Z}_+}$ forms an orthonornal base in $L_2([0,1), rdr)$.

Denote by L_n, $n \in \mathbb{Z}_+$, the one-dimensional subspace of $L_2([0,1), rdr)$, generated by the function $\ell_n(r)$. Note, that for $n = 0$ this definition gives exactly previously defined space L_0. And let

$$L_n^{\oplus} = \bigoplus_{k=0}^{n} L_k$$

be the direct sum of the first $(n+1)$ spaces.

THEOREM 4.10. *The unitary operator $U : L_2(\mathbb{D}) \to L_2([0,1), rdr) \otimes l_2$ maps the space $\mathcal{A}^2_n(\mathbb{D})$ of n-$\overline{\mathcal{D}}$-analytic functions onto the space $L^{\oplus}_{n-1} \otimes l_2^+$.*

PROOF. The space $U(\mathcal{A}_n^2(\mathbb{D}))$ obviously coincides with the set of all sequences from $l_2(L_2([0,1),rdr)) = L_2([0,1),rdr)\otimes l_2$, which satisfy the equation

$$\begin{aligned} U\overline{\mathcal{D}}^n U^{-1}\{c_k(r)\}_{k\in\mathbb{Z}} &= U_2\left\{\left[\frac{r}{2}\left(\frac{\partial}{\partial r}-\frac{k}{r}\right)\right]^n\right\}_{k\in\mathbb{Z}} U_2^{-1}\{c_k(r)\}_{k\in\mathbb{Z}} \\ &= \{\overline{\mathcal{D}}_k^n c_k(r)\}_{k\in\mathbb{Z}} = 0, \end{aligned}$$

where

$$\overline{\mathcal{D}}_k = \begin{cases} \frac{k+1}{2}\, r\, \frac{d}{dr}, & \text{if } \ k\in\mathbb{Z}_+ \\ \frac{|k|+1}{2}\left(\frac{2|k|}{|k|+1}+r\,\frac{d}{dr}\right), & \text{if } \ k\in\mathbb{Z}_- \end{cases}.$$

It is easy to see that the intersection of the general solution of this equation with the space $l_2(L_2([0,1),rdr)) = L_2([0,1),rdr)\otimes l_2$ coincides with the set of all sequences of the form

$$\sum_{m=0}^{n-1}(\log r)^m\,\{\chi_+(k)\,d_k^{(m)}\}_{k\in\mathbb{Z}},$$

where $\{d_k^{(m)}\}_{k\in\mathbb{Z}}\in l_2$, for all $m=\overline{0,n-1}$, or rearranging polynomials on $\log r$, with the set of all sequences

$$\sum_{m=0}^{n-1}\ell_m(r)\{c_k^{(m)}\}_{k\in\mathbb{Z}_+},$$

where $\{c_k^{(m)}\}_{k\in\mathbb{Z}_+}\in l_2^+$, for all $m=\overline{0,n-1}$. □

Introduce the space $\mathcal{A}_{(n)}^2$ of true-n-$\overline{\mathcal{D}}$-analytic functions by

$$\mathcal{A}_{(n)}^2 = \mathcal{A}_n^2 \ominus \mathcal{A}_{n-1}^2,$$

for $n>1$, and by $\mathcal{A}_{(1)}^2=\mathcal{A}_1^2$, for $n=1$, then, of course,

$$\mathcal{A}_n^2 = \bigoplus_{k=1}^{n}\mathcal{A}_{(k)}^2.$$

COROLLARY 4.11. *The unitary operator $U: L_2(\mathbb{D})\to L_2([0,1),rdr)\otimes l_2$ maps the space $\mathcal{A}_{(n)}^2(\mathbb{D})$ of true-n-$\overline{\mathcal{D}}$-analytic functions onto the space $L_{n-1}\otimes l_2^+$.*

Analogously in the $\mathcal{D}$-analytic situations we have the following assertions.

THEOREM 4.12. *The unitary operator $U: L_2(\mathbb{D})\to L_2([0,1),rdr)\otimes l_2$ maps the space $\widetilde{\mathcal{A}}_n^2(\mathbb{D})$ of n-$\mathcal{D}$-analytic functions onto the space $L_{n-1}^{\oplus}\otimes l_2^-$.*

PROOF. We follow all the steps of the proof of Theorem 4.10. The space $U(\widetilde{\mathcal{A}}_n^2(\mathbb{D}))$ obviously coincides with the set of all sequences from $l_2(L_2([0,1),rdr)) = L_2([0,1),rdr)\otimes l_2$, which satisfy the equation

$$\begin{aligned} U\mathcal{D}^n U^{-1}\{c_k(r)\}_{k\in\mathbb{Z}} &= U_2\left\{\left[\frac{r}{2}\left(\frac{\partial}{\partial r}+\frac{k}{r}\right)\right]^n\right\}_{k\in\mathbb{Z}} U_2^{-1}\{c_k(r)\}_{k\in\mathbb{Z}} \\ &= \{\mathcal{D}_k^n c_k(r)\}_{k\in\mathbb{Z}} = 0, \end{aligned}$$

where the operator $\{\mathcal{D}_k\}_{k\in\mathbb{Z}}$ is given by (4.6).

Now the intersection of the general solution of this equation with the space $l_2(L_2([0,1),rdr)) = L_2([0,1),rdr)\otimes l_2$ coincides with the set of all sequences of the form

$$\sum_{m=0}^{n-1} (\log r)^m \, \{\chi_-(k)\, d_k^{(m)}\}_{k\in\mathbb{Z}},$$

where $\{d_k^{(m)}\}_{k\in\mathbb{Z}} \in l_2$, for all $m = \overline{0,n-1}$, or rearranging polynomials on $\log r$, with the set of all sequences

$$\sum_{m=0}^{n-1} \ell_m(r)\{c_k^{(m)}\}_{k\in\mathbb{Z}_-},$$

where $\{c_k^{(m)}\}_{k\in\mathbb{Z}_-} \in l_2^-$, for all $m = \overline{0,n-1}$.

□

Symmetrically, introduce the space $\widetilde{\mathcal{A}}^2_{(n)}$ of true-n-$\mathcal{D}$-analytic functions by

$$\widetilde{\mathcal{A}}^2_{(n)} = \widetilde{\mathcal{A}}^2_n \ominus \widetilde{\mathcal{A}}^2_{n-1},$$

for $n > 1$, and by $\widetilde{\mathcal{A}}^2_{(1)} = \widetilde{\mathcal{A}}^2_1$, for $n = 1$, analogously,

$$\widetilde{\mathcal{A}}^2_n = \bigoplus_{k=1}^{n} \widetilde{\mathcal{A}}^2_{(k)}.$$

Corollary 4.13. *The unitary operator $U : L_2(\mathbb{D}) \to L_2([0,1),rdr)\otimes l_2$ maps the space $\widetilde{\mathcal{A}}^2_{(n)}(\mathbb{D})$ of true-n-$\mathcal{D}$-analytic functions onto the space $L_{n-1}\otimes l_2^-$.*

The above results lead to the following theorem.

Theorem 4.14. *We have the following isometric isomorphisms and decompositions of the spaces:*

(1) *Isomorphic images of poly-$\overline{\mathcal{D}}$-analytic spaces,*

$$\begin{aligned} W &: \mathcal{A}^2_{(n)}(\mathbb{D}) \longrightarrow L_{n-1}\otimes H^2(\mathbb{D}), \\ W &: \mathcal{A}^2_{n}(\mathbb{D}) \longrightarrow \bigoplus_{k=0}^{n-1} L_k\otimes H^2(\mathbb{D}), \\ W &: \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)}(\mathbb{D}) \longrightarrow L_2([0,1),rdr)\otimes H^2(\mathbb{D}). \end{aligned}$$

(2) *Isomorphic images of poly-$\mathcal{D}$-analytic spaces,*

$$\begin{aligned} W &: \widetilde{\mathcal{A}}^2_{(n)}(\mathbb{D}) \longrightarrow L_{n-1}\otimes H^2(\mathbb{D}^-), \\ W &: \widetilde{\mathcal{A}}^2_{n}(\mathbb{D}) \longrightarrow \bigoplus_{k=0}^{n-1} L_k\otimes H^2(\mathbb{D}^-), \\ W &: \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}^2_{(k)}(\mathbb{D}) \longrightarrow L_2([0,1),rdr)\otimes H^2(\mathbb{D}^-). \end{aligned}$$

(3) *Decomposition of the space* $L_2(\mathbb{D})$*,*

$$\begin{aligned} L_2(\mathbb{D}) &= \bigoplus_{k=1}^{\infty} (\mathcal{A}^2_{(k)}(\mathbb{D}) \oplus \widetilde{\mathcal{A}}^2_{(k)}(\mathbb{D})) \\ &= \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)}(\mathbb{D}) \oplus \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}^2_{(k)}(\mathbb{D}). \end{aligned}$$

Here L_n *is the one-dimensional subspace of* $L_2([0,1), rdr)$*, generated by the function* $\ell_n(r)$ *of the form (4.8).*

5. Toeplitz Operators with Commutative Symbol Algebras

Let $\mathcal{A}(\mathbb{D})$ be a C^*-subalgebra of $L_\infty(\mathbb{D})$, where $\mathbb{D}$ is the unit disk, as previously. Given $a \in \mathcal{A}(\mathbb{D})$, denote, as usual, by T_a the Toeplitz operator

$$T_a : \varphi \in \mathcal{A}^2(\mathbb{D}) \longmapsto B_{\mathbb{D}} a\varphi \in \mathcal{A}^2(\mathbb{D}),$$

acting on the space $\mathcal{A}^2(\mathbb{D})$, and denote by $\mathcal{T}(\mathcal{A}(\mathbb{D}))$ the Toeplitz operator algebra, i.e., the algebra generated by all the operators T_a, with $a \in \mathcal{A}(\mathbb{D})$. Furthermore, introduce the algebra $\mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})$, generated by all the operators

$$A = aI + bB_{\mathbb{D}},$$

where $a,\, b \in \mathcal{A}(\mathbb{D})$.

Consider the following statements:

1) For each $a \in \mathcal{A}(\mathbb{D})$ the commutator $[B_{\mathbb{D}}, aI] = B_{\mathbb{D}} aI - aB_{\mathbb{D}}$ is compact.
2) For each pair $a,\, b \in \mathcal{A}(\mathbb{D})$ the semi-commutator $[T_a, T_b) = T_a T_b - T_{ab}$ is compact.
3) For each pair $a,\, b \in \mathcal{A}(\mathbb{D})$ the commutator $[T_a, T_b] = T_a T_b - T_b T_a$ is compact.

We have obviously 1) $\Rightarrow$ 2) $\Rightarrow$ 3). In fact 1) $\Leftrightarrow$ 2), but 3) is certainly stronger then both 1) and 2). A simpler and better known example here is the algebra $\mathcal{A}(\mathbb{D}) = PC(\overline{\mathbb{D}}, L)$ of piece-wise continuous functions, i.e., the functions continuous in $\overline{\mathbb{D}} \setminus L$, and having one-sided limit values at each point of a simple (smooth) line L, which connects two points of the boundary $\partial\, \mathbb{D}$ of the unit disk (see, for example, [**40**]).

The very important question of description of the maximal algebra satisfying the statement 2) was solved in [**45**] (see also [**46**]). One of the main features of algebras having property 2) is that the corresponding Toeplitz operator algebras $\mathcal{T}(\mathcal{A}(\mathbb{D}))$ admit the commutative symbolic calculus, i.e., the symbol (Calkin) algebra $\operatorname{Sym} \mathcal{T}(\mathcal{A}(\mathbb{D})) = \mathcal{T}(\mathcal{A}(\mathbb{D}))/\mathcal{K}$, where $\mathcal{K}$ is the ideal of compact operator, is commutative.

Note, that under the condition 2) the symbol algebra $\operatorname{Sym} \mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}}) = \mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})/\mathcal{K}$ of the algebra $\mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})$ is commutative as well.

At the same time, under the (only) condition 3) the symbol algebra $\operatorname{Sym} \mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}}) = \mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})/\mathcal{K}$ of the algebra $\mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})$ is *non-commutative*, while the symbol algebra $\operatorname{Sym} \mathcal{T}(\mathcal{A}(\mathbb{D})) = \mathcal{T}(\mathcal{A}(\mathbb{D}))/\mathcal{K}$ still remains *commutative*.

The natural questions appear:

– describe functional algebras $\mathcal{A}(\mathbb{D})$ having the property 3),

– how complicated the symbol algebras of corresponding algebras $\mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})$ can be, while Toeplitz operator algebras $\mathcal{T}(\mathcal{A}(\mathbb{D}))$ still admit the commutative symbolic calculus.

To answer these questions, we construct a number of algebras $\mathcal{A}$ with the (only) property 3), as well as algebras with the more strong property: $T_aT_b = T_bT_a$, for each a, b from corresponding $\mathcal{A}$. Moreover, we prove that for each finite set $\Lambda = \langle n_0, n_1, ..., n_m \rangle$, where $1 = n_0 < n_1 < ... < n_m \leq \infty$, and $n_k \in \mathbb{N} \cup \{\infty\}$, there are algebras $\mathcal{A}_\Lambda$ of the above type, such that the symbol algebras $\mathrm{Sym}\,\mathcal{T}(\mathcal{A}_\Lambda)$ of the algebras $\mathcal{T}(\mathcal{A}_\Lambda)$ are *commutative*, while the symbol algebras $\mathrm{Sym}\,\mathcal{R}(\mathcal{A}_\Lambda, B_{\mathbb{D}})$ of the algebras $\mathcal{R}(\mathcal{A}_\Lambda, B_{\mathbb{D}})$ have *irreducible representations exactly of dimensions* n_0, n_1, ..., n_m.

It is worth mentioning that for Toeplitz operators on Hardy spaces functional algebras with the property 2) were known long ago [**33, 34**], and that the Gohberg – Krupnik results of the end of 60^{th} [**17, 18**] gave examples of algebras with only property 3). Nevertheless the above questions still remain open for the case of Toeplitz operators on Hardy spaces.

5.1. On a C^*-algebra generated by orthogonal projections. Recall now the necessary results (for details and proofs see [**42**]) about C^*-algebras generated by a finite set of orthogonal projections.

Let H be a Hilbert space, we will call the orthogonal projections P, Q_1, ..., Q_n, acting on H, *all-but-one*, if

$$\begin{aligned} I &= Q_1 + Q_2 + ... + Q_n, \\ Q_j \cdot Q_k &= \delta_{j,k} Q_j, \quad j,k = \overline{1,n}. \end{aligned}$$

The set of all-but-one projections P, Q_1, ..., Q_n are in *generic position* if for all $k = \overline{1,n}$

$$\mathrm{Im}\, P \;\cap\; \ker Q_k = \{0\}, \tag{5.1}$$

$$\mathrm{Im}\, Q_k \;\cap\; \ker P = \{0\}. \tag{5.2}$$

Given the set of all-but-one generic position orthogonal projections P, Q_1, ..., Q_n, the operators $C_k = PQ_kP : \mathrm{Im}\, P \to \mathrm{Im}\, P$ are positive and injective with $C_1 + ... + C_n = I$, and there exists the unitary operator $V : (\mathrm{Im}\, P)^n \to H$, such that the projections V^*PV and V^*Q_kV, $k = \overline{1,n}$, acting on $(\mathrm{Im}\, P)^n$ are, in matrix form, given as

$$\begin{aligned} V^*PV &= (C_j^{1/2} \cdot C_k^{1/2})_{j,k=1}^n, \\ V^*Q_kV &= \mathrm{diag}\,(0, \ldots, 0, \underset{k-place}{I}, 0, \ldots, 0), \quad k = \overline{1,n}. \end{aligned}$$

Assume now that the operators C_k, $k = \overline{1,n}$, are pairwise commuting, and denote by $\Delta = \Delta(C_1, ..., C_n)$ their joint spectrum. The properties of C_k imply that Δ is a (compact) subset of the standard $(n-1)$–dimensional simplex

$$\Delta_{n-1} = \{t = (t_1, \ldots, t_n) \in \mathbb{R}^n : \, t_k \geq 0, \quad k = \overline{1,n}, \quad \sum_{k=1}^n t_k = 1\}.$$

Denote by $\mathfrak{S}(\Delta_{n-1})$ the C^*-algebra of all $n \times n$ matrix–valued functions continuous on Δ_{n-1} and having the following properties:

(i) they are diagonal at all vertices $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, ..., $(0, \ldots, 0, 1)$ of Δ_{n-1};

(ii) fix a k–tuple $[k]=(i_1,\dots,i_k)$ of indices $1\le i_1<i_2<\dots<i_k\le n$, and let $[n]\backslash[k]$ be the complementary $(n-k)$–tuple of indices; then for each k–tuple, $1\le k\le n-1$

$$\sigma_{ij}(t)=\sigma_{ji}(t)\equiv 0 \quad \text{for} \quad i\in[k], \quad j\neq i$$

and $\sigma_{ii}(t)=\sigma_i=\text{const}$ on $(n-k-1)$–face $t_{i_1}=t_{i_2}=\dots=t_{i_k}=0$ of Δ_{n-1}.

Theorem 5.1. *The C^*-algebra $\mathcal{C}^*(P,Q_1,...,Q_n)$, generated by all-but-one generic position orthogonal projections P, Q_1, ..., Q_n with pairwise commuting C_k, is isomorphic and isometric to the restriction of $\mathfrak{S}(\Delta_{n-1})$ onto the set Δ, i.e., to the algebra*

$$\mathfrak{S}(\Delta)=\{\sigma|_{\Delta}:\sigma\in\mathfrak{S}(\Delta_{n-1})\}.$$

The isomorphism

$$\nu:\mathcal{C}^*(P,Q_1,...,Q_n)\longrightarrow\mathfrak{S}(\Delta)$$

is generated by the following mapping of the projections

$$\nu:P \quad\longmapsto\quad p(t)=(\sqrt{t_j}\sqrt{t_k})_{j,k=1}^{n}, \tag{5.3}$$

$$\nu:Q_k \quad\longmapsto\quad q_k(t)=\operatorname{diag}(0,\dots,0,\underset{k-place}{1},0,\dots,0), \tag{5.4}$$

where $t=(t_1,\dots,t_n)\in\Delta$.

In the sequel we will need the following variant of the above situation. Consider the all-but-one projections P, Q_1, ..., Q_n, which satisfy only property (5.1) (the property (5.2) does not hold). Thus for some indexes k we have

$$\operatorname{Im}Q_k\cap\ker P\neq\{0\}.$$

Denote by m $(0<m\le n)$ the number of these indexes, and reorder the projections Q_k in such a form that

$$\begin{aligned}\operatorname{Im}Q_k &\cap \ker P\neq\{0\}, \qquad k=\overline{1,m},\\ \operatorname{Im}Q_k &\cap \ker P=\{0\}, \qquad k=\overline{m+1,n}.\end{aligned}$$

We continue to assume that the operators $C_k=PQ_kP:\operatorname{Im}P\to\operatorname{Im}P$, $k=\overline{1,n}$, are pairwise commuting, and keep denoting by $\Delta=\Delta(C_1,...,C_n)$ their joint spectrum.

Theorem 5.2. *Under the above conditions the C^*-algebra $\mathcal{C}^*(P,Q_1,...,Q_n)$, generated by the orthogonal projections P, Q_1, ..., Q_n is isomorphic and isometric to a subalgebra of the algebra $\mathfrak{S}(\Delta)\oplus\mathbb{C}^m$. The isomorphic imbedding*

$$\nu:\mathcal{C}^*(P,Q_1,...,Q_n)\longrightarrow\mathfrak{S}(\Delta)\oplus\mathbb{C}^m$$

is generated by the following mapping of the projections

$$\begin{aligned}\nu:P &\quad\longmapsto\quad (p(t),(0,0,\dots,0)),\\ \nu:Q_k &\quad\longmapsto\quad (q_k(t),(0,\dots,0,\underset{k-place}{1},0,\dots,0)), \qquad k=\overline{1,m},\\ \nu:Q_k &\quad\longmapsto\quad (q_k(t),(0,0,\dots,0)), \qquad k=\overline{m+1,n},\end{aligned}$$

where $p(t)$ and $q_k(t)$ are defined in (5.3) and (5.4), $t=(t_1,\dots,t_n)\in\Delta$.

PROOF. Split the space H onto the following direct sum

$$H = \bigoplus_{k=1}^{m} (\operatorname{Im} Q_k \cap \ker P) \oplus H', \tag{5.5}$$

and introduce the projections

$$P' = P|_{H'}, \quad Q'_k = Q_k|_{H'}, \quad k = \overline{1,n}.$$

Corresponding to the splitting (5.5) the initial projections have the following form:

$$\begin{array}{rcl} P & = & (0 \oplus 0 \oplus \ldots \oplus 0) \oplus P', \\ Q_k & = & (0 \oplus \ldots 0 \oplus \underset{k-place}{1} \oplus 0 \oplus \ldots \oplus 0) \oplus Q'_k, \quad k = \overline{1,m}, \\ Q_k & = & (0 \oplus 0 \oplus \ldots \oplus 0) \oplus Q'_k, \quad k = \overline{m+1,n}. \end{array}$$

The all-but-one projections P', Q'_1, ..., Q'_n are in generic position, and the C^*-algebra generated by them is described in Theorem 5.1. Finally, note that $\operatorname{Im} P = \operatorname{Im} P'$, and $P'Q'_kP' = PQ_kP$ on this space. □

5.2. Infinite dimensional representations. Consider the upper half-plane Π, and the Bergman projection B_Π acting on $L_2(\Pi)$. Introduce the C^*-algebra $\mathcal{A}_\infty = \mathbb{C} \otimes L_\infty(\mathbb{R}_+)$ $(\subset L_\infty(\mathbb{R} \times \mathbb{R}_+))$, and consider two operator algebras, the Toeplitz operator algebra $\mathcal{T}(\mathcal{A}_\infty)$, generated by all the operators of the form

$$T_a : \varphi \in \mathcal{A}^2(\Pi) \longmapsto B_\Pi a\varphi \in \mathcal{A}^2(\Pi),$$

where $a = a(v) \in \mathcal{A}_\infty$, and the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$, generated by all the operators of the form

$$A = aI + bB_\Pi,$$

where $a = a(v)$, $b = b(v) \in \mathcal{A}_\infty$, acting on $L_2(\Pi)$.

THEOREM 5.3. *The algebra $\mathcal{T}(\mathcal{A}_\infty)$ is commutative. The isomorphic imbedding*

$$\tau_\infty : \mathcal{T}(\mathcal{A}_\infty) \longrightarrow C_b(\mathbb{R}_+)$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}_\infty)$

$$\tau_\infty : T_a \longmapsto \gamma_a(x) = \int_{\mathbb{R}_+} a(\frac{\eta}{2x})\, \ell_0^2(\eta)\, d\eta, \quad x \in \mathbb{R}_+,$$

where $a = a(v) \in \mathcal{A}_\infty$, and $\ell_0(\eta) = e^{-\eta/2}$.

PROOF. The algebra $\mathcal{T}(\mathcal{A}_\infty)$ is obviously unitary equivalent to the algebra $R\,\mathcal{T}(\mathcal{A}_\infty)\,R^*$, where the operators R and R^* are given by (3.6) and (3.5) respectively. The algebra $R\,\mathcal{T}(\mathcal{A}_\infty)\,R^*$ is generated by the following operators (acting on $L_2(\mathbb{R}_+)$) :

$$\begin{array}{rcl} R\,T_a\,R^* & = & R\,B_\Pi a B_\Pi R^* = R(R^*R)a(R^*R)R^* \\ & = & (RR^*)RaR^*(RR^*) = RaR^* \\ & = & R_0^* U_2 (F \otimes I) a(v) (F^{-1} \otimes I) U_2^{-1} R_0 \\ & = & R_0^* U_2 a(v) U_2^{-1} R_0 \\ & = & R_0^* a(\dfrac{y}{2|x|}) R_0. \end{array}$$

Calculate now

$$(R_0^* a(\frac{y}{2|x|})R_0 f)(x) = \int_{\mathbb{R}_+} a(\frac{\eta}{2|x|})\, f(x)\, \ell_0^2(\eta)\, d\eta = \gamma_a(x)\cdot f(x),$$

where

$$\gamma_a(x) = \int_{\mathbb{R}_+} a(\frac{\eta}{2x})\, \ell_0^2(\eta)\, d\eta, \quad x \in \mathbb{R}_+. \tag{5.6}$$

□

Return now to the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$. The algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$ is naturally isomorphic to the algebra

$$\mathcal{R}_0 = U\,\mathcal{R}(\mathcal{A}_\infty, B_\Pi)\,U^{-1} = U_2(F\otimes I)\mathcal{R}(\mathcal{A}_\infty, B_\Pi)(F^{-1}\otimes I)U_2^{-1},$$

where $U = U_2(F\otimes I)$, and the operator U_2 is given by (3.3). Under this isomorphism the generators of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$ are mapped to the following generators of the algebra $\mathcal{R}_0$

$$\begin{aligned} U\,B_\Pi\,U^{-1} &= \chi_+(x) I\otimes P_0, \\ U\,a(v)\,U^{-1} &= a(\frac{y}{2|x|})I, \end{aligned}$$

which act on the space $L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+)$. The algebra $\mathcal{R}_0$ splits onto the direct integral of the algebras $\mathcal{R}_0(x)$, $x\in\mathbb{R}\setminus\{0\}$, where each algebra $\mathcal{R}_0(x)$ is generated by the following operators, acting on $L_2(\mathbb{R}_+)$,

for $x\in\mathbb{R}_+$:

$$P_0 \quad \text{and} \quad a(\frac{y}{2x})I,$$

for $x\in\mathbb{R}_-$:

$$0 \quad \text{and} \quad a(-\frac{y}{2x})I.$$

The proof of the following lemma is trivial.

Lemma 5.4. *For $x\in\mathbb{R}_-$ all the algebras $\mathcal{R}_0(x)$ are isomorphic to $L_\infty(\mathbb{R}_+)$. The homomorphism*

$$\nu_x : \mathcal{R}(\mathcal{A}_\infty, B_\Pi) \cong \mathcal{R}_0 \longrightarrow \mathcal{R}_0(x) \cong L_\infty(\mathbb{R}_+) \tag{5.7}$$

is generated by the following mapping of generators of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$

$$\begin{aligned} \nu_x &: \ aI \longmapsto a(y), \\ \nu_x &: \ B_\Pi \longmapsto 0. \end{aligned}$$

Lemma 5.5. *For each $x\in\mathbb{R}_+$ the algebra $\mathcal{R}_0(x)$ is irreducible and contains the ideal $\mathcal{K}$ of all compact in $L_2(\mathbb{R}_+)$ operators, and its Calkin algebra $\widehat{\mathcal{R}}_0(x) = \mathcal{R}_0(x)/\mathcal{K}$ is isomorphic to $L_\infty(\mathbb{R}_+)$.*

Proof. To prove the first statement we show that the algebra $\mathcal{R}_0(x)$ does not have any non-trivial invariant subspace. First, observe that all the invariant subspaces of the subalgebra

$$\{a(\frac{y}{2x})I : a\in L_\infty(\mathbb{R}_+)\} \;\cong\; L_\infty(\mathbb{R}_+)$$

of the algebra $\mathcal{R}_0(x)$ have the form

$$X_M = \{\psi_M(y) = \chi_M(y)\psi(y) : \psi(y)\in L_2(\mathbb{R}_+)\},$$

where M is a measurable subset of $\mathbb{R}_+$ having positive measure. Thus the algebra $\mathcal{R}_0(x)$ has an invariant subspace if and only if there exists a measurable set M with positive measure, such that for each $\psi_M(y) \in X_M$ we have $P_0\psi_M(y) \in X_M$. Further,

$$P_0\psi_M = \langle \chi_M\psi, \ell_0\rangle \cdot \ell_0.$$

The function ℓ_0 is always nonzero on $\mathbb{R}_+$, thus $\langle \chi_M\psi, \ell_0\rangle$ has to be zero on M for all $\psi \in L_2(\mathbb{R}_+)$. But for $\psi = \ell_0$ we have

$$\langle \chi_M\ell_0, \ell_0\rangle = \langle \chi_M\ell_0, \chi_M\ell_0\rangle = \|\chi_M\ell_0\|^2.$$

Thus χ_M has to be zero a.e. on $\mathbb{R}_+$, or M must have measure zero. So the algebra $\mathcal{R}_0(x)$ does not have any non-trivial invariant subspace, and thus is irreducible. Now, the algebra $\mathcal{R}_0(x)$ is obviously a *GCR*-algebra, which contains the one-dimensional operator P_0, and thus [**11**] contains the whole ideal $\mathcal{K}$ of compact operators.

The last statement of the lemma is obvious. □

Note, that all the algebras $\mathcal{R}_0(x)$, $x \in \mathbb{R}_+$, are the same, and equal to the algebra $\mathcal{R}(L_\infty(\mathbb{R}_+), P_0)$, which is generated by P_0 and all the multiplication operators aI, $a \in L_\infty(\mathbb{R}_+)$, acting on $L_2(\mathbb{R}_+)$.

LEMMA 5.6. *For each $x \in \mathbb{R}_+$ the homomorphism*

$$\nu_x : \mathcal{R}(\mathcal{A}_\infty, B_\Pi) \cong \mathcal{R}_0 \longrightarrow \mathcal{R}_0(x) \cong \mathcal{R}(L_\infty(\mathbb{R}_+), P_0)$$

gives an (infinite dimensional) irreducible representation of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$, and for different values of $x \in \mathbb{R}_+$ the representations ν_x are not unitary equivalent.

PROOF. The first statement follows directly from Lemma 5.5. Let now $0 < x_1 < x_2 < \infty$, and let $a(y)$ be a function from L_∞, such that $\gamma_a(x_1) \neq \gamma_a(x_2)$, where $\gamma_a(x)$ is given by (5.6). Then for the operator $A = B_\Pi(\gamma_a(x_1) - a(y))B_\Pi \in \mathcal{R}(\mathcal{A}_\infty, B_\Pi)$ we have

$$\begin{aligned} v_{x_1} &: A \longmapsto (\gamma_a(x_1) - \gamma_a(x_1))P_0 = 0, \\ v_{x_2} &: A \longmapsto (\gamma_a(x_1) - \gamma_a(x_2))P_0 \neq 0. \end{aligned}$$

Thus the representations v_{x_1} and v_{x_2} are not unitary equivalent. □

THEOREM 5.7. *The algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$ does not contain any non-zero compact operator. All its infinite dimensional representations are parameterized by the points $x \in \mathbb{R}_+$ and are of the form*

$$\nu_x : \mathcal{R}(\mathcal{A}_\infty, B_\Pi) \cong \mathcal{R}_0 \longrightarrow \mathcal{R}(L_\infty(\mathbb{R}_+), P_0),$$

where

$$\begin{aligned} v_x &: a(y)I \in \mathcal{R}(\mathcal{A}_\infty, B_\Pi) \longmapsto a(\frac{y}{2x})I \in \mathcal{R}(L_\infty(\mathbb{R}_+), P_0), \\ v_x &: B_\Pi \in \mathcal{R}(\mathcal{A}_\infty, B_\Pi) \longmapsto P_0 \in \mathcal{R}(L_\infty(\mathbb{R}_+), P_0). \end{aligned}$$

All finite dimensional irreducible representations of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$ are one-dimensional, and are the composition of the homomorphism (5.7) of Lemma 5.4 with one-dimensional representations of $L_\infty(\mathbb{R}_+)$.

PROOF. Follows directly from Lemmas 5.4, 5.5 and 5.6. □

Let us pass now from the upper half-plane to the unit disk $\mathbb{D}$. The Moebius transformation

$$w = \frac{z+i}{1+iz} \tag{5.8}$$

maps the unit disk $\mathbb{D}$ onto the upper half-plane Π, and generates the unitary operator $U_0 : L_2(\mathbb{D}) \to L_2(\Pi)$ acting by the rule

$$(U_0\varphi)(w) = \frac{2}{(1-iw)^2}\,\varphi\left(\frac{w-i}{1-iw}\right), \tag{5.9}$$

then the inverse (and adjoint) operator $U_0^{-1} : L_2(\Pi) \to L_2(\mathbb{D})$ is given by

$$(U_0^{-1}\varphi)(z) = \frac{2}{(1+iz)^2}\,\varphi\left(\frac{z+i}{1+iz}\right).$$

Denote by $\widetilde{\mathcal{A}}_\infty$ the image of the algebra $\mathcal{A}_\infty$ on the disk, i.e.,

$$\widetilde{\mathcal{A}}_\infty = U_0^{-1}\mathcal{A}_\infty U_0 = \{a(\omega(z)) : a \in \mathcal{A}_\infty\},$$

where $\omega(z)$ is the Moebius transformation (5.8).

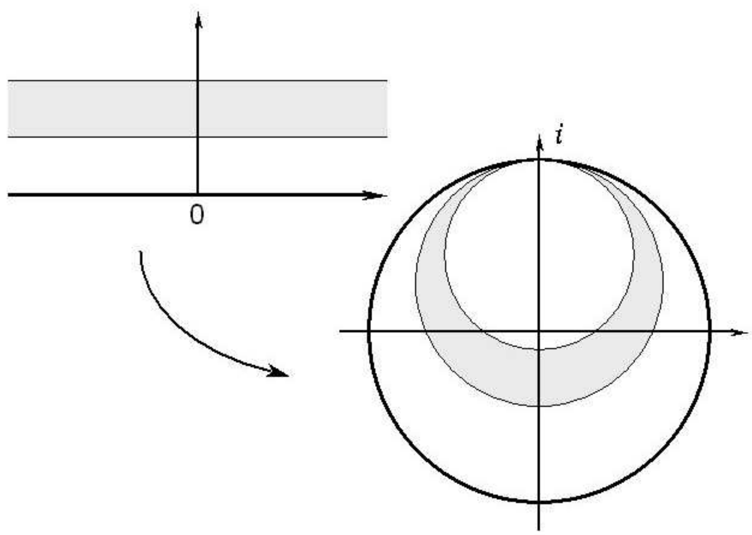

FIGURE 1.

Each horizontal line $\operatorname{Im}\omega = \eta_0 = \text{const}$ goes under the mapping $z = z(\omega)$ (inverse to the mapping (5.8)) to the circle $C(\eta_0)$ having center at the point $i\frac{\eta_0}{1+\eta_0}$ and radius $\frac{1}{1+\eta_0}$, and which is tangent to $\partial\,\mathbb{D}$ at the point i. Figure 1 shows the image of a horizontal strip in the upper half-plane under the mapping $z = z(\omega)$. Note that the functions from $\widetilde{\mathcal{A}}_\infty$ are constant on each circle $C(\eta_0)$, $\eta_0 \in \mathbb{R}_+$.

The algebras $\mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_{\mathbb{D}})$ and $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$, as well as the algebras $\mathcal{T}(\widetilde{\mathcal{A}}_\infty)$ and $\mathcal{T}(\mathcal{A}_\infty)$, are unitary equivalent. Thus Theorems 5.3 and 5.7 lead directly to the following result.

THEOREM 5.8. *The algebra $\mathcal{T}(\widetilde{\mathcal{A}}_\infty)$ is commutative. The isomorphic imbedding*

$$\tilde{\tau}_\infty : \mathcal{T}(\widetilde{\mathcal{A}}_\infty) \longrightarrow C_b(\mathbb{R}_+)$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\widetilde{\mathcal{A}}_\infty)$

$$\tilde{\tau}_\infty : T_{a(\omega(z))} \longmapsto \gamma_a(x) = \int_{\mathbb{R}_+} a(\frac{\eta}{2x})\, \ell_0^2(\eta)\, d\eta, \quad x \in \mathbb{R}_+,$$

where $a = a(\omega(z)) \in \widetilde{\mathcal{A}}_\infty$, and $\ell_0(\eta) = e^{-\eta/2}$.

The algebra $\mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_{\mathbb{D}})$ does not contain any non-zero compact operator. All its infinite dimensional representations are parameterized by the points $x \in \mathbb{R}_+$ and are of the form

$$\tilde{\nu}_x : \mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_{\mathbb{D}}) \cong \mathcal{R}_0 \longrightarrow \mathcal{R}(L_\infty(\mathbb{R}_+), P_0),$$

where

$$\begin{aligned} \tilde{\nu}_x &: a(\omega(z))I \in \mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_{\mathbb{D}}) \longmapsto a(\frac{y}{2x})I \in \mathcal{R}(L_\infty(\mathbb{R}_+), P_0), \\ \tilde{\nu}_x &: B_{\mathbb{D}} \in \mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_{\mathbb{D}}) \longmapsto P_0 \in \mathcal{R}(L_\infty(\mathbb{R}_+), P_0). \end{aligned}$$

All finite dimensional irreducible representations of the algebra $\mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_{\mathbb{D}})$ are one-dimensional, and are the composition of the homomorphism (5.7) of Lemma 5.4 with one-dimensional representations of $L_\infty(\mathbb{R}_+)$.

5.3. Spectra and compactness. We give here applications of results of the previous section to spectral and compactness properties of Toeplitz operators.

The next lemma describes the spectrum of Toeplitz operators with $\mathcal{A}_\infty$ symbols, extending the results [**24, 45, 46**] about the connectedness of spectrum to this class of symbols.

LEMMA 5.9. *For each $a \in \mathcal{A}_\infty$ the spectrum of the Toeplitz operator $T_a \in \mathcal{T}(\mathcal{A}_\infty)$ is connected and coincides with the closure of the image of the function $\gamma_a(x)$.*

For a real valued function $a \in \mathcal{A}_\infty$ we have

$$\operatorname{sp} T_a = [\inf_{x\in\mathbb{R}_+} \gamma_a(x), \sup_{x\in\mathbb{R}_+} \gamma_a(x)].$$

PROOF. Direct corollary of Theorem 5.3. $\square$

Let M be a measurable set in $\mathbb{R}_+$, and $M' = \mathbb{R} + iM$ be the corresponding set in Π. Introduce the characteristic functions χ_M and $\chi_{M'}$ of the sets M and M' respectively.

Consider now one special form of a set $M \subset \mathbb{R}_+$. For $\sigma \in (0,1]$ and $s \in (0,\sigma)$ introduce the sets $\Delta_0 = [1, 1+s]$ $(\subset [1, 1+\sigma))$, $\Delta_n = (1+\sigma)^n \cdot \Delta_0$, where $n \in \mathbb{Z}$, and

$$M(s) = \frac{1}{2} \bigcup_{n\in\mathbb{Z}} \Delta_n. \tag{5.10}$$

The function

$$\gamma_s(x) = \int_{\mathbb{R}_+} \chi_{M(s)}(\frac{\eta}{2x})\, e^{-\eta}\, d\eta = \int_{2x\cdot M(s)} e^{-\eta}\, d\eta = \int_{\bigcup_{n\in\mathbb{Z}} \Delta_n} x e^{-x\eta}\, d\eta$$

is "multiplicative-periodic", i.e., $\gamma_s((1+\sigma)x) = \gamma_s(x)$, for all $x \in \mathbb{R}_+$. Thus to analyze the values of $\gamma_s(x)$ it is sufficient to consider only $x \in [1, 1+\sigma)$.

LEMMA 5.10. *For each $\varepsilon > 0$ there is $\sigma \in (0,1)$ such that for each set $M(s)$ of the form (5.10) we have*

$$\sup_{x\in\mathbb{R}_+} \gamma_s(x) - \inf_{x\in\mathbb{R}_+} \gamma_s(x) < \varepsilon.$$

PROOF. Given $x_1, x_2 \in [1, 1+\sigma)$ consider

$$\begin{aligned}
\gamma_s(x_1) - \gamma_s(x_2) &= \int_{\bigcup_{n\in\mathbb{Z}}\Delta_n} (x_1 e^{-x_1\eta} - x_e^{-x_2\eta})\, d\eta \\
&= \sum_{n\in\mathbb{Z}} \int_{\Delta_n} (x_1 e^{-x_1\eta} - x_2 e^{-x_2\eta})\, d\eta \\
&= s\sum_{n\in\mathbb{Z}} (1+\sigma)^n (x_1 e^{-x_1\eta_n} - x_e^{-x_2\eta_n}),
\end{aligned}$$

for some $\eta_n \in \Delta_n$. Note, that $|\Delta_n| = s\cdot(1+\sigma)^n$.

Further

$$x_1 e^{-x_1\eta_n} - x_2 e^{-x_2\eta_n} = (x_1 - x_2) e^{-\xi_n\eta_n}(1 - \xi_n\eta_n)$$

for some $\xi_n \in (x_1, x_2) \subset [1, 1+\sigma)$.

Now, for $\zeta_n = \xi_n\eta_n \in (1+\sigma)^n \cdot (1, (1+\sigma)^2)$ we have

$$\gamma_s(x_1) - \gamma_s(x_2) = s(x_1 - x_2)\sum_{n\in\mathbb{Z}} (1+\sigma)^n e^{-\zeta_n}(1-\zeta_n),$$

thus

$$\begin{aligned}
|\gamma_s(x_1) - \gamma_s(x_2)| &\le \sigma^2 \sum_{n\in\mathbb{Z}} (1+\sigma)^n e^{-\zeta_n}|1-\zeta_n| \\
&= \sigma^2 \sum_{n=1}^{\infty} (1+\sigma)^n e^{-\zeta_n}|1-\zeta_n| \\
&+ \sigma^2 \sum_{k=0}^{\infty} \frac{1}{(1+\sigma)^k} e^{-\zeta_k}|1-\zeta_k|.
\end{aligned}$$

Estimate each of these sums:

$$\sigma^2 \sum_{k=0}^{\infty} \frac{1}{(1+\sigma)^k} e^{-\zeta_k}|1-\zeta_k| < \sigma^2 \sum_{k=0}^{\infty} \frac{1}{(1+\sigma)^k}\cdot 1\cdot 3 < 6\sigma,$$

$$\begin{aligned}
\sigma^2 \sum_{n=1}^{\infty} (1+\sigma)^n e^{-\zeta_n}|1-\zeta_n| &< \sigma^2 \sum_{n=1}^{\infty} (1+\sigma)^n e^{-\zeta_n}\zeta_n \\
&< \sigma^2 \sum_{n=1}^{\infty} (1+\sigma)^{2(n+1)} e^{-(1+\sigma)^n} \\
&< (1+\sigma)^3\sigma \int_1^{\infty} y\, e^{-y}\, dy \\
&< 8\sigma \int_{\mathbb{R}_+} y\, e^{-y}\, dy = 8\sigma.
\end{aligned}$$

$\square$

THEOREM 5.11. *The spectrum of the Toeplitz operator $T_{\chi_{M'}}$ is given by the formula*

$$\operatorname{sp} T_{\chi_{M'}} = [\alpha, \beta],$$

where $\alpha = \inf_{x\in\mathbb{R}_+} \gamma_{\chi_{M'}}(x) \geq 0$, $\beta = \sup_{x\in\mathbb{R}_+} \gamma_{\chi_{M'}}(x) \leq 1$.

Moreover, for every α, β *with* $0 \leq \alpha < \beta \leq 1$, *there exists a set* M *such that the spectrum of the Toeplitz operator* $T_{\chi_{M'}}$ *is equal to* $[\alpha, \beta]$.

PROOF. The first statement follows directly from Lemma 5.9. Consider now arbitrary α and β with $0 \leq \alpha < \beta \leq 1$. For $\varepsilon = (\beta-\alpha)/2$ fix σ according to Lemma 5.10. It is clear that for each $\lambda \in [0,1]$ there exists s_λ $(\in (0,\sigma))$ such that for the corresponding set $M_\lambda = M(s_\lambda)$ of the form (5.10) we have

$$\sup_{x\in[1,1+\sigma)} \int_{2x\cdot M_\lambda} |\ell_0(y)|^2 dy = \lambda,$$

similarly, there exists a set m_λ of the above form, such that

$$\inf_{x\in[1,1+\sigma)} \int_{2x\cdot m_\lambda} |\ell_0(y)|^2 dy = \lambda.$$

Now for the set $M = (m_\alpha \cap (0,1)) \cup (M_\beta \cap (1,+\infty))$ and

$$\gamma_{\chi_{M'}}(x) = \int_{2x\cdot M} |\ell_0(\eta)|^2 \, d\eta$$

we have

$$\begin{aligned} \inf_{x\in\mathbb{R}_+} \gamma_{\chi_{M'}}(x) &= \liminf_{x\to+\infty} \gamma_{\chi_{M'}}(x) = \alpha, \\ \sup_{x\in\mathbb{R}_+} \gamma_{\chi_{M'}}(x) &= \limsup_{x\to 0} \gamma_{\chi_{M'}}(x) = \beta. \end{aligned}$$

□

REMARK 5.12. All the Toeplitz operators T_a of the above theorem have the function a of the same type – the characteristic function, and with the same range – the set $\{0,1\}$, but their spectra are *quite different*. Recall, that for Toeplitz operators on the Hardy space one *always* has

$$\operatorname{sp} T_\chi = [0,1] \ (= \text{closure of convex hull of } \operatorname{Im}\chi),$$

for any characteristic function χ.

Using the unitary operator U_0 of the form (5.9), the above spectrum characterizations allow immediate reformulation for Toeplitz operator with $\widetilde{\mathcal{A}}_\infty$ symbols, acting on $L_2(\mathbb{D})$.

There are many results (see, for example, [**1, 24, 46**]) describing conditions on functions a for the Toeplitz operator T_a to be compact on $\mathcal{A}^2(\mathbb{D})$. The standard sufficient condition here is a direct corollary of Theorem 2.9, and is as follows.

THEOREM 5.13. *Let the function* $a \in L_\infty(\mathbb{D})$ *be continuous at each point of the boundary* $\partial\,\mathbb{D}$, *and* $a_{|_{\partial\mathbb{D}}} \equiv 0$. *Then the Toeplitz operator* T_a *is compact on* $\mathcal{A}^2(\mathbb{D})$.

We now show how exact the conditions of this theorem are. All the circles $C(\eta)$, $\eta \in \mathbb{R}_+$, are tangent to the unit circle $\partial\,\mathbb{D} = C(0)$ at the point i, and near this point $C(\eta)$ are the graphs of the functions

$$y_\eta(x) = \frac{\eta}{1+\eta} + \sqrt{\frac{1}{(1+\eta)^2} - x^2}.$$

The "degree of adhesion" of $C(\eta)$ to $\partial\,\mathbb{D}$ at the point i can be naturally measured by the following quantity

$$\delta(\eta) = y_0''(0) - y_\eta''(0) = \frac{\eta(2+\eta)}{(1+\eta)^2}.$$

The quantity $\delta(\eta)$, $\eta \in \mathbb{R}_+$, is continuous and strictly increasing, and $\delta(0) = 0$.

We will say that a function $a(z)$ defined on $\overline{\mathbb{D}}$ has ε-limit equal to A at the point i and write

$$\varepsilon\text{-}\lim_{z\to i} a(z) = A,$$

if

$$\lim_{z\to i} a(z)|_{\overline{\mathbb{D}}(\eta)} = A,$$

where $\overline{\mathbb{D}}(\eta)$ is the closed disk with the boundary $C(\eta)$, and such that $\delta(\eta) = \varepsilon$.

The ε-limit has the following obvious properties:

(1) if for any $\varepsilon > 0$ we have $\varepsilon\text{-}\lim_{z\to i} a(z) = 0$, then all non-tangential limit values of the function a at the point i exist and are equal to zero;
(2) if $\varepsilon_1 < \varepsilon_2$ and $\varepsilon_1\text{-}\lim_{z\to i} a(z) = 0$, then $\varepsilon_2\text{-}\lim_{z\to i} a(z) = 0$;
(3) $0\text{-}\lim_{z\to i} a(z) = 0$ if and only if $\lim_{z\to i} a(z) = 0$.

Lemma 5.14. *For each $\varepsilon > 0$ there exists a function $a \in L_\infty(\mathbb{D})$ such that*

(1) *a is continuous at each point of $\partial\,\mathbb{D} \setminus \{i\}$, and $a_{|_{\partial\,\mathbb{D}\setminus\{i\}}} \equiv 0$;*
(2) *$\varepsilon\text{-}\lim_{z\to i} a(z) = 0$;*
(3) *the Toeplitz operator T_a is not compact on $\mathcal{A}^2(\mathbb{D})$.*

Proof. Fix positive ε and consider η having the property $\delta(\eta) = \varepsilon$. Take a function $b \in \mathcal{A}_\infty$ with $\mathbb{R} \times [\eta/3, 2\eta/3] \subset \operatorname{supp} b$ (for example, $b = b(x,y)$ is the characteristic function of the strip $\mathbb{R} \times [\eta/3, 2\eta/3]$, see Figure 1). Then the function $a(z) = (b \cdot \chi_{\mathbb{R}\times[\eta/3,2\eta/3]})(\omega(z))$ satisfies the first two conditions of the lemma. The operator T_a is unitary equivalent to a (non-zero) multiplication operator, and thus is not compact. $\square$

5.4. Finite dimensional representations. Fix a number $n \in \mathbb{N}$. Let Y_k, $k = \overline{1,n}$, be disjoint measurable sets in $\mathbb{R}_+$ having positive measure, and such that $\cup_{k=1}^n Y_k = \mathbb{R}_+$, and let

$$\Pi_k = \mathbb{R} + iY_k, \quad k = \overline{1,n},$$

be the corresponding sets in the upper half-plane $\Pi = \mathbb{R} + i\mathbb{R}_+$. Denote by $\chi_k(y)$ $(\in L_\infty(\mathbb{R}_+))$ the characteristic function of the set Y_k, and by χ_{Π_k} $(\in L_\infty(\Pi))$ the characteristic function of the set Π_k $k = \overline{1,n}$.

Introduce the algebra

$$\mathcal{A}_n = \{a_1\chi_{\Pi_1} + \dots + a_n\chi_{\Pi_n} \;:\; a_k \in \mathbb{C}, \quad k = \overline{1,n}\} \cong \mathbb{C}^n.$$

We will study the algebras $\mathcal{T}(\mathcal{A}_n)$ and $\mathcal{R}(\mathcal{A}_n, B_\Pi)$.

THEOREM 5.15. *The algebra $\mathcal{T}(\mathcal{A}_n)$ is isomorphic and isometric to the algebra $C(\Delta)$, where*

$$\Delta = \Delta(Y_1, ..., Y_n) = \mathrm{clos}\,\{\, (\gamma_{\chi_1}(x), ..., \gamma_{\chi_n}(x)) : x \in \mathbb{R}_+ \,\} \quad \subset \quad \Delta_{n-1}, \tag{5.11}$$

and $\gamma_{\chi_k}(x)$, $k = \overline{1,n}$, are given by

$$\gamma_{\chi_k}(x) = \int_{\mathbb{R}_+} \chi_k(\frac{\eta}{2x})\, |\ell_0(\eta)|^2\, d\eta = \int_{2x\cdot Y_k} |\ell_0(\eta)|^2\, d\eta.$$

The isomorphism

$$\tau_n : \mathcal{T}(\mathcal{A}_n) \longrightarrow C(\Delta)$$

is generated by the following mapping of generators T_a, $a = a_1\chi_{\Pi_1} + ... + a_n\chi_{\Pi_n} \in \mathcal{A}_n$, of the algebra $\mathcal{T}(\mathcal{A}_n)$

$$\tau_n : T_a \longmapsto a_1 t_1 + ... + a_n t_n, \quad t = (t_1, ..., t_n) \in \Delta.$$

PROOF. Follows directly from Theorem 5.3. □

The algebra $\mathcal{R}(\mathcal{A}_n, B_\Pi)$ can be defined by another set of generators. Namely, introduce the orthogonal projections

$$P = B_\Pi, \quad \text{and} \quad Q_k = \chi_{\Pi_k} I, \quad k = \overline{1,n}, \tag{5.12}$$

acting on $L_2(\Pi)$. These projections are obviously all-but-one, and the algebra $\mathcal{R}(\mathcal{A}_n, B_\Pi) = \mathcal{C}^*(P, Q_1, ..., Q_n)$ is generated by them. Thus we have the following theorem.

THEOREM 5.16. *The C^*-algebra $\mathcal{R}(\mathcal{A}_n, B_\Pi)$ is isomorphic and isometric to a subalgebra of the algebra $\mathfrak{S}(\Delta) \oplus \mathbb{C}^n$, where $\Delta = \Delta(Y_1, ..., Y_n)$ is given by (5.11). The isomorphic imbedding*

$$\nu : \mathcal{R}(\mathcal{A}_n, B_\Pi) \longrightarrow \mathfrak{S}(\Delta) \oplus \mathbb{C}^n$$

is generated by the following mapping of the generators of the algebra $\mathcal{R}(\mathcal{A}_n, B_\Pi)$:

$$\begin{aligned} \nu : B_\Pi &\longmapsto (p(t), (0, 0, \ldots, 0)), \\ \nu : \chi_{\Pi_k} I &\longmapsto (q_k(t), (0, \ldots, 0, \underset{k-place}{1}, 0, \ldots, 0)), \quad k = \overline{1,n}, \end{aligned}$$

where $p(t)$ and $q_k(t)$ are defined in (5.3) and (5.4), $t = (t_1, \ldots, t_n) \in \Delta = \Delta(Y_1, ..., Y_n)$.

PROOF. The all-but-one projections (5.12) obviously satisfy (5.1). The analysis of the projections $UB_\Pi U^{-1} = \chi_+(x) I \otimes P_0$ and $U\chi_{\Pi_k} U^{-1} = \chi_{\Pi_k}(\frac{y}{2|x|})I$, $k = \overline{1,n}$, shows that property (5.2) does not hold for the projections (5.12) for all $k = \overline{1,n}$. Thus we are in the situation of Theorem 5.2 with $m = n$. The joint spectrum of the operators $C_k = PQ_kP = T_{\chi_{\Pi_k}}$, $k = \overline{1,n}$, is obviously equal to $\Delta = \Delta(Y_1, ..., Y_n)$. □

REMARK 5.17. It is easy to construct the measurable sets Y_k, $k = \overline{1,n}$, for which the corresponding set $\Delta = \Delta(Y_1, ..., Y_n)$ belongs to the interior of the simplex Δ_{n-1}. In such a case the algebra $\mathfrak{S}(\Delta)$ in Theorem 5.16 is just the algebra $C(\Delta, \mathrm{Mat}_n(\mathbb{C}))$.

5.5. General case. Denote by L_∞^0 the subalgebra of L_∞, which consists of all functions having zero limit at the the point $0 \in R_+$. Also let $\mathcal{A}_\infty^0 = \mathbb{C} \otimes L_\infty^0 \subset \mathcal{A}_\infty$,

$$\widetilde{\mathcal{A}}_\infty^0 = U_0^{-1} \widetilde{\mathcal{A}}_\infty^0 U_0 = \{a(\omega(z)) : a \in \widetilde{\mathcal{A}}_\infty^0\} \subset \widetilde{\mathcal{A}}_\infty.$$

Besides the algebra $\widetilde{\mathcal{A}}_\infty^0$ we will consider its rotated variants. To do this introduce the unitary (rotation) operator on $L_2(\mathbb{D})$: given $t_1, t_2 \in \partial\mathbb{D}$, let

$$(U_{(t_1,t_2)}\varphi)(z) = \varphi(\frac{t_2}{t_1}z),$$

then, of course, $U_{(t_1,t_2)}^{-1} = U_{(t_2,t_1)}$. Denote now $\widetilde{\mathcal{A}}_\infty^0(i) = \widetilde{\mathcal{A}}_\infty^0$, and for each $t \in \partial\mathbb{D}$ introduce

$$\widetilde{\mathcal{A}}_\infty^0(t) = U_{(i,t)}^{-1} \widetilde{\mathcal{A}}_\infty^0(i) U_{(i,t)} = \{a(\frac{i}{t}z) : a \in \widetilde{\mathcal{A}}_\infty^0(i)\}.$$

All the functions from $\widetilde{\mathcal{A}}_\infty^0(t)$ are continuous at each point of $\partial\mathbb{D} \setminus \{t\}$, and their restrictions onto $\partial\mathbb{D} \setminus \{t\}$ are identically zero.

LEMMA 5.18. *Given two points $t_1, t_2 \in \partial\mathbb{D}$, the algebras $\mathcal{R}(\mathbb{C} + \widetilde{\mathcal{A}}_\infty^0(t_1), B_\mathbb{D})$ and $\mathcal{R}(\mathbb{C} + \widetilde{\mathcal{A}}_\infty^0(t_2), B_\mathbb{D})$, as well as the algebras $\mathcal{T}(\mathbb{C} + \widetilde{\mathcal{A}}_\infty^0(t_1))$ and $\mathcal{T}(\mathbb{C} + \widetilde{\mathcal{A}}_\infty^0(t_2))$, are unitary equivalent.*

For all functions $a_k \in \widetilde{\mathcal{A}}_\infty^0(t_k)$, $k = 1, 2$, the operators $T_{a_1 \cdot a_2}$ and $T_{a_1}T_{a_2}$ are compact.

PROOF. The first statement follows from

$$\widetilde{\mathcal{A}}_\infty^0(t_2) = U_{(t_1,t_2)}^{-1} \widetilde{\mathcal{A}}_\infty^0(t_1) U_{(t_1,t_2)}, \qquad B_\mathbb{D} = U_{(t_1,t_2)}^{-1} B_\mathbb{D} U_{(t_1,t_2)}.$$

The operator $T_{a_1 \cdot a_2}$ is compact by Theorem 5.13. Let now $c_{(t_1,t_2)}(z)$ be a continuous function on $\overline{\mathbb{D}}$, equal to 0 on a small neighborhood of the point t_1, and equal to 1 on a small neighborhood of the point t_2. Then the compactness of the operator $T_{a_1}T_{a_2}$ follows from

$$\begin{aligned} T_{a_1}T_{a_2} &= B_\mathbb{D} a_1 B_\mathbb{D}(c_{(t_1,t_2)} + 1 - c_{(t_1,t_2)}) B_\mathbb{D} a_2 B_\mathbb{D} \\ &= B_\mathbb{D} a_1 c_{(t_1,t_2)} B_\mathbb{D} a_2 B_\mathbb{D} + B_\mathbb{D} a_1 B_\mathbb{D}(1 - c_{(t_1,t_2)}) a_2 B_\mathbb{D} + K \\ &= T_{a_1 c_{(t_1,t_2)}} \cdot T_{a_2} + T_{a_1} \cdot T_{(1-c_{(t_1,t_2)})a_2} + K, \end{aligned}$$

where K is compact, and from the compactness of the operators $T_{a_1 c_{(t_1,t_2)}}$ and $T_{(1-c_{(t_1,t_2)})a_2}$. □

THEOREM 5.19. *For each finite (ordered) set $\Lambda = \langle n_0, n_1, ..., n_m \rangle$, where $1 = n_0 < n_1 < ... < n_m \leq \infty$, and $n_k \in \mathbb{N} \cup \{\infty\}$, there is a subalgebra $\mathcal{A}_\Lambda$ of $L_\infty(\mathbb{D})$, such that the symbol (Calkin) algebra $\operatorname{Sym}\mathcal{T}(\mathcal{A}_\Lambda)$ of the algebra $\mathcal{T}(\mathcal{A}_\Lambda)$ is commutative, while the symbol algebra $\operatorname{Sym}\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ of the algebra $\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ has irreducible representations exactly of dimensions n_0, n_1, ..., n_m.*

PROOF. Fix a set $\Lambda = \langle n_0, n_1, ..., n_m \rangle$, and denote by t_1, t_2, ... t_m any m different points on the unit circle $\partial\mathbb{D}$. Now for each k ($k = \overline{1,m}$, if $n_m < \infty$, and $k = \overline{1, m-1}$, if $n_m = \infty$) select the subalgebra $\widetilde{\mathcal{A}}_{n_k}^0(t_k)$ of the algebra $\widetilde{\mathcal{A}}_\infty^0(t_k)$ as follows.

Consider a special case of the algebra $\mathcal{A}_n$ of Section 5.4, the algebra $\mathcal{A}_{n_k}$, generated by sets $Y_{k,1}$, ..., Y_{k,n_k}, such that $Y_{k,n_k} = [0, \eta_k]$, for some positive η_k, and that the corresponding set (5.11) intersects the boundary of the simplex $\Delta_{n_k - 1}$ only by the point $(0, ..., 0, 1)$. The algebra $\mathcal{A}_{n_k}^0 = \mathcal{A}_\infty^0 \cap \mathcal{A}_{n_k} \cong \mathbb{C}^{n_k - 1}$ is generated by the characteristic functions $\chi_{\Pi_{k,j}}$, where $\Pi_{k,j} = \mathbb{R} + iY_{k,j}$, $j = \overline{1, n_k - 1}$, and, of course, $\mathbb{C} + \mathcal{A}^0_{n_k} = \mathcal{A}_{n_k}$. Set now

$$\widetilde{\mathcal{A}}^0_{n_k}(t_k) = U^{-1}_{(i,t_k)} U_0^{-1} \mathcal{A}^0_{n_k} U_0 U_{(i,t_k)} = \{a(\omega(\frac{i}{t_k} z)) : a \in \mathcal{A}^0_{n_k}\} \subset \mathcal{A}^0_\infty(t_k).$$

If $n_m = \infty$, set $\widetilde{\mathcal{A}}^0_{n_k}(t_m) = \widetilde{\mathcal{A}}^0_\infty(t_k)$. Finally define the algebra $\mathcal{A}_\Lambda$ as

$$\mathcal{A}_\Lambda = C(\overline{\mathbb{D}}) + \widetilde{\mathcal{A}}^0_{n_1}(t_1) + \dots + \widetilde{\mathcal{A}}^0_{n_m}(t_m).$$

Now the commutativity of the symbol algebra $\operatorname{Sym} \mathcal{T}(\mathcal{A}_\Lambda) = \mathcal{T}(\mathcal{A}_\Lambda)/\mathcal{K}$ follows from that of $\operatorname{Sym} \mathcal{T}(C(\overline{\mathbb{D}}))$, Theorem 5.3 and Lemma 5.18.

To analyze the symbol algebra $\operatorname{Sym} \mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ of the algebra $\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ we will use the standard local principle (see, for example, [**12, 39**]). The algebra $C(\overline{\mathbb{D}})$ is obviously a central commutative subalgebra of $\widehat{\mathcal{R}} = \operatorname{Sym} \mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$. For each $z_0 \in \overline{\mathbb{D}}$ denote by J_{z_0} the maximal ideal of $C(\overline{\mathbb{D}})$ corresponding to z_0, and by $J(z_0)$ the closed two-sided ideal of the algebra $\widehat{\mathcal{R}}$, generated by J_{z_0}. Then the local algebra at the point z_0 is defined as $\mathcal{R}(z_0) = \widehat{\mathcal{R}}/J(z_0)$, the natural projection

$$\pi_{z_0} : \mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D}) \longrightarrow \operatorname{Sym} \mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D}) \longrightarrow \mathcal{R}(z_0)$$

identifies locally equivalent at the point z_0 elements of the algebra $\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$, and all the irreducible representations of the algebra $\operatorname{Sym} \mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ are the collection of irreducible representations of the local algebras $\mathcal{R}(z_0)$, $z_0 \in \overline{\mathbb{D}}$.

For each $z_0 \in \mathbb{D} = \operatorname{Int} \overline{\mathbb{D}}$ the Bergman projection $B_\mathbb{D}$ is locally equivalent to 0. Thus the local algebra $\mathcal{R}(z_0)$ is commutative, and all the irreducible representations, which correspond to points of $\mathbb{D}$, are just one-dimensional representations of the algebra $\mathcal{A}_\Lambda|_\mathbb{D}$.

Recall that each function $a(z)$ which is continuous at some point z_0, is locally equivalent at this point to its value $a(z_0)$, and that all the functions from $\widetilde{\mathcal{A}}^0_{n_k}(t_k)$ are continuous at each point of $\partial \mathbb{D} \setminus \{t_k\}$, and are identically zero in $\partial \mathbb{D} \setminus \{t_k\}$.

At the points $z_0 \in \partial \mathbb{D} \setminus \{t_1, \dots, t_m\}$, functions $a(z) \in C(\overline{\mathbb{D}})$ are locally equivalent to $a(z_0) \in \mathbb{C}$, while all functions from each $\widetilde{\mathcal{A}}^0_{n_k}(t_k)$ are locally equivalent to 0. The local algebra $\mathcal{R}(z_0)$ in this case is generated by $\mathbb{C}$ and the projection $\pi_{z_0}(B_\mathbb{D})$, and thus is commutative and isomorphic to $\mathbb{C}^2$.

Let finally $z_0 = t_k$ for some $k = \overline{1, m}$. Again, functions $a(z) \in C(\overline{\mathbb{D}})$ are locally equivalent to $a(z_0) \in \mathbb{C}$, but now functions from only $\widetilde{\mathcal{A}}^0_{n_j}(t_j)$, with $j \neq k$, are locally equivalent to 0. Thus the local algebra $\mathcal{R}(t_k)$ is generated by images, under the mapping π_{t_k}, of $B_\mathbb{D}$ and functions from $\mathbb{C} + \widetilde{\mathcal{A}}^0_{n_k}(t_k)$.

Further, the algebra $\mathcal{R}(\mathbb{C} + \widetilde{\mathcal{A}}^0_{n_k}(t_k), B_\mathbb{D})$ is unitary equivalent to $\mathcal{R}(\mathcal{A}_{n_k}, B_\Pi)$, which does not contain any non-zero compact operator. And thus its image on the symbol algebra (which can be described via continuous functions on $\overline{\mathbb{D}}$, valued in local algebras $\mathcal{R}(z_o)$ for each z_0) is isomorphic to the initial algebra $\mathcal{R}(\mathbb{C} + \widetilde{\mathcal{A}}^0_{n_k}(t_k), B_\mathbb{D})$. But $\pi_{z_0}(B_\mathbb{D}) = 0$ for each $z_0 \in \mathbb{D}$, and $\pi_{z_0}(\widetilde{\mathcal{A}}^0_{n_k}(t_k)) = 0$ for each $z_0 \in \partial \mathbb{D} \setminus \{t_k\}$. Thus

$$\mathcal{R}(t_k) \cong \mathcal{R}(\mathbb{C} + \widetilde{\mathcal{A}}^0_{n_k}(t_k), B_\mathbb{D}) \cong \mathcal{R}(\mathcal{A}_{n_k}, B_\Pi),$$

and by Theorem 5.7 (for $n_k = \infty$), or Theorem 5.16 (for $n_k < \infty$) the algebra $\mathcal{R}(t_k)$ has irreducible representations of only dimensions n_k and 1. $\square$

REMARK 5.20. Given Λ, the algebra $\mathcal{A}_\Lambda$ constructed is not unique. In the proof of the theorem we show key steps in constructing such algebras. It is of interest to construct algebras of the type $\mathcal{A}_\Lambda$ having a fundamentally different structure.

Functions a in the algebra $\mathcal{A}_\Lambda = C(\overline{\mathbb{D}}) + \widetilde{\mathcal{A}}^0_{n_1}(t_1) + \dots + \widetilde{\mathcal{A}}^0_{n_m}(t_m)$ are obviously continuous at each point of $\partial\,\mathbb{D} \setminus \{t_1, \dots, t_m\}$, and have well-defined limit values at the points t_k, $k = \overline{1,m}$, over the unit circle $\partial\,\mathbb{D}$:

$$a(t_k) = \lim_{\partial\,\mathbb{D} \ni t \to t_k} a(t). \tag{5.13}$$

Routine verification shows that for each $a \in \mathcal{A}_\Lambda$ and each $k = \overline{1,m}$ there is a unique function $\tilde{a}_k \in \widetilde{\mathcal{A}}^0_{n_k}(t_k)$, such that

$$\lim_{\mathbb{D} \ni z \to t_k} (a(z) - a(t_k) - \tilde{a}_k(z)) = 0.$$

For the functions $\tilde{a}_k(z) \in \widetilde{\mathcal{A}}^0_{n_k}(t_k)$ we will use their "origins" $a_k \in \mathcal{A}^0_{n_k} \subset \mathbb{C} \otimes L^0_\infty(\mathbb{R}_+)$, i.e., the functions

$$a_k(\omega) = \tilde{a}_k(\frac{t_k}{i} z(\omega)), \quad \omega \in \Pi. \tag{5.14}$$

Denote by Γ the disjoint union of the unit circle $\partial\,\mathbb{D}$ and m copies of $(0, +\infty] = (0, +\infty]_k$, $k = \overline{1,m}$, with the points $t_k \in \partial\,\mathbb{D}$ and $+\infty \in (0, +\infty]_k$ identified, for each $k = \overline{1,m}$.

COROLLARY 5.21. *The symbol algebra* $\operatorname{Sym} \mathcal{T}(\mathcal{A}_\Lambda)$ *is isomorphic and isometric to a subalgebra of* $C_b(\Gamma)$. *The homomorphism*

$$\mathrm{sym} : \mathcal{T}(\mathcal{A}_\Lambda) \longrightarrow \operatorname{Sym} \mathcal{T}(\mathcal{A}_\Lambda) \subset C_b(\Gamma)$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}_\Lambda)$

$$\mathrm{sym} : T_a \longmapsto \begin{cases} a(t), & \text{if } \ t \in \partial\,\mathbb{D} \setminus \{t_1, \dots, t_m\} \\ a(t_k), & \text{if } \ t = t_k \\ \gamma_{a_k}(x), & \text{if } \ x \in (0, +\infty)_k \end{cases},$$

where $a \in \mathcal{A}_\Lambda$*, and for the function* a_k*, defined in (5.14), the function* $\gamma_{a_k}(x)$ *is given by (5.6).*

COROLLARY 5.22. *The essential spectrum of each Toeplitz operator* T_a*,* $a \in \mathcal{A}_\Lambda$*, is connected and coincides with the closure of the range of its symbol*

$$\text{ess-sp}\, T_a = \operatorname{clos} \{(\operatorname{sym} T_a)(u) : u \in \Gamma\}.$$

6. Toeplitz Operators on the Unit Disk with Radial Symbols

As it follows, for example, from Theorem 5.13, the Toeplitz operators with radial symbols $a(r)$, continuous at the boundary point 1, are quite trivial, nothing but the compact perturbations of scalar operators, $T_{a(r)} = a(1)I + K$.

One of the principal differences between the study of the Toeplitz operators in the Bergman and Hardy spaces is that in the first case there is an additional direction: "inside the domain". As a consequence, symbols with quite a nice behaviour with respect to the circular direction may have very complicated irregular behaviour with respect to the radial direction. In particular this reflects that Toeplitz operators with radial symbols may have, and as we will see, do have interesting and rich structure.

To analyze the impact of the radial component itself we study Toeplitz operators having pure radial symbols. Note that this topic is not absolutely new. In [**21**], studying the Toeplitz operators with *bounded* radial symbols, B. Korenblum and K. Zhu found two of their important properties: the diagonal form of Toeplitz operators with respect to the standard polynomial basis in $\mathcal{A}^2(\mathbb{D})$, and the criterion for compactness of such operators. The methods used in [**21**] did not permit them to consider *unbounded* symbols, which was left as an open problem. Let us mention as well the papers [**25, 31, 37, 38**], where the compactness of Toeplitz operators with bounded radial symbols were studied in different settings.

It appears that Toeplitz operators with radial symbols possess many interesting properties. In particular there exist *compact* Toeplitz operators whose (radial) symbols are *unbounded* near the unit circle $\partial\mathbb{D}$. The essential spectra of Toeplitz operators with pure radial symbols have sufficiently rich structure, and can even be massive (i.e., have positive plane measure). For bounded operators T_a and T_b whose symbols are unbounded, the operator $T_{a\cdot b}$ may not be bounded at all. That is, contrary to commonly known cases, the set of symbols for which corresponding Toeplitz operators are bounded neither forms an algebra (under the pointwise multiplication) nor admits any natural norm. For more details, see [**19, 43**].

6.1. Toeplitz operators with radial symbols.

THEOREM 6.1. *Let $a = a(r)$ be a measurable function on the segment $[0,1]$. Then the Toeplitz operator T_a acting on $\mathcal{A}^2(\mathbb{D})$ is unitary equivalent to the multiplication operator $\gamma_a I = R\,T_a R^*$, where R and R^* are given by (4.5) and (4.4) respectively, acting on l_2^+. The sequence $\gamma_a = \{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$ is given by*

$$\gamma_a(n) = \int_0^1 a(r^{\frac{1}{2(n+1)}})\,dr, \quad n \in \mathbb{Z}_+. \tag{6.1}$$

PROOF. The operator T_a is obviously unitary equivalent to the operator

$$\begin{aligned} R\,T_a\,R^* &= R\,B_{\mathbb{D}}aB_{\mathbb{D}}R^* = R(R^*R)a(R^*R)R^* \\ &= (RR^*)RaR^*(RR^*) = RaR^* \\ &= R_0^*U_2(I\otimes\mathcal{F})a(r)(I\otimes\mathcal{F}^{-1})U_2^{-1}R_0 \\ &= R_0^*U_2\{a(r)\}U_2^{-1}R_0 \\ &= R_0^*\{a(r^{\frac{1}{n+1}})\}R_0. \end{aligned}$$

Now

$$R_0^*\{a(r^{\frac{1}{n+1}})\}R_0\{c_n\}_{n\in\mathbb{Z}^+} = \left\{\int_0^1 a(r^{\frac{1}{n+1}})\,2\,c_n\,rdr\right\}_{n\in\mathbb{Z}_+} = \{\gamma_a(n)\cdot c_n\}_{n\in\mathbb{Z}_+},$$

where

$$\gamma_a(n) = 2\int_0^1 a(r^{\frac{1}{n+1}})\,rdr = \int_0^1 a(r^{\frac{1}{2(n+1)}})\,dr.$$

$\square$

COROLLARY 6.2. *The Toeplitz operator T_a with measurable radial symbol $a = a(r)$ is bounded on $\mathcal{A}^2(\mathbb{D})$ if and only if*

$$\gamma_a = \{\gamma_a(n)\}_{n\in\mathbb{Z}_+} \in l_\infty,$$

and

$$\|T_a\| = \sup_{n\in\mathbb{Z}_+} |\gamma_a(n)|.$$

The Toeplitz operator T_a is compact if and only if $\gamma_a \in c_0$ that is

$$\lim_{n\to\infty} \gamma_a(n) = 0.$$

The spectrum of bounded Toeplitz operator T_a is given by

$$\mathrm{sp}\, T_a = \overline{\{\gamma_a(n) : n \in \mathbb{Z}_+\}},$$

and its essential spectrum ess-sp T_a *coincides with the set of all limit points of the sequence* $\{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$.

As was proved in [**24**], the spectrum of each Toeplitz operator with a symbol real harmonic on $\mathbb{D}$ is always *connected.* Nevertheless for symbols harmonic in $\mathbb{D}$ except for even a point, the situation can be quite different.

Example 6.3. The general form of a radial function which is harmonic in $\mathbb{D}$ (in $\mathbb{D}\setminus\{0\}$, to be more precise) is as follows:

$$h(r) = c_1 \ln\frac{1}{r} + c_2, \qquad c_1, c_2 \in \mathbb{C}.$$

We have

$$\gamma_h(n) = \int_0^1 h(r^{\frac{1}{2(n+1)}})\, dr = \frac{c_1}{2(n+1)} + c_2,$$

that is, the Toeplitz operator T_h is *bounded* on $\mathcal{A}^2(\mathbb{D})$, and its *discrete* spectrum is given by

$$\mathrm{sp}\, T_h = \{\frac{c_1}{2(n+1)} + c_2\}_{n\in\mathbb{Z}_+} \cup \{c_2\}.$$

The Toeplitz operator T_h is compact if and only if $c_2 = 0$.

To study the Toeplitz operators with radial symbols it is useful first to understand the behaviour of sequences of the type (6.1). We have

$$\gamma_a(n) = \int_0^1 a(r^{\frac{1}{2(n+1)}})\, dr = (n+1)\int_0^1 b(u)u^n\, du, \quad n \in \mathbb{Z}_+,$$

with $b(u) = a(\sqrt{u})$. It is natural to assume that

$$\int_0^1 |b(u)| du < \infty,$$

or equivalently

$$\int_0^1 |a(r)| r dr < \infty.$$

That is, the sequence

$$\eta_b(n) = \frac{1}{n+1}\,\gamma_a(n)$$

forms the sequence of the power momentums of the function $b(u)$.

The following uniqueness result is standard in momentum theory, and is important for us.

Theorem 6.4. *Let*

$$\eta_b(n_k) = 0, \qquad k \in \mathbb{Z}_+,$$

where $n_k = n_0 + dk$, $n_0 \in \mathbb{Z}_+$, $d \in \mathbb{N}$. Then $b(u) = 0$ almost everywhere.

PROOF. We have

$$\int_0^1 b(u)u^{n_0}u^{dk}du = 0.$$

Changing the variable $u^d = s$, we obtain

$$\int_0^1 \left[b(s^{1/d})s^{\frac{n_0+1-d}{d}}\right] s^k ds = 0, \quad k = 0, 1, 2, \ldots$$

Now the function in the square brackets belongs to $L_1(0,1)$, and is orthogonal to all polynomials. Thus this function is equal to zero almost everywhere, so $b(u) = 0$ almost everywhere as well. $\square$

COROLLARY 6.5. *There is no function $b(u) \in L_1(0,1)$ for which $\eta_b(n) \neq 0$ only at a finite number of points.*

The behaviour of a sequence $\gamma_a(n)$, when $n \to \infty$, is completely determined by the behaviour of a function $a(r)$ (or a function b) in a neighborhood of the point $r = 1$. Given $b \in L_1(0,1)$, introduce the function

$$B(s) = \int_s^1 b(u)du.$$

THEOREM 6.6. *If the function $B(s)$ when $s \to 1$ has the form*

$$|B(s)| = O(1-s), \tag{6.2}$$

then

$$\sup_{n\in\mathbb{Z}_+} |\gamma_a(n)| < \infty.$$

If

$$|B(s)| = o(1-s), \tag{6.3}$$

then

$$\lim_{n\to\infty} \gamma_a(n) = 0.$$

PROOF. Let $b(u) = a(\sqrt{u}) \in L_1(0,1)$. Integrating by parts we have for $n \geq 1$

$$\gamma_a(n) = (n+1)n \int_0^1 B(s)s^{n-1}ds.$$

Let $\varepsilon = \varepsilon(n) = n^{-2/3}$. Then assuming (6.2) estimate

$$\begin{aligned}
|\gamma_a(n)| &\leq (n+1)n \int_{1-\varepsilon}^1 |B(s)|s^{n-1}ds + (n+1)n \int_0^{1-\varepsilon} |B(s)|s^{n-1}ds \\
&\leq (n+1)n\, c_\varepsilon \int_{1-\varepsilon}^1 (1-s)s^{n-1}ds + (n+1)n(1-\varepsilon)^{n-1} \int_0^1 |B(s)|ds \\
&\leq (n+1)n\, c_\varepsilon \left(\frac{s^n}{n} - \frac{s^{n+1}}{n+1}\right)\Bigg|_{1-\varepsilon}^1 + \mathrm{const}\,(n+1)n\exp((n-1)\ln(1-\varepsilon)) \\
&\leq c_\varepsilon(n+1)n\left(\frac{1}{(n+1)n} + \frac{\exp((n+1)\ln(1-\varepsilon))}{n+1}\right) \\
&+ \mathrm{const}\,(n+1)n(\exp(-(n-1)\varepsilon + (n-1)O(\varepsilon^2))) \\
&\leq c_\varepsilon(1 + n\exp(-(n+1)\varepsilon + O(\varepsilon^2))) + \mathrm{const}\,n^2\exp(-n^{1/3}) \\
&\leq c_\varepsilon + \mathrm{const}\,n^2\exp(n^{-1/3}),
\end{aligned}$$

where "const" denotes a quantity uniformly bounded in ε. Having (6.2), the quantity c_ε is uniformly bounded on ε, and thus $\gamma_a \in l_\infty$.

Having (6.3), the quantity c_ε can be chosen in such a way that

$$\lim_{\varepsilon \to 0} c_\varepsilon = \lim_{n \to \infty} c_{\varepsilon(n)} = 0,$$

and thus $\gamma_a \in c_0$. □

In fact Theorem 6.6 says that the behaviour near the boundary of a certain average of symbols, rather than the behaviour of the symbols themselves, is responsable for the boundedness and compactness properties of corresponding Toeplitz operators. That is, in spite of bad behaviour of a symbol, which can be even unbounded near the boundary, the corresponding Toeplitz operator can be bounded and even compact.

EXAMPLE 6.7. Let

$$a(r) = (1-r^2)^{-\beta} \sin(1-r^2)^{-\alpha}, \tag{6.4}$$

where $\alpha > 0$, and $\beta < 1$. Consider the corresponding function

$$B(v) = \int_v^1 (1-u)^{-\beta} \sin(1-u)^{-\alpha} du.$$

Changing variables

$$s = (1-u)^{-\alpha}, \quad u = 1 - s^{-1/\alpha},$$

we have

$$B(v) = \frac{1}{\alpha} \int_{(1-v)^{-\alpha}}^{\infty} s^{-\delta} \sin s \, ds, \quad \delta = \frac{1-\beta}{\alpha} + 1.$$

Integrate by parts twice:

$$\begin{aligned} B(v) &= \frac{1}{\alpha}(\cos(1-v)^{-\alpha})(1-v)^{\alpha\delta} + \frac{\delta}{\alpha} \int_{(1-v)^{-\alpha}}^{\infty} s^{-\delta-1} \cos s \, ds \\ &= \frac{1}{\alpha}(\cos(1-v)^{-\alpha})(1-v)^{\alpha\delta} - \frac{\delta}{\alpha}(\sin(1-v)^{-\alpha})(1-v)^{\alpha(\delta+1)} \\ &- \frac{\delta(\delta+1)}{\alpha} \int_{(1-v)^{-\alpha}}^{\infty} s^{-\delta-2} \sin s \, ds. \end{aligned}$$

This implies that

$$B(v) = \frac{\cos(1-v)^{-\alpha}}{\alpha}(1-v)^{\alpha-\beta+1} + O((1-v)^{2\alpha-\beta+1}). \tag{6.5}$$

Thus considering the Toeplitz operator T_a with the radial symbol a of the form (6.4) we have

- for $\alpha \geq \beta$ the sequence $\gamma_a(n)$ is bounded, and thus the Toeplitz operator is bounded on $\mathcal{A}^2(\mathbb{D})$;
- for $\alpha > \beta$ the sequence $\gamma_a(n)$ belongs to c_0 and thus the Toeplitz operator is compact on $\mathcal{A}^2(\mathbb{D})$.

Moreover for $\beta \leq 0$ the symbol (6.4) is bounded, while for $\beta > 0$ the symbol (6.4) is unbounded near the boundary $\partial\mathbb{D}$.

Conditions (6.2) and (6.3) are sufficient for boundedness and compactness of an operator T_a in general. It is known [**21**], that for bounded symbols $a(r) \in L_\infty(0,1)$ the condition (6.3) is necessary and sufficient for compactness of T_a on $\mathcal{A}^2(\mathbb{D})$.

A case when conditions (6.2) and (6.3) are necessary for L_1 symbols is described in the following theorem.

THEOREM 6.8. *Let $b(u) \in L_1(0,1)$, and $b(u) \geq 0$ almost everywhere. Then the conditions (6.2) and (6.3) are necessary and sufficient for $\gamma_a \in l_\infty$ and $\gamma_a \in c_0$, respectively.*

PROOF. Let $n = [(1-s)^{-1}]$, then

$$\gamma_a(n) \geq (n+1)\int_s^1 b(u)u^n du \geq \text{const}\,(n+1)\int_s^1 b(u)du = \text{const}\,(n+1)B(s).$$

Thus

$$B(s) \leq \text{const}\,(1-s)\gamma_a(n).$$

□

EXAMPLE 6.9. Consider the following family of radial symbols

$$a_\alpha(r) = (1-r)^{\alpha-1}, \quad \text{where} \quad \alpha > 0,$$

which scales the (polynomial) growth of symbols near the boundary. We have

$$\gamma_{a_\alpha}(n) = (n+1)\int_0^1 (1-\sqrt{r})^{\alpha-1} r^n dr$$

and

$$B_\alpha(s) = \int_s^1 a_\alpha(\sqrt{r})\,dr = \frac{2}{\alpha}s(1-s)^\alpha + \frac{2}{\alpha(\alpha+1)}(1-s)^{\alpha+1}.$$

By Theorem 6.8 the operator T_{a_α} is bounded if and only if $\alpha \geq 1$, and compact if and only if $\alpha > 1$. That is, in this scale unbounded symbols generate unbounded Toeplitz operators. Moreover, as it will follow from Corollary 6.10, to generate bounded or compact Toeplitz operator its unbounded symbol must necessarily have sufficiently sophisticated oscillating behaviour near the unit circle $\partial\mathbb{D}$.

For a non negative symbol $a(r)$ introduce the function

$$m_a(u) = \inf_{r\in[u,1)} a(r),$$

which is obviously always monotone.

COROLLARY 6.10. *If $\lim_{u\to 1} m_a(u) = +\infty$ (which is equivalent to $\lim_{r\to 1} a(r) = +\infty$), then the Toeplitz operator T_a is unbounded.*

PROOF. Estimate

$$B(s) = \int_s^1 b(u)du \geq \inf_{r\in[s^2,1]} a(r)\cdot(1-s),$$

then $\lim\limits_{t\to\infty}(B(s)/(1-s)) = +\infty$. Thus according to Theorem 6.8 the operator T_a is unbounded. □

Note that for general L_1 symbols the conditions (6.2) and (6.3) fail to be necessary.

EXAMPLE 6.11. Let

$$a(r) = -(1-\gamma)(1-r^2)^{-\gamma}\sin(1-r^2)^{-\alpha} + \alpha(1-r^2)^{-\alpha-\gamma}\cos(1-r^2)^{-\alpha}. \tag{6.6}$$

Then

$$B(u) = (1-u)^{1-\gamma}\sin(1-u)^{-\alpha}.$$

Suppose that

$$0 < \gamma < \alpha \tag{6.7}$$

and

$$\alpha + \gamma < 1.$$

Then $b(r)(= a(\sqrt{r})) \in L_1(0,1)$ but conditions (6.2) and (6.3) are not realized. However we can show that the operator T_a is bounded and compact in $\mathcal{A}^2(\mathbb{D})$. Indeed,

$$\gamma_a(n) = (n+1)\int_0^1 b(r)r^n dr = (n+1)n\int_0^1 B(r)r^{n-1}dr.$$

Integrating by parts once more we have

$$\gamma_a(n) = (n+1)n(n-1)\int_0^1 C(r)r^{n-2}dr,$$

where

$$C(r) = \int_r^1 B(s)ds.$$

Setting $\beta = \gamma - 1$ from (6.5) it follows

$$C(r) = \frac{\cos(1-r)^{-\alpha}}{\alpha}(1-r)^{\alpha-\gamma+2} + O((1-r)^{2\alpha-\gamma+2}).$$

Thus

$$|\gamma_a(n)| \leq \text{const}\cdot n^3\int_0^1 (1-r)^{\alpha-\gamma+2}r^{n-2}dr.$$

Integrating by parts two times we have

$$|\gamma_a(n)| \leq \text{const}\cdot(n+1)\int_0^1 (1-r)^{\alpha-\gamma}r^n dr,$$

and due to (6.7) from Theorem 6.6 we have

$$\lim_{n\to\infty}\gamma_a(n) = 0.$$

In the above example we have used the following fact: second "antiderivative" of the function $b(r)$ in a neighborhood of the point $r=1$ has the following asymptotic

$$C(r) = o((1-r)^2).$$

This observation hints the following generalization of Theorem 6.6.

Given $b(u)$, introduce the functions

$$b^{(-j)}(u) = \int_u^1 b^{(-(j-1))}(s)ds, \quad j = 1, 2, \dots,$$

where $b^{(-0)}(s) = b(s)$.

Note, that $B(u) = b^{(-1)}(u)$, and $C(u) = b^{(-2)}(u)$.

THEOREM 6.12. *Let a function $b^{(-j)}(s)$, $j \in \mathbb{N}$, has the following asymptotic, when $s \to 1$,*

$$|b^{(-j)}(s)| = O((1-s)^j),$$

then

$$\sup_{n \in \mathbb{Z}_+} |\gamma_a(n)| < \infty.$$

If

$$|b^{(-j)}(s)| = o((1-s)^j)$$

then

$$\lim_{n \to \infty} \gamma_a(n) = 0.$$

PROOF. The poof is analogous to that of Theorem 6.6. Note that the function (6.6) of Example 6.11 satisfies the hypothesis of Theorem 6.12 for $j = 2$. □

Another useful characterization of a sequence $\gamma_a(n)$ is given by the following theorem.

THEOREM 6.13. *Let $b(u) \in L_1(0,1)$. Then*

$$\lim_{n \to \infty} (\gamma_a(n) - \gamma_a(n+1)) = 0. \tag{6.8}$$

PROOF. Consider

$$\begin{aligned} \gamma_a(n) - \gamma_a(n+1) &= (n+1)\int_0^1 (1-u)u^n b(u)du - \int_0^1 u^{n+1} b(u)du \\ &= I_1(n) + I_2(n). \end{aligned}$$

To estimate the first summand find the point of maximum of the function $s(u) = (1-u)u^n$. This point is obviously $u_0 = 1 - \frac{1}{n+1}$, thus

$$\sup_{u \in [0,1]} s(u) = s(u_0) = \frac{\left(1 - \frac{1}{n+1}\right)^n}{n+1} \leq \frac{\text{const}}{n+1}.$$

Let $\varepsilon = \varepsilon(n) = \frac{1}{\sqrt{n}}$, then

$$\begin{aligned} |I_1(n)| &\leq (n+1)\int_{1-\varepsilon}^1 (1-u)u^n |b(u)|du + (n+1)\int_0^{1-\varepsilon} (1-u)u^n |b(u)|du \\ &\leq \text{const} \int_{1-\varepsilon}^1 |b(u)|du + (n+1)(1-\varepsilon)^n \int_0^1 |b(u)|du. \end{aligned}$$

Now from

$$\lim_{n \to \infty} \int_{1-\varepsilon}^1 |b(u)|du = 0$$

and

$$(n+1)\left(1 - \frac{1}{\sqrt{n}}\right)^n = (n+1)\exp\left(-n\left(\frac{1}{\sqrt{n}} - \frac{1}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right)\right) \to 0$$

it follows that

$$\lim_{n \to \infty} I_1(n) = 0.$$

Analogously splitting the integral $I_2(n)$ on segments $[1-1/\sqrt{n}, 1]$ and $[0, 1-1/\sqrt{n}]$ one can show that

$$\lim_{n \to \infty} I_2(n) = 0,$$

which finishes the proof of the theorem. □

COROLLARY 6.14. *Let $b(u) \in L_1(0,1)$. Then the set of all limit points of the sequence $\gamma_a(n)$ forms a closed connected subset of $\mathbb{C}$. In particular the sequence $\gamma_a(n)$ can not have a finite or countable set of limit points.*

PROOF. Suppose the set K of limit points is not connected. Then there exist two closed subsets K_1 and K_2 (intersecting K) with a positive distance between them such that $K \subset K_1 \cup K_2$. Without loss of generality we can assume that $\gamma_a(n) \in K_1 \cup K_2$ for each n starting from some N. Thus there exist infinitely many $n_j \in \mathbb{N}$ such that $\gamma_a(n_j) \in K_1$ but, at that time, either $\gamma_a(n_j+1) \in K_2$, or $\gamma_a(n_j-1) \in K_2$, which contradicts (6.8). □

COROLLARY 6.15. *The essential spectrum of a bounded Toeplitz operator with a radial symbol is always connected.*

That is, if an l_∞ sequence $\gamma_a(n)$ does not have a limit, then the essential spectrum of the corresponding Toeplitz operator may be either a compact connected curve, or a compact connected subset of $\mathbb{C}$ having positive plain measure. Let us show that both these cases can be realized.

EXAMPLE 6.16. Unit circle and unit interval.
Let $a_p(r) = \alpha_p(\ln r^{-2})^{ip}$, with $\alpha_p \in \mathbb{C}$, and $p \in \mathbb{R}$. Then

$$\begin{aligned} \gamma_{a_p}(n) &= \alpha_p(n+1)\int_0^1 (\ln u^{-1})^{ip} u^n du \\ &= \alpha_p \int_0^1 (\ln(s^{-1/(n+1)}))^{ip} ds \\ &= (n+1)^{-ip}\left[\alpha_p \int_0^1 (\ln s^{-1})^{ip} ds\right]. \end{aligned}$$

Select now α_p in such a way that the multiple in the square brackets is equal to 1, it is easy to see that $\alpha_p = 1/\Gamma(ip+1)$. Then

$$\gamma_{a_p}(n) = (n+1)^{-ip} = \exp(-ip\ln(n+1)).$$

Thus

$$\operatorname{sp} T_{a_p} = \text{ess-sp}\, T_{a_p} = S^1.$$

If $c_p(r) = \operatorname{Im} \alpha_p(\ln r^{-2})^{ip}$, then

$$\gamma_{c_p}(n) = -\sin(p\ln(n+1)), \tag{6.9}$$

and

$$\operatorname{sp} T_{c_p} = \text{ess-sp}\, T_{c_p} = [-1,1].$$

EXAMPLE 6.17. Square.
Let $a(r) = c_1(r) + ic_{\sqrt{2}}(r)$, then by (6.9) we have

$$\gamma_a(n) = -(\sin\ln(n+1) + i\sin\sqrt{2}\ln(n+1)).$$

Since the number $\sqrt{2}$ is irrational, the points $\{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$ form a dense set in the square, and thus

$$\operatorname{sp} T_a = \text{ess-sp}\, T_a = [-1,1]\times[-1,1].$$

EXAMPLE 6.18. A more complicated curve.
Let $a(r) = c_1(r) + ic_2(r)$, then by (6.9) we have

$$\gamma_a(n) = -(\sin\ln(n+1) + i\sin 2\ln(n+1)).$$

Now the points of this sequence are located on the curve

$$y^2 - 4x^2 + 4x^4 = 0.$$

6.2. Algebras of Toeplitz operators. We consider now the C^*-algebra generated by bounded Toeplitz operators with radial symbols. First observe that our class of symbols, as well as the corresponding Toeplitz C^*-algebra, will have certain peculiarities. In particular, contrary to commonly known and studied cases (see, for example [**46**]), the Toeplitz operator algebra is commutative, but the semicommutators $[T_{a_1}, T_{a_2}) = T_{a_1} \cdot T_{a_2} - T_{a_1 \cdot a_2}$ are not compact in general. Moreover, the symbols under study do not form an algebra (under pointwise multiplication). That is, having two radial symbols $a_1(r)$ and $a_2(r)$, for which the corresponding Toeplitz operators $T_{a_1(r)}$ and $T_{a_2(r)}$ are bounded, the Toeplitz operator $T_{a_1 \cdot a_2}$ which corresponds to the product of these symbols is not necessarily bounded. The natural structure on the set of symbols under consideration is a linear space (in the algebraic sense, i.e., no norm structure assumed).

EXAMPLE 6.19. Let

$$a_1(r) = \sin(1-r^2)^{-\alpha}$$

and

$$a_2(r) = (1-r^2)^{-\beta}\sin(1-r^2)^{-\alpha}$$

where $0 \leq \beta < 1$ and $\beta < \alpha$. Then according to the Example 6.7 both operators T_{a_1} and T_{a_2} are bounded and compact.

The product $a_1 \cdot a_2$ has the form

$$a_1(r) \cdot a_2(r) = \frac{(1-r^2)^{-\beta}}{2} - \frac{(1-r^2)^{-\beta}\cos 2(1-r^2)^{-\alpha}}{2}.$$

Let $\beta = 0$, then $T_{a_1 \cdot a_2} = \frac{1}{2}I - T_{a_3}$, where the operator T_{a_3} with the symbol

$$a_3(r) = \frac{1}{2}\cos 2(1-r^2)^{-\alpha}$$

is compact. The compactness of T_{a_3} can be shown as in Example 6.7.

Now let $\beta > 0$. Then the operator T_{a_4} with the symbol

$$a_4(r) = \frac{1}{2}(1-r^2)^{-\beta}$$

according to Example 6.9 is unbounded. At the same time the operator T_{a_5} with the symbol

$$a_5(r) = \frac{1}{2}(1-r^2)^{-\beta}\cos 2(1-r^2)^{-\alpha}$$

is compact (again analogously to Example 6.7).

That is

(i) for $\beta = 0$ the operators T_{a_1} and T_{a_2} are compact, the operator $T_{a_1 \cdot a_2}$ is bounded but not compact; that is, the semicommutator $[T_{a_1}, T_{a_2}) = T_{a_1} \cdot T_{a_2} - T_{a_1 \cdot a_2}$ is *not compact*;
(ii) for $\beta > 0$ the operators T_{a_1} and T_{a_2} are bounded, but the operator $T_{a_1 \cdot a_2}$ is *not bounded* at all.

Denote by $\mathcal{M}$ the linear space of measurable functions such that for each $a(r) \in \mathcal{M}$ the Toeplitz operator $T_{a(r)}$ is bounded on $L_2(\mathbb{D})$, and denote by $\mathcal{T}(\mathcal{M})$ the C^*-algebra which is generated by all Toeplitz operators T_a with symbols $a \in \mathcal{M}$.

THEOREM 6.20. *The C^*-algebra $\mathcal{T}(\mathcal{M})$ is commutative and isomorphically embedded into the algebra l_∞. This embedding*

$$\nu : \mathcal{T}(\mathcal{M}) \longrightarrow l_\infty$$

is generated by the following mapping

$$\nu : T_a \longmapsto \gamma_a,$$

where $a(r) \in \mathcal{M}$, and the sequence γ_a is given by (6.1).

PROOF. Follows directly from Theorem 6.1 and Corollary 6.2. □

THEOREM 6.21. *Let T_a be the Toeplitz operator with radial symbol $a = a(r)$. Then the corresponding Wick function has the form*

$$\widetilde{a}(z,\zeta) = (1 - z\overline{\zeta})^2 \sum_{n=0}^{\infty} (n+1)(z\overline{\zeta})^n \, \gamma_a(n).$$

PROOF. Calculate

$$\begin{aligned}
\widetilde{a}(z,\zeta) &= \frac{\langle ak_\zeta, k_z\rangle}{\langle k_\zeta, k_z\rangle} = k_\zeta^{-1}(z)\,\langle ak_\zeta, k_z\rangle \\
&= \pi\,(1 - z\overline{\zeta})^2\,\frac{1}{\pi^2}\int_{\mathbb{D}} \frac{a(|\tau|)\;dv(\tau)}{(1-\tau\overline{\zeta})^2\,(1-z\overline{\tau})^2} \\
&= \frac{1}{\pi}\,(1 - z\overline{\zeta})^2 \int_{\mathbb{D}} a(|\tau|) \sum_{k=0}^{\infty}(k+1)(\tau\overline{\zeta})^k \sum_{n=0}^{\infty}(n+1)(z\overline{\tau})^n\,dv(\tau) \\
&= \frac{1}{\pi}\,(1 - z\overline{\zeta})^2 \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(k+1)(n+1)\overline{\zeta}^k z^n \int_0^1 a(r) r^{k+n+1} dr \frac{1}{i}\int_{S^1} t^{k-n-1} dt \\
&= (1 - z\overline{\zeta})^2 \sum_{k=0}^{\infty}(n+1)(z\overline{\zeta})^n\, 2(n+1)\int_0^1 a(r) r^{2n+1} dr \\
&= (1 - z\overline{\zeta})^2 \sum_{n=0}^{\infty}(n+1)(z\overline{\zeta})^n\,\gamma_a(n).
\end{aligned}$$

□

Denote by L_n the one-dimensional subspace of $\mathcal{A}^2(\mathbb{D})$ generated by the base element $e_n(z) = \sqrt{\frac{n+1}{\pi}} z^n$, $n \in \mathbb{Z}_+$. Then the one-dimensional projection P_n of $\mathcal{A}^2(\mathbb{D})$ onto L_n obviously has the form

$$P_n f = \langle f, e_n\rangle\, e_n = \frac{n+1}{\pi}\, z^n \int_{\mathbb{D}} f(\zeta)\,\overline{\zeta}^n d\mu(\zeta).$$

COROLLARY 6.22. *Let T_a be a bounded Toeplitz operator having radial symbol $a(r)$. Then the representation of the operator T_a in the form with the Wick symbol gives the spectral decomposition of the operator T_a*

$$T_a = \sum_{n=0}^{\infty} \gamma_a(n) P_n.$$

COROLLARY 6.23. *Let T_a be bounded Toeplitz operator with radial symbol $a(r)$. Then the Wick symbol of the operator T_a is radial as well, and is given by the formula*

$$\widetilde{a}(r) = (1-r^2)^2 \sum_{n=0}^{\infty} (n+1) r^{2n}\, \gamma_a(n).$$

COROLLARY 6.24. *For any $n \in \mathbb{Z}_+$ the one-dimensional space L_n is an eigenspace for any operator $T \in \mathcal{T}(\mathcal{M})$, and the corresponding eigenvalue $\gamma(n)$ is given by the formula*

$$\gamma(n) = \frac{d^n}{dr^n}\left(\frac{1-r}{(n+1)!}\, \widetilde{a}(\sqrt{r})\right)_{r=0},$$

where $\widetilde{a}$ is the Wick symbol of the operator T.

7. Commutative Toeplitz Operator Algebras and Hyperbolic Geometry

C^*-algebras of Toeplitz operators considered in Sections 5 and 6 are commutative, which is impossible (except of the trivial case of scalar operators) for the algebras of Toeplitz operators acting on the Hardy space. The special features which made this phenomenon possible, were *symbols depending only on imaginary part of a variable* for Toeplitz operators on the upper half-plane, and *radial symbols* for Toeplitz operators on the unit disk. At this stage natural questions appear: *whether there exist other classes of symbols which generate commutative Toeplitz operator C^*-algebras, and how they can be classified.*

The invariance of the Bergman kernel under biholomorphic automorphisms of a domain suggests immediately other classes of symbols, obtained from the initial ones by means of biholomorphic automorphisms. But, surprisingly, the key to understanding the nature of such classes of symbols, and to classifying them lies in the hyperbolic geometry of the unit disk, endowed with the Poincaré metric.

Two different cases, studied in Sections 5 and 6 respectively, can be perfectly treated as the classes of symbols constant on cycles which correspond to the two types of pencils of hyperbolic straight lines: pencil of parallel lines and pencil of lines passing through a given point. Contrary to Euclidean geometry, the hyperbolic case has a third type of pencil, different from the previous ones, the pencil of straight lines orthogonal to a given straight line. And as we will see in this section the symbols which are constant on cycles of this third type of a pencil, will again produce a commutative C^*-algebra of Toeplitz operators.

That is, the sets of symbols which generate commutative C^*-algebras of Toeplitz operators can be classified as follows: each pencil of hyperbolic straight lines, of the first, second or the third type, defines a set of symbols consisting of functions which are constant on the corresponding cycles - the orthogonal trajectories to the lines forming a pencil.

7.1. Bergman metric. We recall here necessary facts about the Bergman metric. For further details see, for example [**10, 35, 46**].

Let D be a (bounded or unbounded) simply connected domain with smooth boundary ∂D in the complex plane $\mathbb{C}$. The Bergman metric of the domain D is defined by the following form

$$ds^2 = \frac{1}{2}\, \frac{\partial^2 \ln K_D(z,\overline{z})}{\partial z\, \partial \overline{z}}\, dv(z),$$

where $K_D(z,\overline{\zeta})$ is the Bergman kernel function of the domain D, and $dv(z)$ is the Euclidean volume element. The element of length on the Bergman metric is given by

$$d\ell = \left(\frac{1}{2}\,\frac{\partial^2 \ln K_D(z,\overline{z})}{\partial z\,\partial \overline{z}}\right)^{\frac{1}{2}} dl,$$

where dl is the Euclidean length element. Denote by $\beta_D(z_1,z_2)$ the distance in the Bergman metric between two points z_1 and z_2 of the domain D. A very important feature of the Bergman metric is its invariance under the biholomorphic mappings: *if the function $\omega = \alpha(z)$ gives a biholomorphic mapping of the domain D onto a domain G, then*

$$\beta_D(z_1,z_2) = \beta_G(\alpha(z_1),\alpha(z_2)).$$

Denote by $\Delta_D(z_0,r)$ the (open) Bergman metric disk with the center $z_0 \in D$ and radius $r > 0$, i.e.,

$$\Delta_D(z_0,r) = \{z \in D : \beta_D(z_0,z) < r\}.$$

The boundary

$$\mathcal{S}_D(z_0,r) = \{z \in D : \beta_D(z_0,z) = r\}$$

of the disk $\Delta_D(z_0,r)$ is called the Bergman circle with the center z_0 and radius r.

For the case of the unit disk $\mathbb{D}$ the Bergman metric is defined by the form

$$ds^2 = \frac{1}{(1-|z|^2)^2}\, dv(z),$$

and coincides with the hyperbolic or the Poincaré metric, realizing the non-Euclidean hyperbolic (Lobachevsky) geometry on the disk $\mathbb{D}$. The Bergman distance between two points z_1 and z_2 in $\mathbb{D}$ is given by

$$\beta_{\mathbb{D}}(z_1,z_2) = \frac{1}{2}\ln\frac{|1-z_1\overline{z}_2| + |z_1-z_2|}{|1-z_1\overline{z}_2| - |z_1-z_2|},$$

and in particular, for $z \in \mathbb{D}$,

$$\beta_{\mathbb{D}}(0,z) = \frac{1}{2}\,\ln\frac{1+|z|}{1-|z|}.$$

Note, that the Bergman disk $\Delta_{\mathbb{D}}(z_0,r)$ is actually an Euclidean disk whose (Euclidean) center and radius are as follows

$$C = \frac{1-s^2}{1-s^2|z_0|^2}\, z_0, \qquad\qquad R = \frac{1-|z_0|^2}{1-s^2|z_0|^2}\, s,$$

where

$$s = \frac{e^r - e^{-r}}{e^r + e^{-r}} = \tanh r \in (0,1).$$

Note also that the Bergman and the Euclidean centers of $\Delta_{\mathbb{D}}(z_0,r)$ coincide only for disks centered at the origin.

7.2. Commutative algebras of the Toeplitz operators on the unit disk. Recall first some basic facts from hyperbolic geometry in the unit disk (see, for example [**30, 32**]). A straight (geodesic or hyperbolic) line here is the part of Euclidean circle orthogonal to the boundary $\partial\mathbb{D}$. There are three types of pencils of lines in the hyperbolic geometry:

(1) All lines passing through a given point $z_0 \in \mathbb{D}$.
(2) All lines parallel to a given line (and tending to the same point $z_0 \in \partial\mathbb{D}$).

(3) All lines orthogonal to a given line (with endpoints $z_1, z_2 \in \partial\mathbb{D}$).

Further, the orthogonal trajectories to elements of a pencil are called *cycles*, and have the following characterization for each of the above cases:

(1) Each cycle is a Bergman circle centered at the point z_0.
(2) Each cycle, called in this case *horocycle*, is an Euclidean circle tangent to $\partial\mathbb{D}$ at the point z_0.
(3) Each cycle, called in this case *hypercycle*, is the arc of an Euclidean circle connecting the points z_1 and z_2.

In Figure 2, Figure 3, and Figure 4, illustrating the above cases, the cycles are drawn in bold. Note that in each case the set of all cycles for a fixed pencil can be characterized as the set of all trajectories equidistant in the Bergman metric to a given cycle.

It turns out that there is a deep connection between the structure of the commuting algebras of Toeplitz operators and geometric properties of pencils of hyperbolic lines.

With each pencil of lines $\mathcal{P}$ we associate the class $\mathcal{A}_{\mathcal{P}}$ of measurable functions (symbols) $a = a(z)$, which are constant on cycles, and for which the corresponding Toeplitz operators T_a are bounded on the Bergman space $\mathcal{A}^2(\mathbb{D})$. In particular, the class $\mathcal{A}_{\mathcal{P}}$ contains all $L_\infty(\mathbb{D})$-functions which are constant on cycles. For the simplicity we restrict ourself to the case of bounded symbols. That is, in what follows we mean by $\mathcal{A}_{\mathcal{P}}$ the algebra of all $L_\infty(\mathbb{D})$-functions which are constant on cycles of the pencil $\mathcal{P}$.

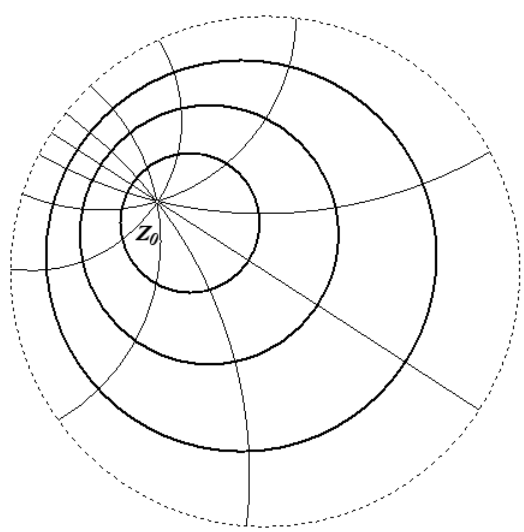

FIGURE 2. Pencil of the first type and corresponding cycles.

We will show that each Toeplitz operator C^*-algebra $\mathcal{T}(\mathcal{A}_{\mathcal{P}})$ which is generated by all operators T_a with $a \in \mathcal{A}_{\mathcal{P}}$, is commutative, and we will describe its structure. Moreover, we conjecture that these are the only possible commutative Toeplitz operator algebras on the unit disk.

Start with a pencil $\mathcal{P} = \mathcal{P}(z_0)$ of the first type, which consists of the all (hyperbolic straight) lines passing through a point $z_0 \in \mathbb{D}$. The functions $a = a(z)$ from $\mathcal{A}(z_0) = \mathcal{A}_{\mathcal{P}(z_0)}$ now are constant on each Bergman circle $\mathcal{S}(z_0, r)$, $r > 0$.

The simplest case of the above situation is the case of $z_0 = 0$. The pencil $\mathcal{P}(0)$ consists of all (Euclidean) diameters of the unit circle $\mathbb{D}$, the set of cycles coincides with the set of all (Euclidean) circles having common center $z_0 = 0$, and the class of symbols $\mathcal{A}(0)$ is the class of radial symbols, studied in detail in Section 6.

Passing to the general case of an arbitrary point $z_0 \in \mathbb{D}$, introduce the Moebius transformation

$$\alpha_{z_0}(z) = \frac{z_0 - z}{1 - z\overline{z}_0}$$

of $\mathbb{D}$ onto itself, which maps the point z_0 to the point 0. Note that this mapping is self-inverse: $\alpha_{z_0}^{-1}(z) = \alpha_{z_0}(z)$. The biholomorphic invariance of the Bergman metric implies that $\mathcal{P}(0) = \alpha_{z_0}(\mathcal{P}(z_0))$ and

$$\mathcal{A}(0) = \{a(\alpha_{z_0}(z)) : a(z) \in \mathcal{A}(z_0)\}.$$

Define the unitary operator $U_{z_0} = U_{z_0}^{-1} : L_2(\mathbb{D}) \longrightarrow L_2(\mathbb{D})$ as follows:

$$(U_{z_0}\varphi)(z) = \alpha'_{z_0}(z)\varphi[\alpha_{z_0}(z)].$$

Then by Lemma 2.10 for each $a(z) \in \mathcal{A}(z_0)$ we have

$$T_{a(z)} = U_{z_0} T_{a(\alpha_{z_0}(z))} U_{z_0}.$$

Thus the properties of Toeplitz operators with $\mathcal{A}(z_0)$-symbols follow directly from the results of Section 6. Let us formulate here just one theorem.

THEOREM 7.1. *The C^*-algebra $\mathcal{T}(\mathcal{A}(z_0))$ is commutative and isomorphically embedded into the algebra l_∞. This embedding*

$$\nu_{z_0} : \mathcal{T}(\mathcal{A}(z_0)) \longrightarrow l_\infty$$

is generated by the following mapping

$$\nu_{z_0} : T_a \longmapsto \gamma_a^{(z_0)},$$

where $a(z) \in \mathcal{A}(z_0)$, and the sequence $\gamma_a^{(z_0)}$ is given by formula

$$\gamma_a^{(z_0)}(n) = (n+1) \int_0^1 a(\sqrt{\alpha_{z_0}(r)}) r^n dr, \qquad n \in \mathbb{Z}_+.$$

Each Toeplitz operator T_a with $a(z) \in \mathcal{A}(z_0)$ admits the following spectral decomposition

$$T_a = \sum_{n=0}^{\infty} \gamma_a^{(z_0)}(n) P_n^{(z_0)},$$

where $P_n^{(z_0)}$ is the one-dimensional projection of $\mathcal{A}^2(\mathbb{D})$ onto the one-dimensional space generated by the function

$$\ell_n^{(z_0)}(z) = \sqrt{\frac{n+1}{\pi}} \left(\frac{z_0 - z}{1 - z\overline{z}_0}\right)^n \frac{|z_0|^2 - 1}{(1 - z\overline{z}_0)^2}.$$

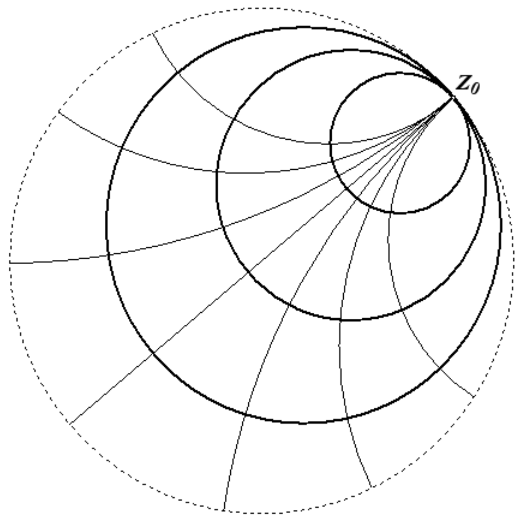

FIGURE 3. Pencil of the second type and corresponding horocycles.

Consider now a pencil $\mathcal{P} = \mathcal{P}(z_0)$ of the second type, which consists of all (straight) lines parallel to a given line, and tending to a same point $z_0 \in \partial\mathbb{D}$. The functions $a = a(z)$ from $\mathcal{A}(z_0) = \mathcal{A}_{\mathcal{P}(z_0)}$ now are constant on each horocycle, i.e., on each Euclidean circle tangent to $\partial\mathbb{D}$ at the point z_0.

Introduce the Moebius transformation

$$w = \alpha_{z_0}(z) = i\frac{z_0 + z}{z_0 - z} \tag{7.1}$$

of the unit disk $\mathbb{D}$ onto the upper half-plane Π, which maps the point $z_0 \in \partial\mathbb{D}$ to the point $\infty \in \Pi$. Then the pencil $\mathcal{P}(\infty) = \alpha_{z_0}(\mathcal{P}(z_0))$ on the upper half-plane Π consists of all semi-lines which are parallel in the Euclidean sense to the positive semi-axis $\{w = 0 + iv : v \in \mathbb{R}_+\}$, the set of all horocycles coincides with the set of all Euclidean straight lines parallel to the real axis being the boundary $\mathbb{R} = \partial\Pi$ of the upper half-plain, and the set

$$\mathcal{A}'(\infty) = \{a(\alpha_{z_0}^{-1}(w)) : a(z) \in \mathcal{A}(z_0)\}.$$

consists of functions depending only on v, the imaginary part of $w = u + iv \in \Pi$. Thus we are in the situation studied in Section 5.

The unitary operator $V_{z_0} : L_2(\Pi) \longrightarrow L_2(\mathbb{D})$, where

$$(V_{z_0}\varphi)(z) = i\frac{2z_0}{(z_0 - z)^2}\,\varphi\left(i\frac{z_0 + z}{z_0 - z}\right),$$

provides the following unitary equivalence of the Toeplitz operators

$$T_{a(z)} = V_{z_0}^{-1} T_{a(\alpha_{z_0}^{-1}(w))} V_{z_0}$$

for each $a(z) \in \mathcal{A}(z_0)$, and where $\alpha_{z_0}^{-1}(w)$ is the mapping inverse to (7.1). Thus as a direct corollary of Theorem 5.3 we have

THEOREM 7.2. *The algebra $\mathcal{T}(\mathcal{A}(z_0))$ is commutative. The isomorphic imbedding*

$$\tau_{z_0} : \mathcal{T}(\mathcal{A}(z_0)) \longrightarrow C_b(\mathbb{R}_+)$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}(z_0))$

$$\tau_{z_0} : T_a \longmapsto \gamma_a^{(z_0)}(x) = \int_{\mathbb{R}_+} a(\alpha_{z_0}^{-1}(\frac{\eta}{2x}))\, e^{-\eta}\, d\eta, \quad x \in \mathbb{R}_+,$$

where $a = a(z) \in \mathcal{A}(z_0)$, and $\alpha_{z_0}^{-1}(w)$ is the mapping inverse to (7.1).

Consider finally a pencil $\mathcal{P} = \mathcal{P}(z_1, z_2)$ of the third type, which consists of all (hyperbolic straight) lines orthogonal to a given line with endpoint z_1, $z_2 \in \partial\mathbb{D}$. The functions $a = a(z)$ from $\mathcal{A}(z_1, z_2) = \mathcal{A}_{\mathcal{P}(z_1,z_2)}$ now are constant on each hypercycle, i.e., on each Euclidean arc connecting the points z_1 and z_2.

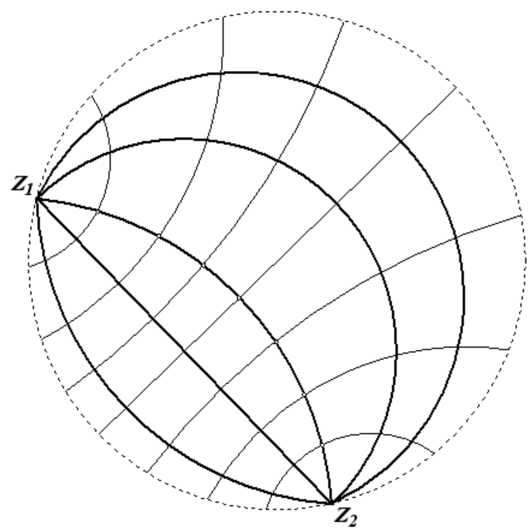

FIGURE 4. Pencil of the third type and corresponding hypercycles.

Introduce the following Moebius transformation of the unit disk $\mathbb{D}$ onto the upper half-plane Π:

$$w = \alpha_{z_1,z_2}(z) = \sqrt{\frac{z_2}{z_1}}\,\frac{z_1 - z}{z - z_2}, \tag{7.2}$$

where the value of the square root is selected to be in the upper half-plane. This transformation maps the points z_1 and z_2 on $\partial\mathbb{D}$ to the points 0 and ∞ on Π, respectively. Then the pencil $\mathcal{P}''(\infty) = \alpha_{z_1,z_2}(\mathcal{P}(z_1, z_2))$ on the upper half-plane Π consists of all Euclidean semi-circles centered at the origin, the set of all hypercycles coincides with the set of all Euclidean rays outgoing from the origin, and the set

$$\mathcal{A}''(\infty) = \{a(\alpha_{z_1,z_2}^{-1}(w)) : a(z) \in \mathcal{A}(z_1, z_2)\},$$

where α_{z_1,z_2}^{-1} is the mapping inverse to (7.2), consists of functions homogeneous of order zero.

Introduce the unitary operator $W_{z_1,z_2} : L_2(\Pi) \longrightarrow L_2(\mathbb{D})$ as follows:

$$(W_{z_1,z_2}\varphi)(z) = \sqrt{\frac{z_2}{z_1}} \, \frac{z_2 - z_1}{(z - z_2)^2} \, \varphi\left(\sqrt{\frac{z_2}{z_1}} \, \frac{z_1 - z}{z - z_2}\right),$$

then the Toeplitz operator algebra $\mathcal{T}(\mathcal{A}(z_1, z_2))$ we are interested in is unitary equivalent to the algebra

$$\mathcal{T}(\mathcal{A}''(\infty)) = W_{z_1,z_2}^{-1} \mathcal{T}(\mathcal{A}(z_1, z_2)) W_{z_1,z_2}.$$

THEOREM 7.3. *The algebra $\mathcal{T}(\mathcal{A}''(\infty))$ is commutative. The isomorphic imbedding*

$$\tau''_\infty : \mathcal{T}(\mathcal{A}''(\infty)) \longrightarrow C_b(\mathbb{R})$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}''(\infty))$

$$\tau''_\infty : T_a \longmapsto \gamma''_a(\lambda) = \frac{2\lambda}{1 - e^{-2\lambda\pi}} \int_0^\pi a(e^{i\theta}) \, e^{-2\lambda\theta} \, d\theta, \quad \lambda \in \mathbb{R},$$

where $a \in \mathcal{A}''(\infty)$

We will prove this theorem in the next subsection.

COROLLARY 7.4. *The algebra $\mathcal{T}(\mathcal{A}(z_1, z_2))$ is commutative. The isomorphic imbedding*

$$\tau_{(z_1,z_2)} : \mathcal{T}(\mathcal{A}(z_1, z_2)) \longrightarrow C_b(\mathbb{R})$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}(z_1, z_2))$

$$\tau_{(z_1,z_2)} : T_a \longmapsto \gamma_a^{(z_1,z_2)}(\lambda) = \frac{2\lambda}{1 - e^{-2\lambda\pi}} \int_0^\pi a(\alpha_{z_1,z_2}^{-1}(e^{i\theta})) \, e^{-2\lambda\theta} \, d\theta, \quad \lambda \in \mathbb{R},$$

where $a(z) \in \mathcal{A}(z_1, z_2)$, and $\alpha_{z_1,z_2}^{-1}(w)$ is the mapping inverse to (7.2).

The pencils of the first and the second types are uniquely defined by a point z_0. This point belongs to $\mathbb{D}$ and is a common point of all the lines, in the first case, and belongs to $\partial\mathbb{D}$ and is a point (at infinity) to with the parallel lines tend, in the second case. The pencil of the third type is uniquely defined by two points z_1, $z_2 \in \partial\mathbb{D}$. These points determine the straight line to which all lines of a pencil are orthogonal. Thus, reformulating the results of this section, we can say each point $z_0 \in \overline{\mathbb{D}}$ or each two points z_1, $z_2 \in \partial\mathbb{D}$ determine uniquely a class of symbols for which the corresponding Toeplitz operator C^*-algebra $\mathcal{T}$ is commutative.

The structure of this algebra $\mathcal{T}$ depends on where these points are situated. For the interior points $z_0 \in \mathbb{D}$ the corresponding algebra $\mathcal{T}$ is isomorphic to the algebra of *sequences*, while for points on the boundary $\partial\mathbb{D}$ the corresponding algebra $\mathcal{T}$ is isomorphic to the algebra of *functions*. Thus, for example, there exist non-zero compact operators in the first case, and no such operators in the second case; each operator has eigenvalues and one-dimensional eigenspaces in the first case, and only continuous spectra in the second case.

Note finally that using biholomorphic mappings the results of this section can be easily extended to the case of an arbitrary simply connected domain in $\mathbb{C}$ with a smooth boundary.

7.3. Proof of Theorem 7.3. On the space $L_2(\Pi)$ introduce the operators

$$(S_\Pi\varphi)(w) = -\frac{1}{\pi}\int_\Pi \frac{\varphi(\omega)}{(\omega - w)^2}\,dv(\omega) \quad \text{and} \quad (S_\Pi^*\varphi)(w) = -\frac{1}{\pi}\int_\Pi \frac{\varphi(\omega)}{(\overline{\omega} - \overline{w})^2}\,dv(\omega).$$

LEMMA 7.5. *The Bergman projection $B_\Pi : L_2(\Pi) \to \mathcal{A}^2(\Pi)$ on the upper half-plane admits the following representation*

$$B_\Pi = I - S_\Pi S_\Pi^*.$$

PROOF. It is easy to check that for the unitary operators $U : L_2(\Pi) \to L_2(\mathbb{D})$ and $W : L_2(\Pi) \to L_2(\Pi)$ defined by

$$(U\varphi)(z) = \frac{2i}{(1-z)^2}\,\varphi\left(i\frac{z+1}{1-z}\right) \quad \text{and} \quad (W\varphi)(w) = \left(\frac{w+i}{\overline{w}-i}\right)^2 \varphi(w),$$

we have $U^{-1}S_\mathbb{D}U = S_\Pi W$. Thus the equality (see Statement 2.6) $B_\mathbb{D} = I - S_\mathbb{D}S_\mathbb{D}^*$ implies

$$\begin{aligned} B_\Pi = U^{-1}B_\mathbb{D}U &= I - U^{-1}S_\mathbb{D}UU^{-1}S_\mathbb{D}^*U \\ &= I - S_\Pi WW^*S_\Pi^* = I - S_\Pi S_\Pi^*. \end{aligned}$$

□

We introduce a number of unitary operators:
The Fourier transform $F : L_2(\mathbb{R}^2) \longrightarrow L_2(\mathbb{R}^2)$ given by

$$(F\varphi)(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\cdot\xi}\varphi(x)dx.$$

The Mellin transform $M : L_2(\mathbb{R}_+, rdr) \longrightarrow L_2(\mathbb{R})$ defined as

$$(M\varphi)(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_+} r^{-i\lambda}\varphi(r)dr.$$

Following [**29**], for $u(t) \in C^\infty(\mathbb{T})$, $\lambda \in \mathbb{C}$, $\lambda \neq ik$, $k = 1, 2, \ldots$, $\operatorname{Im}\lambda > 0$, we define the operator

$$(E(\lambda)u)(t) = \gamma(\lambda)\int_\mathbb{T}(-t\cdot\omega + i0)^{-i\lambda-1}u(\omega)d\omega \tag{7.3}$$

where

$$\gamma(\lambda) = \frac{1}{2\pi}\Gamma(1+i\lambda)e^{\frac{\pi}{2}(i-\lambda)} \tag{7.4}$$

and the construction $(t \pm i0)^\mu$ for $t \in \mathbb{R}$ and $\mu \in \mathbb{C}$ is understood in the sense of distributions (see, for example, [**16**]):

$$(t \pm i0)^\mu = t_+^\mu + e^{\pm i\pi\mu}t_-^\mu, \qquad \mu \neq -1, -2, \ldots$$

$$(t \pm i0)^k = t^k \pm (-1)^k\frac{i\pi}{(k-1)!}\delta^{(k-1)}(t), \qquad k = -1, -2, \ldots$$

For $\operatorname{Im}\lambda \leq 0$ we understand the integral (7.3) in the sense of analytic continuation, since the function

$$\lambda \longrightarrow E(\lambda)u(t), \quad u(t) \in C^\infty(\mathbb{T})$$

admits analytic continuation in the complex plane minus the poles

$$\lambda = ik, \quad k = 1, 2, \ldots$$

of the Γ-function in (7.4) (see [**29**]). In what follows we will consider only $\lambda \in \mathbb{R}$. Let finally $V : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R})$ be the reflection operator

$$(Vf)(\lambda) = f(-\lambda).$$

Passing to polar coordinates in the plane, we have

$$L_2(\mathbb{R}^2) = L_2(\mathbb{R}_+, rdr) \otimes L_2(S^1),$$

as well as the following representation of the Fourier transform:

PROPOSITION 7.6. ([**20, 29**]) *The Fourier transform admits the following decomposition:*

$$F = (M^{-1} \otimes I)(V \otimes I)(I \otimes E(\lambda))(M \otimes I).$$

Let $a = a(x)$ be a homogeneous function of order zero on $\mathbb{R}^2$ whose restriction $a(t)$ to the unit circle S^1 belongs to $L_\infty(S^1)$, and let $b = b(\xi)$ be a homogeneous function of order zero on $\mathbb{R}^2$ having a continuous restriction to the unit circle S^1. The following result has been established in [**20**]

PROPOSITION 7.7. *Passing to the polar coordinates and by means of the Mellin transform, the operators $a(x)I$ and $F^{-1}b(\xi)F$ are unitary equivalent to the following operators*

$$\begin{aligned}
(M \otimes I)a(x)I(M^{-1} \otimes I) &= I \otimes a(t)I, \\
(M \otimes I)F^{-1}b(\xi)F(M^{-1} \otimes I) &= I \otimes E(\lambda)^{-1}b(\omega)E(\lambda) \\
&= I \otimes (b(it)P_{S^1}^{+} + (b(-it)P_{S^1}^{-} + K(\lambda)),
\end{aligned}$$

where $K(\lambda)$ is compact for each $\lambda \in \mathbb{R}$, $P_{S^1}^{\pm} = \frac{1}{2}(I \pm S_{S^1})$ and

$$(S_{S^1}f)(t) = \frac{1}{\pi i}\int_{S^1} \frac{f(\tau)\,d\tau}{\tau - t}.$$

Applying this result to the operator

$$(F^{-1}\frac{\overline{\xi}}{\xi}F\varphi)(z) = (S_{\mathbb{R}^2}\varphi)(w) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\varphi(\omega)}{(w-\omega)^2}\,dv(\omega)$$

we have

PROPOSITION 7.8. ([**23**]) *The following equality holds*

$$(M \otimes I)S_{\Pi}(M^{-1} \otimes I) = I \otimes (K(\lambda) - I)t^{-2}I,$$

where the compact operator $K(\lambda)$ is norm-continuous on $\lambda \in \mathbb{R}$, is diagonal in the standard base $e_n(t) = \frac{1}{\sqrt{2\pi}}t^n$, $n \in \mathbb{Z}$, of the space $L_2(S^1)$, and

$$K(\lambda)e_n(t) = \frac{-2i\lambda}{n - i\lambda + 1}\,e_n(t), \tag{7.5}$$

for each $\lambda \in \mathbb{R}$.

Let $\chi = \chi(z)$ be the characteristic function of the upper half-plane Π. It is a homogeneous function of order zero, and we will preserve the notation $\chi = \chi(t)$ for its restriction to the unit circle, which is the characteristic function of the upper semi-circle S_+^1.

Proposition 7.9. ([**23**]) *The Bergman projection B_Π on the upper half-plane admits the following representation*

$$(M \otimes I)B_\Pi(M^{-1} \otimes I) = I \otimes B(\lambda), \tag{7.6}$$

where

$$B(\lambda) = \chi(K(\lambda) + K(\lambda)^* - K(\lambda)\chi K(\lambda)^*)\chi I$$

is a finite dimensional projection acting on $L_2(S^1_+)$ for each $\lambda \in \mathbb{R}$, and the operator $K(\lambda)$ is given by (7.5).

Proof. By Lemma 7.5 and Proposition 7.8 we have

$$\begin{aligned}
(M \otimes I)B_\Pi(M^{-1} \otimes I) &= (M \otimes I)(I - S_\Pi S_\Pi^*)(M^{-1} \otimes I) \\
&= \chi I - (M \otimes I)\chi S_{\mathbb{R}^2}\chi(M^{-1} \otimes I)(M \otimes I)\chi S^*_{\mathbb{R}^2}\chi(M^{-1} \otimes I) \\
&= \chi I - \chi(M \otimes I)S_{\mathbb{R}^2}(M^{-1} \otimes I)\chi(M \otimes I)S^*_{\mathbb{R}^2}(M^{-1} \otimes I)\chi I \\
&= \chi I - I \otimes (\chi(K(\lambda) - I)t^{-2}\chi t^2(K(\lambda)^* - I)\chi I) \\
&= I \otimes \chi(K(\lambda) + K(\lambda)^* - K(\lambda)\chi K(\lambda)^*)\chi I = I \otimes B(\lambda).
\end{aligned}$$

Finally, for each $\lambda \in \mathbb{R}$ the operator $B(\lambda)$ is compact and is a projection (because B_Π is), thus it has to be finite dimensional. □

Proposition 7.10. ([**23**]) *For each $f \in \mathcal{A}^2(\Pi)$ and $f_\lambda \in L_2(S^1_+)$, $\lambda \in \mathbb{R}$, such that*

$$(M \otimes I)f = I \otimes f_\lambda$$

we have $B(\lambda)f_\lambda = f_\lambda$.

Furthermore, for the elements

$$h_n(w) = \frac{2i}{\sqrt{2\pi}} \frac{(w-i)^n}{(w+i)^{n+2}}, \qquad n \in \mathbb{Z},$$

of an orthonormal base in $\mathcal{A}^2(\Pi)$ the corresponding functions $h_n(\lambda, t) \in L_2(S^1_+)$, where $\lambda \in \mathbb{R}$ and $t \in S^1_+$, have the form

$$h_n(\lambda, t) = c_n(\lambda)\, t^{i\lambda - 1},$$

where $c_n(\lambda)$ is given by (7.7).

Proof. Using (7.6) calculate

$$\begin{aligned}
I \otimes B(\lambda)f_\lambda &= (I \otimes B(\lambda))(I \otimes f_\lambda) \\
&= (M \otimes I)B_\Pi(M^{-1} \otimes I)(I \otimes f_\lambda) \\
&= (M \otimes I)B_\Pi f = (M \otimes I)f \\
&= I \otimes f_\lambda.
\end{aligned}$$

To find the form of $h_n(\lambda, t)$ set $t = e^{i\theta}$, $\theta \in [0, \pi]$, then

$$\begin{aligned}
((M \otimes I)h_n)(\lambda, t) &= \frac{2i}{\sqrt{2\pi}} \int_{\mathbb{R}_+} r^{-i\lambda} \frac{(re^{i\lambda} - i)^n}{(re^{i\lambda} + i)^{n+2}}\, dr \\
&= \frac{2i}{\sqrt{2\pi}} \int_{\mathbb{R}_+} r^{(-i\lambda+1)-1} \frac{(re^{i\lambda} - i)^n}{(re^{i\lambda} + i)^{n+2}}\, dr.
\end{aligned}$$

Then by formula **2.29** from [**27**] we have

$$((M \otimes I)h_n)(\lambda, t) = \frac{2i}{\sqrt{2\pi}} (-1)^n\, i^{-i\lambda-1}(e^{i\theta})^{i\lambda-1} B(i\lambda+1, -i\lambda+1)\, {}_2F_1(-n, -i\lambda+1, 2; 2),$$

where B is the beta function and ${}_2F_1(a,b,c;z)$ is the Gauss hypergeometric function. Thus

$$((M\otimes I)h_n)(\lambda,t) = c_n(\lambda)\, t^{i\lambda-1},$$

where

$$c_n(\lambda) = \frac{(-1)^n}{\sqrt{2\pi}}\, i^{-i\lambda}\Gamma(i\lambda+1)\Gamma(-i\lambda+1)_2F_1(-n,-i\lambda+1,2;2). \tag{7.7}$$

$\square$

Corollary 7.11. ([**23**]) *For each $\lambda\in\mathbb{R}$ the operator $B(\lambda)$ is the orthogonal one-dimensional projection of the space $L_2(S^1_+)$ onto the one-dimensional space generated by the function*

$$g_\lambda(t) = \sqrt{\frac{2\lambda}{1-e^{-2\lambda\pi}}}\, t^{i\lambda-1},$$

that is

$$(B(\lambda)f)(t) = g_\lambda(t)\int_{S^1_+} f(\tau)\,\overline{g_\lambda(\tau)}\,\frac{d\tau}{i\tau}.$$

Let finally $a(w)\in L_\infty(\Pi)$ be a homogeneous function of order zero. Then the Toeplitz operator $T_a = B_\Pi a(w) B_\Pi$ is unitary equivalent to the following operator

$$\begin{aligned}(M\otimes I)T_a(M^{-1}\otimes I) &= I\otimes[B(\lambda)((M\otimes I)a(w)(M^{-1}\otimes I))B(\lambda)]\\ &= I\otimes[B(\lambda)a(t)B(\lambda)]\\ &= I\otimes\gamma''_a(\lambda)B(\lambda),\end{aligned}$$

where

$$\begin{aligned}\gamma''_a(\lambda) &= \int_{S^1_+} a(t)|g_\lambda(t)|^2\frac{dt}{it}\\ &= \frac{2\lambda}{1-e^{-2\lambda\pi}}\int_0^\pi a(e^{i\theta})\,e^{-2\lambda\theta}\,d\theta.\end{aligned}$$

Thus the C^*-algebra generated by the above Toeplitz operators T_a is isomorphic and isometric to the algebra generated by the corresponding functions $\gamma''_a(\lambda)$, $\lambda\in\mathbb{R}$. This proves Theorem 7.3.

References

[1] S. Axler. *Bergman spaces and their operators, Surveys on Some Recent Results in Operator Theory, Vol. 1*, pages 1–50. Pitman Research Notes in Math. Vol. 171, 1988.

[2] M. B. Balk. *Polyanalytic functions and their generalizations, Complex analysis I. Encycl. Math. Sci*, volume 85, pages 195–253. Springer Verlag, 1997.

[3] M. B. Balk and M. F. Zuev. On polyanalytic functions. *Russ. Math. Surveys*, 25(5):201–223, 1970.

[4] Harry Bateman. *Table of Integral Transformas, Vol. I.* McGraw-Hill, 1954.

[5] Harry Bateman and Arthur Erdélyi. *Higher transcendental functions, Vol. 2.* McGraw-Hill.

[6] F. A. Berezin. Wick and anti–Wick symbols of operators. *Math. USSR Sbornik*, 84:578–610, 1971.

[7] F. A. Berezin. Covariant and contravariant symbols of operators. *Math. USSR Izvestia*, 6:1117–1151, 1972.

[8] F. A. Berezin. *Method of Second Quantization.* "Nauka", Moscow, 1988.

[9] F. A. Berezin and M. A. Shubin. *The Schroedinger equation.* Kluwer Academic Publishers, Dordrecht, Boston, 1991.

[10] S. Bergman. *The Kernel Function and Conformal Mapping.* American Mathematical Society, New York, 1950.

[11] Jacques Dixmier. *Les C^*-algebres et Leurs Représentations.* Gauthier-Villars, Paris, 1964.

[12] Ronald G. Douglas. *Banach Algebra Techniques in Operator Theory.* Academic Press, New York, 1972.

[13] A. Dzhuraev. On the theory of systems of singular integral equations over a bounded plane domain. *Soviet Math. Dokl.*, 20(6):1178–1182, 1979.

[14] A. Dzhuraev. Multikernel functions of a domain, kernel operators, singular integral operators. *Soviet Math. Dokl.*, 32(1):251–253, 1985.

[15] A. Dzhuraev. *Methods of Singular Integral Equations.* Longman Scientific & Technical, 1992.

[16] I. M. Gelfand and G. E. Shilov. *Generalized Functions. Volume 1: Properties and Operations.* Academic Press, New York, 1964.

[17] I. Gohberg and N. Krupnik. On the algebra generated by Toeplitz matrices. *Funct. Anal. Appl*, 3:119–127, 1969.

[18] I. Gohberg and N. Krupnik. On the algebra generated by one-dimensional singular integral operators with piecewise continuous coefficients. *Funct. Anal. Appl*, 4:193–201, 1970.

[19] S. Grudsky and N. Vasilevski. Bergman-Toeplitz operators: Radial component influence. Reporte Interno #273, Departamento de Matemáticas, CINVESTAV del I.P.N., Mexico City, 2000. To appear in *Integr. Equat. Oper. Th.*

[20] A. N. Karapetyants, V. S. Rabinovich, and N. L. Vasilevski. On algebras of two-dimensional singular integral operators with homogeneous discontinuities in symbols. Reporte Interno #272, Departamento de Matemáticas, CINVESTAV del I.P.N., Mexico City, 2000. To appear in *Integr. Equat. Oper. Th.*

[21] B. Korenblum and K. Zhu. An application of Tauberian theorems to Toeplitz operators. *J. Operator Theory*, 339:353–361, 1995.

[22] St. G. Krantz. *Function Theory of Several Complex Variables.* Willey and Sons, New York, 1982.

[23] Maribel Loaiza Leyva. *Algebras Genetared by the Bergman Projection and Multiplication Operators by Piecewise Comtinuous Functions.* Ph.D. Dissertation, CINVESTAV del I.P.N., Department of Mathematics, 2000.

[24] G. McDonald and C. Sundberg. Toeplitz operators on the disk. *Indiana Math. J.*, 28:595–611, 1979.

[25] Jie Miao. Toeplitz operators with bounded radial symbols on the harmonic Bergman space of the unit ball. *Acta Sci. Math. (Szeged)*, 63:639–645, 1997.

[26] S. G. Mikhlin and S. Proessdorf. *Singular Integral Operators.* Springer Verlag, Berlin, Heidelberg, 1986.

[27] F. Oberhettinger. *Tables of Mellin Transforms.* Springer Verlag, Berlin, Heidelberg, New York, 1974.

[28] L. Peng, R. Rochberg, and Z. Wu. Orthogonal polynomials and middle Hankel operators on Bergman spaces. *Studia Math.*, 102(1):57–75, 1992.

[29] B. A. Plamenevsky. *Algebras of Pseudodifferential Operators.* Kluwer Academic Publishers, 1989.

[30] I. I. Privalov. *Introduction to the Theory of Function of a Complex Variable.* "Nauka", Moscow, 1977.

[31] Sangadji and K. Stroethoff. Compact Toeplitz operators on generalized Fock spaces. *Acta Sci. Math. (Szeged)*, 64:657–669, 1998.

[32] Giovanni Sansone and Johan Gerrtsen. *Lectures on the Theory of Functions of a Complex Variable*, volume II. Geometric Theory. Wolters-Noordhoff Publishing Groningen, The Netherlands, 1969.

[33] D. Sarason. Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.*, 207:391–405, 1975.

[34] D. Sarason. Toeplitz operators with piecewise quasicontinuous symbols. *Indiana Univ. Math. J.*, 26(5):817–838, 1977.

[35] B. V. Shabat. *Introduction to the Complex Analysis,* Part I, and Part II. "Nauka", Moscow, 1976.

[36] I. B. Simonenko. A new general method for studing linear operator equations of singular integral eguation type, I, II. *Izv. Akad. Nauk SSSR, Ser. Math.*, 29:567–586, 757–782, 1965.

[37] K. Stroethoff. Compact Toeplitz operators on Bergman spaces. *Math. Poc. Camb. Phil. Soc.*, 124:151–160, 1998.

[38] K. Stroethoff. Compact Toeplitz operators on weighted harmonic Bergman spaces. *J. Austal. Math. Soc. (Series A)*, 64:136–148, 1998.

[39] J. Varela. *Duality of C^*-algebras, Memories Amer. Math. Soc.*, volume 148, pages 97–108. AMS, Providence, Rhode Island, 1974.

[40] N. L. Vasilevski. Banach algebras generated by two-dimensional integral operators with Bergman kernel and piece-wise continuous coefficients. I. *Soviet Math. (Izv. VUZ)*, 30(3):14–24, 1986.

[41] N. L. Vasilevski. Convolution operators on standard CR–manifold. II. Algebras of convolution operators on the Heisenberg group. *Integr. Equat. Oper. Th.*, 19:327–348, 1994.

[42] N. L. Vasilevski. C^*-algebras generated by orthogonal projections and their applications. *Integr. Equat. Oper. Th.*, 31:113–132, 1998.

[43] N. L. Vasilevski. Bergman type spaces on the unit disk and Toeplitz operators with radial symbols. Reporte Interno #245, Departamento de Matemáticas, CINVESTAV del I.P.N., Mexico City, 1999.

[44] N. L. Vasilevski. On the structure of Bergman and poly–Bergman spaces. *Integr. Equat. Oper. Th.*, 33:471–488, 1999.

[45] Kehe Zhu. VMO, ESV, and Toeplitz operators on the Bergman space. *Trans. Amer. Math. Soc.*, 302:617–646, 1987.

[46] Kehe Zhu. *Operator Theory in Function Spaces.* Marcel Dekker, Ins., 1990.

Departamento de Matemáticas, CINVESTAV del I.P.N., Mexico, D.F.
E-mail address: nvasilev@math.cinvestav.mx