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Bargmann Projection, Three–Valued Functions and Corresponding Toeplitz Operators

Enrique Ramírez de Arellano Nikolai Vasilevski

Abstract

We describe the C^*-algebra generated by the Bargmann projection and the multiplication operators by three–valued functions. The key point of this investigation is the description of one important specific case of the C^*-algebra generated by three orthogonal projections.

The algebra generated by Toeplitz operators with three–valued symbols acting on the Fock space, and spectra of Toeplitz operators are considered as well.

KEYWORDS: *Orthogonal projections, Toeplitz operators, Fock space, operator algebras*

AMS SUBJECT CLASSIFICATION: Primary 47B35; Secondary 47D25

1 Introduction

1.1 The investigation of the C^*-algebra generated by an orthogonal projection and piecewise continuous functions (in the case when this projection commutes with multiplication operators by continuous functions up to a compact operator) is quite standard (see, for example, [8, 13]) and splits basically into the two following steps: the application of the local principle for C^*-algebras [5, 12] and the description of model (or local) C^*-algebras generated by an orthogonal projection and the multiplication operator by the characteristic function of an appropriate measurable set. The last step is normally based on the well known description of the C^*-algebra generated by two orthogonal projections [7, 9, 14].

However, if one would like to change piecewise continuous functions for functions having three cluster points on some set, the second step fails. There is no general description of the C^*-algebra generated by three orthogonal projections, even in the needed specific case: the product of two of them is equal to zero. In Section 2 we study one important special case of the C^*-algebra generated by three orthogonal projections. This allows us to carry out the desired step. In Section 3 we describe the C^*-algebra generated by the Bargmann projection and two others: the multiplication operators by characteristic functions of two different measurable sets. In Section 4 we show how the structure of the algebra depends on these measurable sets. The corresponding Toeplitz operator algebra and the spectrum of Toeplitz operators are considered in Section 5.

1.2 We use the following standard notation: $z = x + iy = (z_1, \ldots, z_n) \in \mathbb{C}^n$; $\overline{z} = (\overline{z}_1, \ldots, \overline{z})$ with the usual notion of the complex conjugation; for $z, w \in \mathbb{C}^n$, $z \cdot w = z_1 w_1 + \ldots + z_n w_n$; $|z|^2 = |z_1|^2 + \ldots + |z_n|^2 (= z \cdot \overline{z})$; $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$. Denote by $d\mu_n(z)$ the following Gaussian measure over $\mathbb{C}^n$:

$$d\mu_n(z) = \pi^{-n} e^{-z \cdot \overline{z}} dv(z),$$

where $dv(z) = dx\, dy$ is the usual Euclidean volume measure on $\mathbb{C}^n = \mathbb{R}^{2n}$.

Denote by $L_2(\mathbb{C}^n, d\mu_n)$ the Hilbert space of μ square–integrable functions on $\mathbb{C}^n$ with the following inner product:

$$\langle f, \varphi \rangle = \int_{\mathbb{C}^n} f(z) \overline{\varphi(z)}\, d\mu_n(z).$$

The closed subspace $F^2(\mathbb{C}^n)$ of $L_2(\mathbb{C}^n, d\mu_n)$ consisting of all entire functions is usually called the Fock [2, 6] or the Segal–Bargmann space [1, 11].

The orthogonal Bargmann projection

$$P_n : L_2(\mathbb{C}^n, d\mu_n) \to F_2(\mathbb{C}^n)$$

is given [3] by the formula

$$(P_n \varphi)(z) = \int_{\mathbb{C}^n} \varphi(\zeta) e^{\overline{\zeta} \cdot z} d\mu_n(\zeta). \tag{1.1}$$

2 Special case of the algebra generated by three orthogonal projections

2.1 Let H be a Hilbert space, $l_0 \in H$ with $\| l_0 \| = 1$. Denote by L_0 the one–dimensional subspace in H generated by l_0, and let

$$P\varphi = \langle \varphi, l_0 \rangle l_0 : H \to L_0$$

be the one–dimensional orthogonal projection of H onto L_0.

Consider two (other) orthogonal projections Q_1 and Q_2 on H with the property

$$Q_1 \cdot Q_2 = Q_2 \cdot Q_1 = 0 \tag{2.1}$$

We are going to describe the C^*–algebra $\mathcal{R} = \mathcal{R}(\mathbb{C}; P, Q_1, Q_2)$ with identity $I : H \to H$, generated by the three orthogonal projections: the one–dimensional projection P and projections Q_1, Q_2 with the property (2.1).

2.2 Introduce $Q_3 = I - Q_1 - Q_2$ and denote by M_1, M_2, M_3 the images of Q_1, Q_2, Q_3 respectively. Then,

$$H = M_1 \oplus M_2 \oplus M_3.$$

The condition $L_0 \subset M_k$ is equivalent to $PQ_k = P$. If this is the case, we have some additional relations between generators of the algebra $\mathcal{R}$. Thus, for all $k = \overline{1,3} := 1, ..., 3$

$$L_0 \cap M_k = \{0\}.$$

Analogously the condition $M_k \subset L_0^\perp$ leads to the additional relation $Q_k(I - P) = Q_k$.

Split the space H onto the following direct sum:

$$H = (L_0^\perp \cap M_1) \oplus (L_0^\perp \cap M_2) \oplus (L_0^\perp \cap M_3) \oplus \mathfrak{M}.$$

We have obviously $L_0 \subset \mathfrak{M}$.

Let

$$P' = P|_{\mathfrak{M}}, \quad Q'_k = Q_k|_{\mathfrak{M}}, \quad k = \overline{1,3}, \quad I' = I|_{\mathfrak{M}}.$$

Each M_k does not belong to $L_0^\perp$, thus all these restrictions onto $\mathfrak{M}$ are not trivial, and

$$I' = Q'_1 + Q'_2 + Q'_3.$$

Furthermore $\operatorname{Im} P' = L_0$, $\operatorname{Im} Q'_k = M_k \cap \mathfrak{M} = M'_k$, and

$$\mathfrak{M} = M'_1 \oplus M'_2 \oplus M'_3.$$

A simple Halmos–like [7] argument shows that the operator $P|_{M'_k} : M'_k \to L_0$ is a bijection, thus all spaces M'_k are one–dimensional. Let $M'_k = \{c \cdot m_k : c \in \mathbb{C}\}$, then the space $\mathfrak{M}$ is three dimensional and thus

$$l_0 = \alpha_1 \cdot m_1 + \alpha_2 \cdot m_2 + \alpha_3 \cdot m_3$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$.

If some $\alpha_k = 0$, then $M_k' \subset L_0^\perp$, which leads to $M_k \subset L_0^\perp$. Thus all $\alpha_k \neq 0$ and $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1$. Represent H as the following direct sum:

$$H = (L_0^\perp \cap M_1) \oplus (L_0^\perp \cap M_2) \oplus (L_0^\perp \cap M_3) \oplus (M_1' \oplus M_2' \oplus M_3'),$$

then each of the projections has the following form:

$$\begin{aligned}
Q_1 &= I + 0 + 0 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
Q_2 &= 0 + I + 0 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
Q_3 &= 0 + 0 + I + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
P &= 0 + 0 + 0 + \begin{pmatrix} \alpha_1\overline{\alpha_1} & \alpha_1\overline{\alpha_2} & \alpha_1\overline{\alpha_3} \\ \alpha_2\overline{\alpha_1} & \alpha_2\overline{\alpha_2} & \alpha_2\overline{\alpha_3} \\ \alpha_3\overline{\alpha_1} & \alpha_3\overline{\alpha_2} & \alpha_3\overline{\alpha_3} \end{pmatrix}.
\end{aligned}$$

2.3 Theorem *Let $L_0^\perp \cap M_k \neq \{0\}$, $k = \overline{1,3}$, then the algebra $\mathcal{R} = \mathcal{R}(\mathbb{C}; P, Q_1, Q_2)$ is isomorphic to $\mathbb{C}^3 \oplus \mathrm{Mat}_3(\mathbb{C})$.*

The isomorphism

$$\nu : \quad \mathcal{R} \longrightarrow \mathbb{C}^3 \oplus \mathrm{Mat}_3(\mathbb{C})$$

is generated by the following mapping of the projections

$$\begin{aligned}
\nu : P &\longmapsto \left((0,0,0), \begin{pmatrix} \gamma_1 & \sqrt{\gamma_1\gamma_2} & \sqrt{\gamma_1\gamma_3} \\ \sqrt{\gamma_1\gamma_2} & \gamma_2 & \sqrt{\gamma_2\gamma_3} \\ \sqrt{\gamma_1\gamma_3} & \sqrt{\gamma_2\gamma_3} & \gamma_3 \end{pmatrix}\right), \\
\nu : Q_1 &\longmapsto \left((1,0,0), \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right), \\
\nu : Q_2 &\longmapsto \left((0,1,0), \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right), \\
\nu : Q_3 &\longmapsto \left((0,0,1), \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right),
\end{aligned}$$

where $\gamma_1 = \| Q_1 l_0 \|^2$, $\gamma_2 = \| Q_2 l_0 \|^2$, $\gamma_3 = 1 - \gamma_1 - \gamma_2$.

PROOF. Introduce the following unitary operator on H:

$$W : I \oplus I \oplus I \oplus \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix},$$

where $\omega_k = \alpha_k / |\alpha_k|$.

Then we have obviously

$$\begin{aligned} \widetilde{P} &= W^{-1} P W = 0 + 0 + 0 + \begin{pmatrix} |\alpha_1|^2 & |\alpha_1 \alpha_2| & |\alpha_1 \alpha_3| \\ |\alpha_2 \alpha_1| & |\alpha_2|^2 & |\alpha_2 \alpha_3| \\ |\alpha_3 \alpha_1| & |\alpha_3 \alpha_2| & |\alpha_3|^2 \end{pmatrix}, \\ \widetilde{Q}_1 &= W^{-1} Q_1 W = I = 0 + 0 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \widetilde{Q}_2 &= W^{-1} Q_2 W = 0 + I + 0 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \widetilde{Q}_3 &= W^{-1} Q_3 W = 0 + 0 + I + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

and the algebra $\mathcal{R} = \mathcal{R}(\mathbb{C}; P, Q_1, Q_2)$ is naturally isomorphic to the algebra $\widetilde{\mathcal{R}} = W^{-1} \mathcal{R} W = \mathcal{R}(\mathbb{C}; \widetilde{P}, \widetilde{Q}_1, \widetilde{Q}_2)$ generated by the projections $\widetilde{P}$, $\widetilde{Q}$, and $\widetilde{Q}_2$ (and of course, by I or $\widetilde{Q}_3$). The last algebra is naturally isomorphic to $\mathbb{C}^3 \oplus \mathrm{Mat}_3(\mathbb{C})$.

To finish the proof one needs to follow the correspondence of generators and to use new quantities $\gamma_1 = |\alpha_1|^2 = \| Q_1 l_0 \|^2$, $\gamma_2 = |\alpha_2|^2 = \| Q_2 l_0 \|^2$ and $\gamma_3 = |\alpha_3|^2 = 1 - \gamma_1 - \gamma_2$.

3 Algebra generated by the Bargmann projection and three–valued functions

3.1 Let X_1 and X_2 be two disjoint measurable sets in $\mathbb{R}^n = \mathbb{R}^n_x$, and let

$$X'_k = X_k + i\mathbb{R}^n_y, \quad k = 1, 2$$

be the corresponding sets in $\mathbb{C}^n = \mathbb{R}^n_x + i\mathbb{R}^n_y$.

Introduce the characteristic functions $\chi_{X'_k}$ of the sets X'_k and the orthogonal projections $Q_k = \chi_{X'_k} I$ acting on $L_2(\mathbb{C}^n, d\mu_n)$, here $k = 1, 2$.

Furthermore let

$$Q_3 = I - Q_1 - Q_2.$$

The aim of this section is to describe the C^*–algebra $\mathcal{R} = \mathcal{R}(X_1, X_2)$ with identity I generated by the Bargmann projection $P = P_n$ (1.1) and the projections Q_1, Q_2 acting on $L_2(\mathbb{C}^n, d\mu_n)$.

We have obviously

$$Q_k \cdot Q_j = \delta_{kj} Q_k, \quad k, j = \overline{1, 3},$$

and

$$I = Q_1 + Q_2 + Q_3.$$

3.2 Introduce the following operators:

The unitary operator

$$U : L_2(\mathbb{C}^n, d\mu_n) \to L_2(\mathbb{R}^{2n}) = L_2(\mathbb{R}^{2n}, dxdy),$$

defined by

$$(U\varphi)(z) = \pi^{-\frac{n}{2}} e^{-\frac{z \cdot \overline{z}}{2}} \varphi(z),$$

or

$$(U\varphi)(x, y) = \pi^{-\frac{n}{2}} e^{-\frac{x^2+y^2}{2}} \varphi(x + iy);$$

the Fourier transformation $F : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$, where

$$(Ff)(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\eta \cdot y} f(\eta) d\eta;$$

and the isomorphism

$$W = W^* = W^{-1} : L_2(\mathbb{R}^{2n}) \to L_2(\mathbb{R}^{2n}),$$

where

$$(Wf)(x, y) = f(\frac{1}{\sqrt{2}}(x + y), \frac{1}{\sqrt{2}}(x - y)).$$

Then the unitary operator $K = W(I \otimes F)U$ is an isomorphism

$$K : L_2(\mathbb{C}^n, d\mu_n) \to L_2(\mathbb{R}^n, dx) \otimes L_2(\mathbb{R}^n, dy).$$

We have for the generators of the algebra $\mathcal{R}(X_1, X_2)$:

$$KPK^{-1} = I \otimes P_0,$$

where
$$(P_0 f)(y) = \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(y^2+u^2)} f(u)du$$
is the one–dimensional orthogonal projection of $L_2(\mathbb{R}^n, dy)$ onto the one–dimensional space L_0 generated by $l_0(y) = \pi^{-\frac{n}{4}} e^{-\frac{y^2}{2}}$;

$$\begin{aligned}
& KQ_k K^{-1} = \\
= \ & W(I \otimes F) U \chi_{X'_k} U^{-1} (I \otimes F^{-1}) W = \\
= \ & W(I \otimes F) \chi_{X_k}(x) (I \otimes F^{-1}) W = \\
= \ & W \chi_{X_k}(x) W = \chi_{X_k}(\frac{1}{\sqrt{2}}(x+y)) I = \\
= \ & \chi_{\sqrt{2}X_k - x}(y) I, \quad k = \overline{1,3},
\end{aligned}$$

where $X_3 = \mathbb{R}^n \backslash (X_2 \cup X_2)$.

Thus the algebra $K\mathcal{R}K^{-1}$ splits into the direct integral of algebras $\mathcal{R}(x)$, $x \in \mathbb{R}^n$, generated by the orthogonal projections

$$P(x) = P_0, \quad Q_k(x) = \chi_{\sqrt{2}X_k - x}(y) I, \quad k = 1, 2,$$

acting on $L_2(\mathbb{R}^n, dy)$.

3.3 Lemma *The algebra $\mathcal{R}(x)$, $x \in \mathbb{R}^n$, is isomorphic to $\mathbb{C}^3 \oplus \mathrm{Mat}_3(\mathbb{C})$. The isomorphism*

$$\nu_x : \quad \mathcal{R}(x) \to \mathbb{C}^3 \oplus \mathrm{Mat}_3(\mathbb{C})$$

is generated by the following mapping of the generators of the algebra $\mathcal{R}(x)$:

$$\begin{aligned}
\nu_x : \quad P(x) \ &\longmapsto \ ((0,0,0), \begin{pmatrix} \gamma_1(x) & \sqrt{\gamma_1(x)\gamma_2(x)} & \sqrt{\gamma_1(x)\gamma_3(x)} \\ \sqrt{\gamma_1(x)\gamma_2(x)} & \gamma_2(x) & \sqrt{\gamma_2(x)\gamma_3(x)} \\ \sqrt{\gamma_1(x)\gamma_3(x)} & \sqrt{\gamma_2(x)\gamma_3(x)} & \gamma_3(x) \end{pmatrix}), \\
\nu_x : \quad Q_1(x) \ &\longmapsto \ ((1,0,0), \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}), \\
\nu_x : \quad Q_2(x) \ &\longmapsto \ ((0,1,0), \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}),
\end{aligned}$$

where $\gamma_1(x) = \| Q_1(x) l_0 \|^2$, $\gamma_2(x) = \| Q_2(x) l_0 \|^2$, $\gamma_3(x) = 1 - \gamma_1(x) - \gamma_2(x)$.

Proof. Follows directly from Theorem 2.3.

3.4 Denote by $\mathfrak{S}(X_1, X_2)$ the C^*–algebra with identity generated by the following 3×3 matrix–valued functions

$$\begin{aligned} p(x) &= \begin{pmatrix} \gamma_1(x) & \sqrt{\gamma_1(x)\gamma_2(x)} & \sqrt{\gamma_1(x)\gamma_3(x)} \\ \sqrt{\gamma_1(x)\gamma_2(x)} & \gamma_2(x) & \sqrt{\gamma_2(x)\gamma_3(x)} \\ \sqrt{\gamma_1(x)\gamma_3(x)} & \sqrt{\gamma_2(x)\gamma_3(x)} & \gamma_3(x) \end{pmatrix}, \\ q_1(x) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q_2(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where $x \in \mathbb{R}^n$ and $\gamma_1(x)$, $\gamma_2(x)$, $\gamma_3(x)$ are as in Lemma 3.3.

3.5 Corollary *The algebra $\mathcal{R} = \mathcal{R}(X_1, X_2)$ is isomorphic to the algebra $\mathbb{C}^3 \oplus \mathfrak{S}(X_1, X_2)$. The isomorphism*

$$\nu_0 : \mathcal{R} \longmapsto \mathbb{C}^3 \oplus \mathfrak{S}(X_1, X_2)$$

is generated by the following mapping of the generators of the algebra $\mathcal{R}$:

$$\begin{aligned} \nu_0 : \quad P &\longmapsto ((0,0,0),\, p(x)), \\ \nu_0 : \quad Q_1 &\longmapsto ((1,0,0),\, q_1(x)), \\ \nu_0 : \quad Q_2 &\longmapsto ((0,1,0),\, q_2(x)), \end{aligned}$$

where $x \in \mathbb{R}^n$.

3.6 To describe the algebra $\mathfrak{S}(X_1, X_2)$ we need to introduce some kind of "maximal" algebra. Let

$$\Delta_{max} = \{t = (t_1, t_2) \in \mathbb{R}^2 : \ 0 \le t_1 \le 1, \ 0 \le t_2 \le 1, \ 0 \le t_1 + t_2 \le 1\}$$

be the standard two–dimensional simplex.

Denote by $\mathfrak{S}(\Delta_{max})$ the C^*–algebra with identity generated by the following matrix–valued functions

$$\begin{aligned} p(t) &= \begin{pmatrix} t_1 & \sqrt{t_1 t_2} & \sqrt{t_1 t_3} \\ \sqrt{t_1 t_2} & t_2 & \sqrt{t_2 t_3} \\ \sqrt{t_1 t_3} & \sqrt{t_2 t_3} & t_3 \end{pmatrix}, \\ q_1(t) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q_2(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where $t = (t_1, t_2) \in \Delta_{max}$, $t_3 = 1 - t_1 - t_2$.

Now denote by $\mathfrak{S}_{max}$ the C^*–algebra (with identity) of all 3×3 matrix–valued functions $\sigma(t) = (\sigma_{kj}(t))_{k,j=1}^3$ continuous on Δ_{max} and having the following properties:

(i) they are diagonal at the vertices $(0,0)$, $(0,1)$ and $(1,0)$ of Δ_{max}:

(ii) $\sigma_{12}(t) = \sigma_{13}(t) = \sigma_{21}(t) = \sigma_{31}(t) \equiv 0$, $\sigma_{11}(t) \equiv$ const on the side $t_1 = 0$ of the triangle Δ_{max};

(iii) $\sigma_{21}(t) = \sigma_{23}(t) = \sigma_{12}(t) = \sigma_{32}(t) \equiv 0$, $\sigma_{22}(t) \equiv$ const on the side $t_2 = 0$ of the triangle Δ_{max};

(iv) $\sigma_{31}(t) = \sigma_{32}(t) = \sigma_{13}(t) = \sigma_{23}(t) \equiv 0$, $\sigma_{33}(t) \equiv$ const on the side $t_1 + t_2 = 1$ $(t_3 = 0)$ of Δ_{max}.

Standard arguments lead directly to the following lemma.

3.7 Lemma *Up to unitary equivalence all irreducible representations of the algebra $\mathfrak{S}_{max}$ are included in the following list:*

(i) the family $\{\pi_t\}$ of three–dimensional representations parametrized by the points $t \in \operatorname{Int}\Delta_{max}$ and defined by the rule

$$\pi_t : \sigma \in \mathfrak{S}_{max} \longmapsto \sigma(t) \in \mathrm{Mat}_3(\mathbb{C});$$

(ii) the three families $\{\pi_t^{(1)}\}$, $\{\pi_t^{(2)}\}$ and $\{\pi_t^{(3)}\}$ of two–dimensional representations parametrized by the interior parts of the sides $t_1 = 0$, $t_2 = 0$ and $t_1 + t_2 = 1$ $(t_3 = 0)$ of the triangle Δ_{max} respectively; these representations (for the corresponding points t) are defined by the rule

$$\begin{aligned}
\pi_t^{(1)} : \sigma \in \mathfrak{S}_{max} &\longmapsto \begin{pmatrix} \sigma_{22}(t) & \sigma_{23}(t) \\ \sigma_{32}(t) & \sigma_{33}(t) \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{C}), \\
\pi_t^{(2)} : \sigma \in \mathfrak{S}_{max} &\longmapsto \begin{pmatrix} \sigma_{11}(t) & \sigma_{13}(t) \\ \sigma_{31}(t) & \sigma_{33}(t) \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{C}), \\
\pi_t^{(3)} : \sigma \in \mathfrak{S}_{max} &\longmapsto \begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{C});
\end{aligned}$$

(iii) the six one–dimensional representations defined as follows:

$$\begin{aligned}
\pi_{(1)} : \sigma \in \mathfrak{S}_{max} &\longmapsto \sigma_{11}(1,0), \\
\pi_{(2)} : \sigma \in \mathfrak{S}_{max} &\longmapsto \sigma_{11}(0,\tau), \\
\pi_{(3)} : \sigma \in \mathfrak{S}_{max} &\longmapsto \sigma_{22}(0,1), \\
\pi_{(4)} : \sigma \in \mathfrak{S}_{max} &\longmapsto \sigma_{22}(\tau,0), \\
\pi_{(5)} : \sigma \in \mathfrak{S}_{max} &\longmapsto \sigma_{33}(0,0), \\
\pi_{(6)} : \sigma \in \mathfrak{S}_{max} &\longmapsto \sigma_{33}(\tau,1-\tau),
\end{aligned}$$

where $t = (t_1, t_2) \in \partial\Delta_{max}$.

3.8 Theorem *The algebra* $\mathfrak{S}(\Delta_{max})$ *coincides with the* C^**–algebra* $\mathfrak{S}_{max}$.

Proof. The algebra $\mathfrak{S}(\Delta_{max})$ is obviously a C^*–subalgebra of $\mathfrak{S}_{max}$. We will prove that $\mathfrak{S}(\Delta_{max})$ is a massive [4] subalgebra of $\mathfrak{S}_{max}$, and then by the Stone–Weierstrass theorem for GCR–algebras [4] we will have $\mathfrak{S}(\Delta_{max}) = \mathfrak{S}_{max}$. To prove that, we need to check two properties:

(i) for each irreducible representation π of $\mathfrak{S}_{max}$ the representation $\pi|_{\mathfrak{S}(\Delta_{max})}$ is irreducible;

(ii) for each two nonequivalent irreducible representations π_1 and π_2 of $\mathfrak{S}_{max}$ their restrictions $\pi_1|_{\mathfrak{S}(\Delta_{max})}$ and $\pi_2|_{\mathfrak{S}(\Delta_{max})}$ are nonequivalent as well.

We check these two statements for three–dimensional representations only.

Fix a point $t_0 \in \Delta_{max}$ and consider the corresponding representation π_{t_0}. To prove the first statement it is sufficient to find for each matrix $m = (m_{kj})_{k,j=1}^3 \in \mathrm{Mat}_3(\mathbb{C})$ an element $a \in \mathfrak{S}(\Delta_{max})$ such that

$$\pi_{t_0}(a) = m.$$

It is easy to see that we can take a to be of the form

$$a = (m_{kj}\frac{1}{(p(t_0))_{kj}}(q_k p(t_0) q_j)_{kj})_{k,j=1}^3.$$

To prove the second statement it is sufficient to find for each two fixed different points $t' = (t'_1, t'_2)$ and $t'' = (t''_1, t''_2)$ of $\mathrm{Int}\,\Delta_{max}$ an element $a \in \mathfrak{S}(\Delta_{max})$ such that

$$\pi_{t'}(a) = 0, \quad \pi_{t''}(a) \neq 0.$$

The corresponding element, for example, is

$$a = q_1(p - t'_2 e)q_1 + q_2(p - t'_2 e)q_2.$$

3.9 Introduce the set

$$\Delta = \Delta(X_1, X_2) =$$
$$= \mathrm{clos}\{t = (t_1, t_2) \in \Delta_{max} : \exists x \in \mathbb{R}^n \ \text{ with } \ t_1 = \gamma_1(x), \ \ t_2 = \gamma_2(x)\},$$

and denote by $\mathfrak{S}(\Delta)$ the restriction of the algebra $\mathfrak{S}_{max} = \mathfrak{S}(\Delta_{max})$ to the set Δ; i.e.,

$$\mathfrak{S}(\Delta) = \{\sigma|_\Delta : \sigma \in \mathfrak{S}_{max}\}.$$

3.10 Theorem *The algebra $\mathfrak{S}(X_1, X_2)$ is isomorphic to the algebra $\mathfrak{S}(\Delta)$.*

PROOF. Denote for a moment by $\mathfrak{S}_\Delta$ the C^*–algebra with identity generated by $p(t)$, $q_1(t)$ and $q_2(t)$, $t \in \Delta$.

The set $\Delta = \Delta(X_1, X_2)$ is obviously the set of maximal ideals of the C^*–algebra with identity $\mathcal{R}(\mathbb{C}; \gamma_1, \gamma_2)$ generated by the two functions $\gamma_1(x)$ and $\gamma_2(x)$, $x \in \mathbb{R}^n$. By the Gelfand transformation, the algebra $\mathcal{R}(\mathbb{C}, \gamma_1, \gamma_2)$ is isomorphic to $C(\Delta)$, and t_k is the image of $\gamma_k(x)$, $k = 1, 2$, under this transformation.

Further, the algebras $\mathcal{R}(\mathbb{C}; \gamma_1, \gamma_2) \otimes \mathrm{Mat}_3(\mathbb{C})$ and $C(\Delta) \otimes \mathrm{Mat}_3(\mathbb{C})$ are naturally isomorphic, and their subalgebras $\mathfrak{S}(X_1, X_2)$ and $\mathfrak{S}_\Delta$ are isomorphic as well.

Finally, the Stone–Weierstrass theorem shows, as in the proof of Theorem 3.8, that $\mathfrak{S}_\Delta = \mathfrak{S}(\Delta)$.

3.11 Corollary *The algebra $\mathcal{R}(X_1, X_2)$ is isomorphic to the algebra $\mathbb{C}^3 \oplus \mathfrak{S}(\Delta)$. The isomorphism*

$$\nu : \mathcal{R}(X_1, X_2) \longrightarrow \mathbb{C}^3 \oplus \mathfrak{S}(\Delta)$$

is generated by the following mapping of the projections:

$$\begin{aligned} \nu \ &: \ P \longmapsto ((0,0,0), p), \\ \nu \ &: \ Q_1 \longmapsto ((1,0,0), q_1), \\ \nu \ &: \ Q_2 \longmapsto ((0,1,0), q_2), \end{aligned}$$

where

$$p = p(t) = \begin{pmatrix} t_1 & \sqrt{t_1 t_2} & \sqrt{t_1 t_3} \\ \sqrt{t_1 t_2} & t_2 & \sqrt{t_2 t_3} \\ \sqrt{t_1 t_3} & \sqrt{t_2 t_3} & t_3 \end{pmatrix},$$

$$q_1 = q_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q_2 = q_2(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$t = (t_1, t_2) \in \Delta = \Delta(X_1, X_2), \quad t_3 = 1 - t_1 - t_2.$

3.12 Denote by $PC(X_1, X_2)$ the algebra of all piecewise constant functions of the form

$$a(x) = \alpha_1 \chi_1(x) + \alpha_2 \chi_2(x) + \alpha_3 \chi_3(x),$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$; i.e., the algebra of all three–valued functions constant on the sets X_k, $k = \overline{1,3}$.

Now the algebra $\mathcal{R}(X_1, X_2)$ can be introduced as the algebra generated by the Bargmann projection P and piecewise constant functions from $PC(X_1, X_2)$, i.e., by all operators of the form

$$A = a(x)P + b(x)(I - P),$$

with $a(x) = \sum_1^3 \alpha_k \chi_k(x), b(x) = \sum_1^3 \beta_k \chi_k(x) \in PC(X_1, X_2)$.

It is easy to see that the image of A under the isomorphism

$$\nu : \mathcal{R}(X_1, X_2) \longrightarrow \mathbb{C}^3 \oplus \mathfrak{S}(\Delta)$$

is

$$\nu \quad : \quad A \longmapsto \ ((\beta_1, \beta_2, \beta_3),\ a(t)),$$

where

$$a(t) = \begin{pmatrix} \alpha_1 t_1 + \beta_1(1-t_1) & (\alpha_1 - \beta_1)\sqrt{t_1 t_2} & (\alpha_1 - \beta_1)\sqrt{t_1 t_3} \\ (\alpha_2 - \beta_2)\sqrt{t_1 t_2} & \alpha_2 t_2 + \beta_2(1-t_2) & (\alpha_2 - \beta_2)\sqrt{t_2 t_3} \\ (\alpha_3 - \beta_3)\sqrt{t_1 t_3} & (\alpha_3 - \beta_3)\sqrt{t_2 t_3} & \alpha_3 t_3 + \beta_3(1-t_3) \end{pmatrix},$$

with $t = (t_1, t_2) \in \Delta = \Delta(X_1, X_2), \quad t_3 = 1 - t_1 - t_2$.

4 The set $\Delta = \Delta(X_1, X_2)$

4.1 Recall that by definition

$$\Delta = \Delta(X_1, X_2) =$$

$$= \text{clos}\{t = (t_1, t_2) \in \Delta_{max} : \exists x \in \mathbb{R}^n \quad \text{with} \quad t_1 = \gamma_1(x), \quad t_2 = \gamma_2(x)\},$$

where $\gamma_k(x) = \| Q_k(x) l_0 \|^2 = \int_{\sqrt{2}X_k - x} |l_0(y)|^2 dy$, $k = 1, 2$.

The functions $\gamma_k(x)$, $k = 1, 2$, are continuous on $\mathbb{R}^n$ and thus Δ is a connected compact subset of Δ_{max}.

4.2 Let us give some examples of sets X_1, X_2 and the corresponding set $\Delta = \Delta(X_1, X_2)$. For simplicity we will consider the two–dimensional case ($X_k \subset \mathbb{R}^2$) only.

EXAMPLE 1. A complete simplex Δ_{max}.

$$X_k = \{x = re^{i\varphi} \in \mathbb{R}^2 : \quad r \in \mathbb{R}_+,\ \varphi \in [\frac{2(k-1)\pi}{3}, \frac{2k\pi}{3})\}, \quad k = 1, 2.$$

It is easy to see that $\Delta = \Delta_{max}$.

EXAMPLE 2. A polygon.

First recall [10] how to construct for given $0 \le \alpha < \beta \le 1$ the set $X \subset \mathbb{R}$ with the properties

$$\inf_{x \in \mathbb{R}} \int_{\sqrt{2}X - x} |l_0(y)|^2 dy \quad = \quad \alpha,$$
$$\sup_{x \in \mathbb{R}} \int_{\sqrt{2}X - x} |l_0(y)|^2 dy \quad = \quad \beta.$$

It is clear that for each $\gamma \in [0,1]$ there exists $a_\gamma \in [0,1]$ such that for the set

$$Q_\gamma = \frac{1}{\sqrt{2}} \bigcup_{n\in\mathbb{Z}} [-a_\gamma + 2n, a_\gamma + 2n]$$

we have

$$\int_{\sqrt{2}Q_\gamma} |l_0(y)|^2 dy = \gamma.$$

For the set $X = ((\mathbb{R} \setminus Q_{1-\alpha}) \bigcap (-\infty, 0)) \bigcup (Q_\beta \bigcap (0, +\infty))$ and

$$\gamma(x) = \int_{\sqrt{2}X-x} |l_0(y)|^2 dy$$

we have

$$\begin{aligned} \inf_{x\in\mathbb{R}} \gamma(x) &= \liminf_{x\to-\infty} \gamma(x) = \alpha, \\ \sup_{x\in\mathbb{R}} \gamma(x) &= \limsup_{x\to+\infty} \gamma(x) = \beta. \end{aligned}$$

Now, for given $\gamma \in (0, 1-\alpha)$ let $Y \subset \mathbb{R}$ be a set (constructed as above) with

$$\inf_{x\in\mathbb{R}} \int_{\sqrt{2}X-x} |l_0(y)|^2 dy = \liminf_{x\to-\infty} \int_{\sqrt{2}X-x} |l_0(y)|^2 dy = \gamma.$$

Introduce:

$$\begin{aligned} X_1 &= ((\mathbb{R} \setminus X) \times \mathbb{R}_+) \cup (Y \times (\mathbb{R} \setminus \mathbb{R}_+)), \\ X_2 &= (\mathbb{R} \setminus (X \cup Y)) \times (\mathbb{R} \setminus \mathbb{R}_+). \end{aligned}$$

Then, if $\gamma < \alpha$ we have that Δ is the trapezoid bounded by the lines

$$\begin{aligned} x_1 + x_2 = \alpha, &\qquad x_1 + x_2 = \beta, \\ x_2 = 0, &\qquad x_1 = \gamma; \end{aligned}$$

if $\alpha \le \gamma$ we have that Δ is the triangle bounded by the lines

$$x_1 = \gamma, \quad x_2 = 0, \quad x_1 + x_2 = \beta.$$

Of course, one can modify this example to move this trapezoid or triangle in Δ_{max} and to change Δ for a pentagon or a hexagon.

EXAMPLE 3. A closed curve.

Let

$$
\begin{aligned}
X_1 &= (\bigcup_{k=-\infty}^{\infty} [k, k+\frac{1}{3})) \times \mathbb{R}, \\
X_2 &= (\bigcup_{k=-\infty}^{\infty} [k+\frac{1}{3}, k+\frac{2}{3})) \times \mathbb{R}.
\end{aligned}
$$

Then the set Δ is a simple closed curve passing through the following points of Δ_{max}:

$$
(\alpha, \gamma), \quad (\gamma, \alpha), \quad (\delta, \beta), \quad (\beta, \delta),
$$

where

$$
\begin{aligned}
\alpha = \gamma_1(\frac{2}{3}) = \gamma_2(0), \qquad & \beta = \gamma_1(\frac{1}{6}) = \gamma_2(\frac{1}{2}), \\
\gamma = \gamma_2(\frac{2}{3}) = \gamma_1(0), \qquad & \delta = \gamma_1(\frac{1}{6}) = \gamma_2(\frac{1}{2}).
\end{aligned}
$$

EXAMPLE 4. A double spiral.

A small change of the X_k, $k = 1, 2$, in the previous Example with

$$
\begin{aligned}
X_1 &= (\bigcup_{k=-\infty}^{\infty} [k, k+\frac{k(2+\operatorname{sign} k)}{3(2k+1)})) \times \mathbb{R}, \\
X_2 &= (\bigcup_{k=-\infty}^{\infty} [k+\frac{k(2+\operatorname{sign} k)}{3(2k+1)}, k+\frac{2}{3})) \times \mathbb{R}.
\end{aligned}
$$

changes Δ to a double spiral.

5 Algebra of Toeplitz operators

5.1 Denote by $\mathcal{T}(PC(X_1, X_2))$ the C^*–algebra generated by all the Toeplitz operators

$$
T_a : \quad \varphi \;\to\; Pa\varphi
$$

with symbols $a(x) = \sum_1^3 \alpha_k \chi_k(x) \in PC(X_1, X_2)$, acting on the Fock space $F^2(\mathbb{C}^n)$

Corollary 3.11 and Subsection 3.12, together with standard considerations, lead directly to the following theorem.

5.2 Theorem *The algebra $\mathcal{T}(PC(X_1, X_2))$ is isomorphic and isometric to the algebra $C(\Delta(X_1, X_2))$. The isomorphism*

$$\tau : \ \mathcal{T}(PC(X_1, X_2)) \ \longrightarrow \ C(\Delta(X_1, X_2))$$

is generated by the following mapping:

$$\tau : \ T_a \ \longmapsto \ \alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3,$$

where $t = (t_1, t_2) \in \Delta = \Delta(X_1, X_2), \quad t_3 = 1 - t_1 - t_2$.

5.3 Corollary *The spectrum of the Toeplitz operator T_a is calculated by the formula*

$$\mathrm{sp}\, T_a = \{\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 : \quad (t_1, t_2) \in \Delta(X_1, X_2), \quad t_3 = 1 - t_1 - t_2\}.$$

5.4 Remark The examples for the set $\Delta(X_1, X_2)$ given in Section 4 show how complicated $\mathrm{sp}\, T_a$ can be for general X_1, X_2 and arbitrary $a(x) = \sum_1^3 \alpha_k \chi_k(x) \in PC(X_1, X_2)$. But if we restrict ourselves to two–valued symbols ($\alpha_2 = \alpha_3$), then matters simplify greatly and we come to the known result:

The spectrum of T_a is a line segment

$$\mathrm{sp}\, T_a = \{\alpha_1 t_1 + \alpha_2 (1 - t_1) : \quad t_1 \in [\alpha, \beta]\, \},$$

where $\alpha = \inf_{x \in \mathbb{R}^n} \gamma_1(x), \quad \beta = \sup_{x \in \mathbb{R}^n} \gamma_1(x)$.

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