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ON BERGMAN–TOEPLITZ OPERATORS WITH COMMUTATIVE SYMBOL ALGEBRAS*

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Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, $\mathcal{A}^2(\mathbb{D})$ be the Bergman space, consisting of all analytic functions from $L_2(\mathbb{D})$, and $B_{\mathbb{D}}$ be the Bergman projection of $L_2(\mathbb{D})$ onto $\mathcal{A}^2(\mathbb{D})$. We construct C^*-algebras $\mathcal{A} \subset L_\infty(\mathbb{D})$, for functions of which the commutator of Toeplitz operators $[T_a, T_b] = T_aT_b - T_bT_a$ is compact, and, at the same time, the semi-commutator $[T_a, T_b) = T_aT_b - T_{ab}$ is not compact.

It is proved, that for each finite set $\Lambda = \langle n_0, n_1, ..., n_m \rangle$, where $1 = n_0 < n_1 < ... < n_m \leq \infty$, and $n_k \in \mathbb{N} \cup \{\infty\}$, there are algebras $\mathcal{A}_\Lambda$ of the above type, such that the symbol algebras $\mathrm{Sym}\,\mathcal{T}(\mathcal{A}_\Lambda)$ of Toeplitz operator algebras $\mathcal{T}(\mathcal{A}_\Lambda)$ are *commutative*, while the symbol algebras $\mathrm{Sym}\,\mathcal{R}(\mathcal{A}_\Lambda, B_{\mathbb{D}})$ of the algebras $\mathcal{R}(\mathcal{A}_\Lambda, B_{\mathbb{D}})$, generated by multiplication operators aI, $a \in \mathcal{A}_\Lambda$ and $B_{\mathbb{D}}$, have *irreducible representations exactly of dimensions* n_0, n_1, ..., n_m.

1 Introduction

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, introduce the space $L_2(\mathbb{D})$, with usual Lebesgue volume measure $d\mu(z) = dxdy$, $z = x + iy$, and its subspace $\mathcal{A}^2(\mathbb{D})$, the Bergman space, consisting of all functions analytic in $\mathbb{D}$. It is well known that the orthogonal projection $B_{\mathbb{D}}$ of $L_2(\mathbb{D})$ onto $\mathcal{A}^2(\mathbb{D})$, the so-called Bergman projection, is given by

$$(B_{\mathbb{D}}\varphi)(z) = \int_{\mathbb{D}} K_{\mathbb{D}}(z, \overline{\zeta})\varphi(\zeta)\, d\mu(\zeta),$$

where the Bergman kern-function $K_{\mathbb{D}}(z, \overline{\zeta})$ of the unit disk is given by

$$K_{\mathbb{D}}(z, \overline{\zeta}) = \frac{1}{\pi} \cdot \frac{1}{(1 - z\overline{\zeta})^2}.$$

Let $\mathcal{A}(\mathbb{D})$ be a C^*-subalgebra of $L_\infty(\mathbb{D})$. Given $a \in \mathcal{A}(\mathbb{D})$, denote, as usual, by T_a the following Toeplitz operator

$$T_a : \varphi \in \mathcal{A}^2(\mathbb{D}) \;\longmapsto\; B_{\mathbb{D}}a\varphi \in \mathcal{A}^2(\mathbb{D}),$$

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acting on the space $\mathcal{A}^2(\mathbb{D})$, and denote by $\mathcal{T}(\mathcal{A}(\mathbb{D}))$ the Toeplitz operator algebra, i.e., the algebra generated by all the operators T_a, with $a \in \mathcal{A}(\mathbb{D})$. Furthermore, introduce the algebra $\mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})$, generated by all the operators

$$A = aI + bB_{\mathbb{D}},$$

where a, $b \in \mathcal{A}(\mathbb{D})$.

Consider the following statements:

1) For each $a \in \mathcal{A}(\mathbb{D})$ the commutator $[B_{\mathbb{D}}, aI] = B_{\mathbb{D}}aI - aB_{\mathbb{D}}$ is compact.

2) For each pair a, $b \in \mathcal{A}(\mathbb{D})$ the semi-commutator $[T_a, T_b) = T_aT_b - T_{ab}$ is compact.

3) For each pair a, $b \in \mathcal{A}(\mathbb{D})$ the commutator $[T_a, T_b] = T_aT_b - T_bT_a$ is compact.

We have obviously 1) $\Rightarrow$ 2) $\Rightarrow$ 3). A bit of work shows that, in fact, 1) $\Leftrightarrow$ 2), but 3) is certainly stronger then both 1) and 2). The more simple and known example here is the algebra $\mathcal{A}(\mathbb{D}) = PC(\overline{\mathbb{D}}, L)$ of piece-wise continuous functions, i.e., the functions continuous in $\overline{\mathbb{D}} \setminus L$, and having one-sided limit values at each point of simple (smooth) line L, which connects two points of the boundary $\partial\,\mathbb{D}$ of the unit dick (see, for example, [10]).

The very important question of description of the maximal algebra satisfying the statement 2) was solved in [13] (see also [14]). One of the main features of algebras having property 2) is that the corresponding Toeplitz operator algebras $\mathcal{T}(\mathcal{A}(\mathbb{D}))$ admit the commutative symbolic calculus, i.e., the symbol (Calkin) algebra $\operatorname{Sym}\mathcal{T}(\mathcal{A}(\mathbb{D})) = \mathcal{T}(\mathcal{A}(\mathbb{D}))/\mathcal{K}$, where $\mathcal{K}$ is the ideal of compact operator, is commutative.

Note, that under the condition 2) the symbol algebra $\operatorname{Sym}\mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}}) = \mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})/\mathcal{K}$ of the algebra $\mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})$ is commutative as well.

At the same time, under the (only) condition 3) the symbol algebra $\operatorname{Sym}\mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}}) = \mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})/\mathcal{K}$ of the algebra $\mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})$ is *non-commutative*, while the symbol algebra $\operatorname{Sym}\mathcal{T}(\mathcal{A}(\mathbb{D})) = \mathcal{T}(\mathcal{A}(\mathbb{D}))/\mathcal{K}$ still remains *commutative*.

The natural questions appear:

- describe functional algebras $\mathcal{A}(\mathbb{D})$ having the property 3),
- how complicated the symbol algebras of corresponding algebras $\mathcal{R}(\mathcal{A}(\mathbb{D}), B_{\mathbb{D}})$ can be, while Toeplitz operator algebras $\mathcal{T}(\mathcal{A}(\mathbb{D}))$ still admit the commutative symbolic calculus.

Answering to these questions, we construct a number of algebras $\mathcal{A}$ with the (only) property 3), as well as algebras with the more strong property: $T_aT_b = T_bT_a$, for each a, b from corresponding $\mathcal{A}$.

Moreover, we prove that for each finite set $\Lambda = \langle n_0, n_1, ..., n_m\rangle$, where $1 = n_0 < n_1 < ... < n_m \leq \infty$, and $n_k \in \mathbb{N} \cup \{\infty\}$, there are algebras $\mathcal{A}_\Lambda$ of the above type, such that the symbol algebras $\operatorname{Sym}\mathcal{T}(\mathcal{A}_\Lambda)$ of the algebras $\mathcal{T}(\mathcal{A}_\Lambda)$ are *commutative*, while the symbol algebras $\operatorname{Sym}\mathcal{R}(\mathcal{A}_\Lambda, B_{\mathbb{D}})$ of the algebras $\mathcal{R}(\mathcal{A}_\Lambda, B_{\mathbb{D}})$ have *irreducible representations exactly of dimensions n_0, n_1, ..., n_m*.

It is worth mentioning that for Toeplitz operators on Hardy spaces functional algebras with the property 2) were known a long time ago [7], [8], and that the Gohberg – Krupnik results of the end of 60^{th} [4], [5] gave the examples of algebras with the only property 3). Nevertheless the above questions still remain open for the case of Toeplitz operators on Hardy spaces.

2 Preliminaries

2.1 The Moebius transformation

$$w = \frac{z+i}{1+iz} \tag{2.1}$$

maps the unit disk $\mathbb{D}$ onto the upper half-plane Π, and generates the unitary operator $U_0 : L_2(\mathbb{D}) \to L_2(\Pi)$ acting by the rule

$$(U_0\varphi)(w) = \frac{2}{(1-iw)^2}\,\varphi\left(\frac{w-i}{1-iw}\right),$$

then the inverse (and adjoint) operator $U_0^{-1} : L_2(\Pi) \to L_2(\mathbb{D})$ is given by

$$(U_0^{-1}\varphi)(z) = \frac{2}{(1+iz)^2}\,\varphi\left(\frac{z+i}{1+iz}\right).$$

The operator U_0 maps the Bergman space $\mathcal{A}^2(\mathbb{D})$ onto the Bergman space $\mathcal{A}^2(\Pi)$, and the orthogonal (Bergman) projection

$$B_\Pi = U_0\, B_{\mathbb{D}}\, U_0^{-1} : L_2(\Pi) \to \mathcal{A}^2(\Pi)$$

has the form

$$(B_\Pi\varphi)(w) = \int_\Pi K_\Pi(w,\overline{\omega})\,\varphi(\omega)\,d\mu(\omega) = -\frac{1}{\pi}\int_\Pi \frac{\varphi(\omega)\,d\mu(\omega)}{(w-\overline{\omega})^2}.$$

Introduce the unitary operators

$$U_1 = F\otimes I : \ L_2(\Pi) = L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+),$$

where $F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is the Fourier transform

$$(Ff)(u) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-iu\xi} f(\xi)\,d\xi,$$

and

$$U_2 : L_2(\Pi) = L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+)$$

by the rule

$$U_2 : \varphi(u,v) \longmapsto \frac{1}{\sqrt{2|x|}}\,\varphi(x, \frac{y}{2|x|}).$$

Then the inverse operator $U_2^{-1} = U_2^* : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ is given by

$$U_2^{-1} : \varphi(x,y) \longmapsto \sqrt{2|u|}\,\varphi(u, 2|u|\cdot v).$$

Let $\ell_0(y) = e^{-y/2}$, we have $\ell_0(y) \in L_2(\mathbb{R}_+)$ and $\|\ell_0(y)\| = 1$. Denote by L_0 the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_0(y)$, then the one-dimensional projection P_0 of $L_2(\mathbb{R}_+)$ onto L_0 has the form

$$(P_0\psi)(y) = \langle \psi, \ell_0\rangle \cdot \ell_0 = e^{-y/2} \int_{\mathbb{R}_+} \psi(\nu)\, e^{-\nu/2}\, d\nu. \tag{2.2}$$

The following results were obtained in [12].

Theorem 2.1 *The unitary operator $U = U_2U_1$ gives an isometric isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ under which*

1. *the Bergman space $\mathcal{A}^2(\Pi)$ is mapped onto $L_2(\mathbb{R}_+) \otimes L_0$*

$$U : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+) \otimes L_0,$$

where L_0 is the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by $\ell_0 = e^{-y/2}$,

2. *the Bergman projection B_Π is unitary equivalent to the following one*

$$U\, B_\Pi\, U^{-1} = \chi_+ I \otimes P_0,$$

where P_0 is the one-dimensional projection (2.2) of $L_2(\mathbb{R}_+)$ onto L_0.

Introduce the isometric imbedding

$$R_0 : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

by the rule

$$(R_0 f)(x,y) = \chi_+(x)\, f(x)\, \ell_0(y).$$

The adjoint operator $R_0^* : L_2(\Pi) \to L_2(\mathbb{R}_+)$ is given by

$$(R_0^*\varphi)(x) = \chi_+(x) \int_{\mathbb{R}_+} \varphi(x,\eta)\, \ell_0(\eta)\, d\eta,$$

and

$$\begin{aligned} R_0^* R_0 = I \quad &: \quad L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+) \\ R_0 R_0^* = \chi_+ I \otimes P_0 \quad &: \quad L_2(\Pi) \longrightarrow L_2(\mathbb{R}_+) \otimes L_0. \end{aligned}$$

Now the operator $R = R_0^* U$ maps the space $L_2(\Pi)$ onto $L_2(\mathbb{R}_+)$, and the restriction

$$R|_{\mathcal{A}^2(\Pi)} : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+)$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : L_2(\mathbb{R}_+) \longrightarrow \mathcal{A}^2(\Pi) \subset L_2(\Pi)$$

is an isometric isomorphism of $L_2(\mathbb{R}_+)$ onto the subspace $\mathcal{A}^2(\Pi)$ of the space $L_2(\Pi)$.

Remark 2.2 We have

$$
\begin{aligned}
R\,R^* = I \quad &: \quad L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+) \\
R^*R = B_\Pi \quad &: \quad L_2(\Pi) \longrightarrow \mathcal{A}^2(\Pi)
\end{aligned}
$$

Theorem 2.3 *The isometric isomorphism*

$$R^* = U^* R_0 : L_2(\mathbb{R}_+) \longrightarrow \mathcal{A}^2(\Pi)$$

is given by

$$(R^* f)(z) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} \sqrt{\xi}\, f(\xi)\, e^{iz\cdot\xi}\, d\xi.$$

Corollary 2.4 *The inverse isomorphism*

$$R : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+)$$

is given by

$$(R\varphi)(x) = \sqrt{x}\,\frac{1}{\sqrt{\pi}} \int_{\Pi} \varphi(w)\, e^{-i\overline{w}\cdot x}\, d\mu(w).$$

2.2 Recall now the necessary results (for details and proofs see [11]) about C^*-algebras generated by a finite set of orthogonal projections.

Let H be a Hilbert space, we will call the orthogonal projections P, Q_1, ..., Q_n, acting on H, *all-but-one*, if

$$
\begin{aligned}
I &= Q_1 + Q_2 + \ldots + Q_n, \\
Q_j \cdot Q_k &= \delta_{j,k} Q_j, \quad j,k = \overline{1,n}.
\end{aligned}
$$

The set of all-but-one projections P, Q_1, ..., Q_n are in *generic position*, if for all $k = \overline{1,n}$

$$\operatorname{Im} P \ \cap \ \ker Q_k = \{0\}, \tag{2.3}$$

$$\operatorname{Im} Q_k \ \cap \ \ker P = \{0\}. \tag{2.4}$$

Given the set of all-but-one generic position orthogonal projections P, Q_1, ..., Q_n, the operators $C_k = PQ_kP : \operatorname{Im} P \to \operatorname{Im} P$ are positive and injective with $C_1 + \ldots + C_n = I$, and there exists the unitary operator $V : (\operatorname{Im} P)^n \to H$, such that the projections V^*PV and V^*Q_kV, $k = \overline{1,n}$, acting on $(\operatorname{Im} P)^n$ are, in matrix form, given as

$$
\begin{aligned}
V^*PV &= (C_j^{1/2} \cdot C_k^{1/2})_{j,k=1}^n, \\
V^*Q_kV &= \operatorname{diag}\,(0, \ldots, 0, \underset{k-place}{I}, 0, \ldots, 0), \quad k = \overline{1,n}.
\end{aligned}
$$

Assume now that the operators C_k, $k = \overline{1,n}$, are pairwise commuting, and denote by $\Delta = \Delta(C_1, ..., C_k)$ their joint spectrum. The properties of C_k imply that Δ is a (compact) subset of the standard $(n-1)$–dimensional simplex

$$\Delta_{n-1} = \{t = (t_1, \ldots, t_n) \in \mathbb{R}^n \,:\, t_k \geq 0, \quad k = \overline{1,n}, \quad \sum_{k=1}^n t_k = 1\}.$$

Denote by $\mathfrak{S}(\Delta_{n-1})$ the C^*-algebra of all $n \times n$ matrix–valued functions continuous on Δ_{n-1} and having the following properties:

(i) they are diagonal at all vertices $(1,0,\ldots,0)$, $(0,1,\ldots,0)$, ..., $(0,\ldots,0,1)$ of Δ_{n-1};

(ii) fix a k–tuple $[k]=(i_1,\ldots,i_k)$ of indices $1\le i_1<i_2<\ldots<i_k\le n$, and let $[n]\backslash[k]$ be the complementary $(n-k)$–tuple of indices; then for each k–tuple, $1\le k\le n-1$

$$\sigma_{ij}(t)=\sigma_{ji}(t)\equiv 0 \quad \text{for} \quad i\in[k], \quad j\ne i$$

and $\sigma_{ii}(t)=\sigma_i=\text{const}$ on $(n-k-1)$–face $t_{i_1}=t_{i_2}=\ldots=t_{i_k}=0$ of Δ_{n-1}.

Theorem 2.5 *The C^*-algebra $\mathcal{C}^*(P,Q_1,...,Q_n)$, generated by all-but-one generic position orthogonal projections P, Q_1, ..., Q_n with pairwise commuting C_k, is isomorphic and isometric to the restriction of $\mathfrak{S}(\Delta_{n-1})$ onto the set Δ, i.e., to the algebra*

$$\mathfrak{S}(\Delta)=\{\sigma|_\Delta : \sigma\in\mathfrak{S}(\Delta_{n-1})\}.$$

The isomorphism

$$\nu:\mathcal{C}^*(P,Q_1,...,Q_n)\longrightarrow\mathfrak{S}(\Delta)$$

is generated by the following mapping of the projections

$$\nu:P \longmapsto p(t)=(\sqrt{t_j}\sqrt{t_k})_{j,k=1}^n, \tag{2.5}$$
$$\nu:Q_k \longmapsto q_k(t)=\operatorname{diag}(0,\ldots,0,\underset{k-place}{1},0,\ldots,0), \tag{2.6}$$

where $t=(t_1,\ldots,t_n)\in\Delta$.

In the sequel we will need the following variant of the above situation. Consider the all-but-one projections P, Q_1, ..., Q_n, which satisfy the only property (2.3) (the property (2.4) does not hold). Thus for some indexes k we have

$$\operatorname{Im}Q_k\cap\ker P\ne\{0\}.$$

Denote by m $(0<m\le n)$ the number of these indexes, and reorder the projections Q_k in such a form that

$$\begin{aligned}\operatorname{Im}Q_k &\cap \ker P\ne\{0\}, \qquad k=\overline{1,m},\\ \operatorname{Im}Q_k &\cap \ker P=\{0\}, \qquad k=\overline{m+1,n}.\end{aligned}$$

We keep assuming that the operators $C_k=PQ_kP:\operatorname{Im}P\to\operatorname{Im}P$, $k=\overline{1,n}$, are pairwise commuting, and keep denoting by $\Delta=\Delta(C_1,...,C_n)$ their joint spectrum.

Theorem 2.6 *Under the above conditions the C^*-algebra $\mathcal{C}^*(P,Q_1,...,Q_n)$, generated by the orthogonal projections P, Q_1, ..., Q_n is isomorphic and isometric to a subalgebra of the algebra $\mathfrak{S}(\Delta)\oplus\mathbb{C}^m$. The isomorphic imbedding*

$$\nu:\mathcal{C}^*(P,Q_1,...,Q_n)\longrightarrow\mathfrak{S}(\Delta)\oplus\mathbb{C}^m$$

is generated by the following mapping of the projections

$$\begin{aligned}\nu:P &\longmapsto (p(t),(0,0,\ldots,0)),\\ \nu:Q_k &\longmapsto (q_k(t),(0,\ldots,0,\underset{k-place}{1},0,\ldots,0)), \quad k=\overline{1,m},\\ \nu:Q_k &\longmapsto (q_k(t),(0,0,\ldots,0)), \quad k=\overline{m+1,n},\end{aligned}$$

where $p(t)$ and $q_k(t)$ are defined in (2.5) and (2.6), $t=(t_1,\ldots,t_n)\in\Delta$.

Proof. Split the space H onto the following direct sum

$$H = \bigoplus_{k=1}^{m} (\operatorname{Im} Q_k \cap \ker P) \oplus H', \tag{2.7}$$

and introduce the projections

$$P' = P|_{H'}, \qquad Q'_k = Q_k|_{H'}, \qquad k = \overline{1,n}.$$

Corresponding to the splitting (2.7) the initial projections have the following form

$$\begin{aligned}
P &= (0 \oplus 0 \oplus \ldots \oplus 0) \oplus P', \\
Q_k &= (0 \oplus \ldots 0 \oplus \underset{k-place}{1} \oplus 0 \oplus \ldots \oplus 0) \oplus Q'_k, \quad k = \overline{1,m}, \\
Q_k &= (0 \oplus 0 \oplus \ldots \oplus 0) \oplus Q'_k, \quad k = \overline{m+1,n}.
\end{aligned}$$

The all-but-one projections P', Q'_1, ..., Q'_n are in generic position, and the C^*-algebra generated by them is described in Theorem 2.5. Finally, note that $\operatorname{Im} P = \operatorname{Im} P'$, and $P'Q'_kP' = PQ_kP$ on this space. □

3 Infinite dimensional representations

Consider the upper half-plaine Π, and the Bergman projection B_Π, acting on $L_2(\Pi)$. Introduce the C^*-algebra $\mathcal{A}_\infty = \mathbb{C} \otimes L_\infty(\mathbb{R}_+)$ $(\subset L_\infty(\mathbb{R} \times \mathbb{R}_+))$, and consider two operator algebras, the Toeplitz operator algebra $\mathcal{T}(\mathcal{A}_\infty)$, generated by all the operators of the form

$$T_a : \varphi \in \mathcal{A}^2(\Pi) \longmapsto B_\Pi a\varphi \in \mathcal{A}^2(\Pi),$$

where $a = a(v) \in \mathcal{A}_\infty$, and the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$, generated by all the operators of the form

$$A = aI + bB_\Pi,$$

where $a = a(v)$, $b = b(v) \in \mathcal{A}_\infty$, acting on $L_2(\Pi)$.

Theorem 3.1 *The algebra $\mathcal{T}(\mathcal{A}_\infty)$ is commutative. The isomorphic imbedding*

$$\tau_\infty : \mathcal{T}(\mathcal{A}_\infty) \longrightarrow C_b(\mathbb{R}_+)$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}_\infty)$

$$\tau_\infty : T_a \longmapsto \gamma_a(x) = \int_{\mathbb{R}_+} a(\frac{\eta}{2x})\, \ell_0^2(\eta)\, d\eta, \quad x \in \mathbb{R}_+,$$

where $a = a(v) \in \mathcal{A}_\infty$, and $\ell_0(\eta) = e^{-\eta/2}$.

PROOF. The algebra $\mathcal{T}(\mathcal{A}_\infty)$ is obviously unitary equivalent to the algebra $R\,\mathcal{T}(\mathcal{A}_\infty)\,R^*$, which is generated by the following operators (acting on $L_2(\mathbb{R}_+)$) :

$$\begin{aligned}
R\,T_a\,R^* &= R\,B_\Pi a B_\Pi R^* = R(R^*R)a(R^*R)R^* \\
&= (RR^*)RaR^*(RR^*) = RaR^* \\
&= R_0^* U_2(F\otimes I)a(v)(F^{-1}\otimes I)U_2^{-1}R_0 \\
&= R_0^* U_2 a(v) U_2^{-1} R_0 \\
&= R_0^* a(\frac{y}{2|x|})R_0.
\end{aligned}$$

Calculate now

$$(R_0^* a(\frac{y}{2|x|})R_0 f)(x) = \int_{\mathbb{R}_+} a(\frac{\eta}{2|x|})\, f(x)\, \ell_0^2(\eta)\, d\eta = \gamma_a(x)\cdot f(x),$$

where

$$\gamma_a(x) = \int_{\mathbb{R}_+} a(\frac{\eta}{2x})\, \ell_0^2(\eta)\, d\eta, \quad x\in\mathbb{R}_+. \tag{3.1}$$

□

Return now to the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$. The algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$ is naturally isomorphic to the algebra

$$\mathcal{R}_0 = U\,\mathcal{R}(\mathcal{A}_\infty, B_\Pi)\,U^{-1} = U_2(F\otimes I)\mathcal{R}(\mathcal{A}_\infty, B_\Pi)(F^{-1}\otimes I)U_2^{-1}.$$

And under this isomorphism the generators of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$ are mapped to the following generators of the algebra $\mathcal{R}_0$

$$\begin{aligned}
U\,B_\Pi\,U^{-1} &= \chi_+(x)I\otimes P_0, \\
U\,a(v)\,U^{-1} &= a(\frac{y}{2|x|})I,
\end{aligned}$$

acting on the space $L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+)$. The algebra $\mathcal{R}_0$ splits onto the direct integral of the algebras $\mathcal{R}_0(x)$, $x\in\mathbb{R}\setminus\{0\}$, where each algebra $\mathcal{R}_0(x)$ is generated by the following operators, acting on $L_2(\mathbb{R}_+)$:

for $x\in\mathbb{R}_+$

$$P_0 \quad \text{and} \quad a(\frac{y}{2x})I,$$

for $x\in\mathbb{R}_-$

$$0 \quad \text{and} \quad a(-\frac{y}{2x})I.$$

The proof of the following lemma is trivial.

Lemma 3.2 *For $x\in\mathbb{R}_-$ all the algebras $\mathcal{R}_0(x)$ are isomorphic to $L_\infty(\mathbb{R}_+)$. The homomorphism*

$$\nu_x : \mathcal{R}(\mathcal{A}_\infty, B_\Pi) \cong \mathcal{R}_0 \longrightarrow \mathcal{R}_0(x) \cong L_\infty(\mathbb{R}_+) \tag{3.2}$$

is generated by the following mapping of generators of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$

$$\begin{aligned} \nu_x &: aI \longmapsto a(y), \\ \nu_x &: B_\Pi \longmapsto 0. \end{aligned}$$

Lemma 3.3 *For each* $x \in \mathbb{R}_+$ *the algebra* $\mathcal{R}_0(x)$ *is irreducible, contains the ideal* $\mathcal{K}$ *of all compact in* $L_2(\mathbb{R}_+)$ *operators, and its Calkin algebra* $\widehat{\mathcal{R}}_0(x) = \mathcal{R}_0(x)/\mathcal{K}$ *is isomorphic to* $L_\infty(\mathbb{R}_+)$.

PROOF. To prove the first statement show that the algebra $\mathcal{R}_0(x)$ does not have any non-trivial invariant subspace. First, observe that all the invariant subspaces of the subalgebra

$$\{a(\frac{y}{2x})I : a \in L_\infty(\mathbb{R}_+)\} \cong L_\infty(\mathbb{R}_+)$$

of the algebra $\mathcal{R}_0(x)$ have the form

$$X_M = \{\psi_M(y) = \chi_M(y)\psi(y) : \psi(y) \in L_2(\mathbb{R}_+)\},$$

where M is a measurable subset of $\mathbb{R}_+$ having positive measure. Thus the algebra $\mathcal{R}_0(x)$ has an invariant subspace if and only if there exists a measurable set M with positive measure, such that for each $\psi_M(y) \in X_M$ we have $P_0\psi_M(y) \in X_M$. Further

$$P_0\psi_M = \langle \chi_M\psi, \ell_0\rangle \cdot \ell_0,$$

the function ℓ_0 is always nonzero on $\mathbb{R}_+$, thus $\langle \chi_M\psi, \ell_0\rangle$ has to be zero on M, for all $\psi \in L_2(\mathbb{R}_+)$, but for $\psi = \ell_0$ we have

$$\langle \chi_M\ell_0, \ell_0\rangle = \langle \chi_M\ell_0, \chi_M\ell_0\rangle = ||\chi_M\ell_0||^2.$$

Thus χ_M has to be zero e.a. on $\mathbb{R}_+$, or M must have measure zero. So the algebra $\mathcal{R}_0(x)$ does not have any non-trivial invariant subspace, and thus is irreducible. Now, the algebra $\mathcal{R}_0(x)$ is obviously GCR-algebra, contains the one-dimensional operator P_0, and thus [2] contains the whole ideal $\mathcal{K}$ of compact operators.

The last statement of the lemma is obvious. □

Note, that all the algebras $\mathcal{R}_0(x)$, $x \in \mathbb{R}_+$, are the same, and equal to the algebra $\mathcal{R}(L_\infty(\mathbb{R}_+), P_0)$, which is generated by P_0, and all the multiplication operators aI, $a \in L_\infty(\mathbb{R}_+)$, acting on $L_2(\mathbb{R}_+)$.

Lemma 3.4 *For each* $x \in \mathbb{R}_+$ *the homomorphism*

$$\nu_x : \mathcal{R}(\mathcal{A}_\infty, B_\Pi) \cong \mathcal{R}_0 \longrightarrow \mathcal{R}_0(x) \cong \mathcal{R}(L_\infty(\mathbb{R}_+), P_0)$$

gives the (infinite dimensional) irreducible representation of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$, *and for different values of* $x \in \mathbb{R}_+$ *the representations* ν_x *are not unitary equivalent.*

PROOF. The first statement follows directly from Lemma 3.3. Let now $0 < x_1 < x_2 < \infty$, and let $a(y)$ be a function from L_∞, such that $\gamma_a(x_1) \neq \gamma_a(x_2)$, where $\gamma_a(x)$ is given by (3.1). Then for the operator $A = \gamma_a(x_1)P_0 - P_0 a(y) P_0 \in \mathcal{R}(\mathcal{A}_\infty, B_\Pi)$ we have

$$\begin{aligned} v_{x_1} \ &: \ A \longmapsto (\gamma_a(x_1) - \gamma_a(x_1))P_0 = 0, \\ v_{x_2} \ &: \ A \longmapsto (\gamma_a(x_1) - \gamma_a(x_2))P_0 \neq 0. \end{aligned}$$

Thus the representations v_{x_1} and v_{x_2} are not unitary equivalent. □

Theorem 3.5 *The algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$ does not contain any non-zero compact operator. All its infinite dimensional representations are parameterized by the points $x \in \mathbb{R}_+$ and are of the form*

$$\nu_x : \mathcal{R}(\mathcal{A}_\infty, B_\Pi) \cong \mathcal{R}_0 \longrightarrow \mathcal{R}(L_\infty(\mathbb{R}_+), P_0),$$

where

$$\begin{aligned} v_x \ &: \ a(y)I \in \mathcal{R}(\mathcal{A}_\infty, B_\Pi) \longmapsto a(\frac{y}{2x})I \in \mathcal{R}(L_\infty(\mathbb{R}_+), P_0), \\ v_x \ &: \ B_\Pi \in \mathcal{R}(\mathcal{A}_\infty, B_\Pi) \longmapsto P_0 \in \mathcal{R}(L_\infty(\mathbb{R}_+), P_0). \end{aligned}$$

All finite dimensional irreducible representations of the algebra $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$ are one-dimensional, and are the composition of the homomorphism (3.2) of Lemma 3.2 with one-dimensional representations of $L_\infty(\mathbb{R}_+)$.

PROOF. Follows directly from Lemmas 3.2, 3.3 and 3.4. □

Return now from the upper half-plane to the unit disk $\mathbb{D}$. Denote by $\widetilde{\mathcal{A}}_\infty$ the image of the algebra $\mathcal{A}_\infty$ on the disk, i.e.,

$$\widetilde{\mathcal{A}}_\infty = U_0^{-1} \mathcal{A}_\infty U_0 = \{a(\omega(z)) : a \in \mathcal{A}_\infty\},$$

where $\omega(z)$ is the Moebius transformation (2.1).

Each horizontal line $\operatorname{Im} \omega = \eta_0 = \text{const}$ goes under the mapping $z = z(\omega)$ (inverse to the mapping (2.1)) to the circle $C(\eta_0)$ having center at the point $i\frac{\eta_0}{1+\eta_0}$ and radius $\frac{1}{1+\eta_0}$, and which is tangent to $\partial\mathbb{D}$ at the point i. Thus the functions from $\widetilde{\mathcal{A}}_\infty$ are constant on each circle $C(\eta_0)$, $\eta_0 \in \mathbb{R}_+$.

The algebras $\mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_\mathbb{D})$ and $\mathcal{R}(\mathcal{A}_\infty, B_\Pi)$, as well as the algebras $\mathcal{T}(\widetilde{\mathcal{A}}_\infty)$ and $\mathcal{T}(\mathcal{A}_\infty)$, are unitary equivalent. Thus Theorems 3.1 and 3.5 lead up directly to the following result.

Theorem 3.6 *The algebra $\mathcal{T}(\widetilde{\mathcal{A}}_\infty)$ is commutative. The isomorphic imbedding*

$$\tilde{\tau}_\infty : \mathcal{T}(\widetilde{\mathcal{A}}_\infty) \longrightarrow C_b(\mathbb{R}_+)$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\widetilde{\mathcal{A}}_\infty)$

$$\tilde{\tau}_\infty : T_{a(\omega(z))} \longmapsto \gamma_a(x) = \int_{\mathbb{R}_+} a(\frac{\eta}{2x})\, \ell_0^2(\eta)\, d\eta, \quad x \in \mathbb{R}_+,$$

where $a = a(\omega(z)) \in \widetilde{\mathcal{A}}_\infty$, *and* $\ell_0(\eta) = e^{-\eta/2}$.

The algebra $\mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_{\mathbb{D}})$ *does not contain any non-zero compact operator. All its infinite dimensional representations are parameterized by the points* $x \in \mathbb{R}_+$ *and are of the form*

$$\tilde{\nu}_x : \mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_{\mathbb{D}}) \cong \mathcal{R}_0 \longrightarrow \mathcal{R}(L_\infty(\mathbb{R}_+), P_0),$$

where

$$\begin{aligned} \tilde{\nu}_x \quad &: \quad a(\omega(z))I \in \mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_{\mathbb{D}}) \longmapsto a(\frac{y}{2x})I \in \mathcal{R}(L_\infty(\mathbb{R}_+), P_0), \\ \tilde{\nu}_x \quad &: \quad B_{\mathbb{D}} \in \mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_{\mathbb{D}}) \longmapsto P_0 \in \mathcal{R}(L_\infty(\mathbb{R}_+), P_0). \end{aligned}$$

All finite dimensional irreducible representations of the algebra $\mathcal{R}(\widetilde{\mathcal{A}}_\infty, B_{\mathbb{D}})$ *are one-dimensional, and are the composition of the homomorphism (3.2) of Lemma 3.2 with one-dimensional representations of* $L_\infty(\mathbb{R}_+)$.

4 Spectra and compactness

We give here applications of results of the previous section to spectral and compactness properties of Toeplitz operators.

4.1 The next lemma describes the spectrum of Toeplitz operators with $\mathcal{A}_\infty$ symbols, extending the results [6], [13], [14] about the connectedness of spectrum to this class of symbols.

Lemma 4.1 *For each* $a \in \mathcal{A}_\infty$ *the spectrum of the Toeplitz operator* $T_a \in \mathcal{T}(\mathcal{A}_\infty)$ *is connected and coincides with the closure of the image of the function* $\gamma_a(x)$.

For a real valued function $a \in \mathcal{A}_\infty$ *we have*

$$\mathrm{sp}\, T_a = [\inf_{x\in\mathbb{R}_+} \gamma_a(x), \sup_{x\in\mathbb{R}_+} \gamma_a(x)].$$

PROOF. Direct corollary of Theorem 3.1. □

Let M be a measurable set in $\mathbb{R}_+$, and $M' = \mathbb{R} + iM$ be the corresponding set in Π. Introduce the characteristic functions χ_M and $\chi_{M'}$ of the sets M and M' respectively.

Consider now one special form of a set $M \subset \mathbb{R}_+$. For $\sigma \in (0,1]$ and $s \in (o,\sigma)$ introduce the sets $\Delta_0 = [1, 1+s]$ $(\subset [1, 1+\sigma))$, $\Delta_n = (1+\sigma)^n \cdot \Delta_0$, where $n \in \mathbb{Z}$, and

$$M(s) = \frac{1}{2} \bigcup_{n\in\mathbb{Z}} \Delta_n. \tag{4.1}$$

The function

$$\gamma_s(x) = \int_{\mathbb{R}_+} \chi_{M(s)}(\frac{\eta}{2x})\, e^{-\eta}\, d\eta = \int_{2x\cdot M(s)} e^{-\eta}\, d\eta = \int_{\bigcup_{n\in\mathbb{Z}}\Delta_n} x e^{-x\eta}\, d\eta$$

is "multiplicative-periodic", i.e., $\gamma_s((1+\sigma)x) = \gamma_s(x)$, for all $x \in \mathbb{R}_+$. Thus to analyze the values of $\gamma_s(x)$ it is sufficient to consider only $x \in [1, 1+\sigma)$.

Lemma 4.2 *For each $\varepsilon > 0$ there is $\sigma \in (0,1)$ such that for each set $M(s)$ of the form (4.1) we have*

$$\sup_{x\in\mathbb{R}_+} \gamma_s(x) - \inf_{x\in\mathbb{R}_+} \gamma_s(x) < \varepsilon.$$

PROOF. Given $x_1, x_2 \in [1, 1+\sigma)$ consider

$$\begin{aligned}
\gamma_s(x_1) - \gamma_s(x_2) &= \int_{\bigcup_{n\in\mathbb{Z}} \Delta_n} (x_1 e^{-x_1\eta} - x_e^{-x_2\eta})\, d\eta \\
&= \sum_{n\in\mathbb{Z}} \int_{\Delta_n} (x_1 e^{-x_1\eta} - x_2 e^{-x_2\eta})\, d\eta \\
&= s \sum_{n\in\mathbb{Z}} (1+\sigma)^n (x_1 e^{-x_1\eta_n} - x_e^{-x_2\eta_n}),
\end{aligned}$$

for some $\eta_n \in \Delta_n$. Note, that $|\Delta_n| = s \cdot (1+\sigma)^n$.

Further

$$x_1 e^{-x_1\eta_n} - x_2 e^{-x_2\eta_n} = (x_1 - x_2) e^{-\xi_n\eta_n}(1 - \xi_n\eta_n)$$

for some $\xi_n \in (x_1, x_2) \subset [1, 1+\sigma)$.

Now, for $\zeta_n = \xi_n\eta_n \in (1+\sigma)^n \cdot (1, (1+\sigma)^2)$ we have

$$\gamma_s(x_1) - \gamma_s(x_2) = s(x_1 - x_2) \sum_{n\in\mathbb{Z}} (1+\sigma)^n e^{-\zeta_n}(1 - \zeta_n),$$

thus

$$\begin{aligned}
|\gamma_s(x_1) - \gamma_s(x_2)| &\leq \sigma^2 \sum_{n\in\mathbb{Z}} (1+\sigma)^n e^{-\zeta_n} |1 - \zeta_n| \\
&= \sigma^2 \sum_{n=1}^{\infty} (1+\sigma)^n e^{-\zeta_n} |1 - \zeta_n| \\
&+ \sigma^2 \sum_{k=0}^{\infty} \frac{1}{(1+\sigma)^k} e^{-\zeta_k} |1 - \zeta_k|.
\end{aligned}$$

Estimate each of these sums

$$\sigma^2 \sum_{k=0}^{\infty} \frac{1}{(1+\sigma)^k} e^{-\zeta_k} |1 - \zeta_k| < \sigma^2 \sum_{k=0}^{\infty} \frac{1}{(1+\sigma)^k} \cdot 1 \cdot 3 < 6\sigma,$$

$$\begin{aligned}
\sigma^2 \sum_{n=1}^{\infty} (1+\sigma)^n e^{-\zeta_n} |1 - \zeta_n| &< \sigma^2 \sum_{n=1}^{\infty} (1+\sigma)^n e^{-\zeta_n} \zeta_n \\
&< \sigma^2 \sum_{n=1}^{\infty} (1+\sigma)^{2(n+1)} e^{-(1+\sigma)^n} \\
&< (1+\sigma)^3 \sigma \int_1^{\infty} y\, e^{-y}\, dy \\
&< 8\sigma \int_{\mathbb{R}_+} y\, e^{-y}\, dy = 8\sigma.
\end{aligned}$$

$\square$

Theorem 4.3 *The spectrum of the Toeplitz operator $T_{\chi_{M'}}$ is given by the formula*

$$\operatorname{sp} T_{\chi_{M'}} = [\alpha, \beta],$$

where $\alpha = \inf_{x\in\mathbb{R}_+} \gamma_{\chi_{M'}}(x) \geq 0$, $\beta = \sup_{x\in\mathbb{R}_+} \gamma_{\chi_{M'}}(x) \leq 1$.

Moreover, for every α, β with $0 \leq \alpha < \beta \leq 1$, there exists a set M such that the spectrum of the Toeplitz operator $T_{\chi_{M'}}$ is equal to $[\alpha, \beta]$.

Note, that all the Toeplitz operators T_a of the above theorem have the function a of the same type – the characteristic function, and with the same range – the set $\{0, 1\}$, but contrary to the case of Toeplitz operators on the Hardy space, their spectra are quite different.

Proof. The first statement follows directly from Lemma 4.1. Consider now arbitrary α and β with $0 \leq \alpha < \beta \leq 1$. For $\varepsilon = (\beta - \alpha)/2$ fix σ according to Lemma 4.2. It is clear that for each $\lambda \in [0, 1]$ there exists s_λ ($\in (0, \sigma)$) such that for the corresponding set $M_\lambda = M(s_\lambda)$ of the form (4.1) we have

$$\sup_{x\in[1,1+\sigma)} \int_{2x\cdot M_\lambda} |\ell_0(y)|^2 dy = \lambda,$$

similarly, there exists a set m_λ of the above form, such that

$$\inf_{x\in[1,1+\sigma)} \int_{2x\cdot m_\lambda} |\ell_0(y)|^2 dy = \lambda.$$

Now for the set $M = (m_\alpha \cap (0, 1)) \cup (M_\beta \cap (1, +\infty))$ and

$$\gamma_{\chi_{M'}}(x) = \int_{2x\cdot M} |\ell_0(\eta)|^2 \, d\eta$$

we have

$$\begin{aligned} \inf_{x\in\mathbb{R}_+} \gamma_{\chi_{M'}}(x) &= \liminf_{x\to+\infty} \gamma_{\chi_{M'}}(x) = \alpha, \\ \sup_{x\in\mathbb{R}_+} \gamma_{\chi_{M'}}(x) &= \limsup_{x\to 0} \gamma_{\chi_{M'}}(x) = \beta. \end{aligned}$$

$\square$

Using the unitary operator U_0, the above spectrum characterizations allow immediate reformulation for Toeplitz operator with $\widetilde{\mathcal{A}}_\infty$ symbols, acting on $L_2(\mathbb{D})$.

4.2 There are many results (see, for example, [6], [1], [14]) describing the conditions on functions a for the Toeplitz operator T_a to be compact on $\mathcal{A}^2(\mathbb{D})$. The more wide sufficient condition can be formulated as follows.

Theorem 4.4 *Let a function $a \in L_\infty(\mathbb{D})$ is continuous at each point of the boundary $\partial\,\mathbb{D}$, and $a_{|\partial\,\mathbb{D}} \equiv 0$, then the Toeplitz operator T_a is compact on $\mathcal{A}^2(\mathbb{D})$.*

Show now how exact the conditions of this theorem are. All the circles $C(\eta)$, $\eta \in \mathbb{R}_+$, are tangent to the unit circle $\partial\mathbb{D} = C(0)$ at the point i, and near this point $C(\eta)$ are the graphs of the functions

$$y_\eta(x) = \frac{\eta}{1+\eta} + \sqrt{\frac{1}{(1+\eta)^2} - x^2}.$$

The "degree of adhesion" of $C(\eta)$ to $\partial\mathbb{D}$ at the point i can be naturally measured by the following quantity

$$\delta(\eta) = y_0''(0) - y_\eta''(0) = \frac{\eta(2+\eta)}{(1+\eta)^2},$$

as a function the quantity $\delta(\eta)$, $\eta \in \mathbb{R}_+$, is continuous, strictly increasing and $\delta(0) = 0$.

We will say that a function $a(z)$, defined on $\overline{\mathbb{D}}$, has the ε-limit equal to A at the point i:

$$\varepsilon\text{-}\lim_{z\to i} a(z) = A,$$

if

$$\lim_{z\to i} a(z)|_{\overline{\mathbb{D}}(\eta)} = A,$$

where $\overline{\mathbb{D}}(\eta)$ is the closed disk with the boundary $C(\eta)$, and such that $\delta(\eta) = \varepsilon$.

The ε-limit has the following obvious properties:

1. if for any $\varepsilon > 0$ we have $\varepsilon\text{-}\lim_{z\to i} a(z) = 0$, then all non-tangent limit values of the function a at the point i exist and are equal to zero;
2. if $\varepsilon_1 < \varepsilon_2$ and $\varepsilon_1\text{-}\lim_{z\to i} a(z) = 0$, then $\varepsilon_2\text{-}\lim_{z\to i} a(z) = 0$;
3. $0\text{-}\lim_{z\to i} a(z) = 0$ if and only if $\lim_{z\to i} a(z) = 0$.

Lemma 4.5 *For each $\varepsilon > 0$ there exists a function $a \in L_\infty(\mathbb{D})$ such that*

1. *a is continuous at each point of $\partial\mathbb{D} \setminus \{i\}$, and $a_{|\partial\mathbb{D}\setminus\{i\}} \equiv 0$;*
2. *$\varepsilon\text{-}\lim_{z\to i} a(z) = 0$;*
3. *the Toeplitz operator T_a is not compact on $\mathcal{A}^2(\mathbb{D})$.*

PROOF. Fix positive ε and consider η having the property $\delta(\eta) = \varepsilon$. Take a function $b \in \mathcal{A}_\infty$ with $\mathbb{R} \times [\eta/3, 2\eta/3] \subset \operatorname{supp} b$, then the function $a(z) = (b \cdot \chi_{\mathbb{R}\times[\eta/3,2\eta/3]})(\omega(z))$ satisfy the first two conditions of the lemma. The operator T_a is unitary equivalent to a (non-zero) multiplication operator, and thus is not compact. $\square$

5 Finite dimensional representations

Fix a number $n \in \mathbb{N}$. Let Y_k, $k = \overline{1,n}$, be disjoint measurable sets in $\mathbb{R}_+$ having positive measure, and such that $\cup_{k=1}^n Y_k = \mathbb{R}_+$, and let

$$\Pi_k = \mathbb{R} + iY_k, \quad k = \overline{1,n},$$

be the corresponding sets in the upper half-plane $\Pi = \mathbb{R}+i\mathbb{R}_+$. Denote by $\chi_k(y)$ $(\in L_\infty(\mathbb{R}_+))$ the characteristic function of the set Y_k, and by χ_{Π_k} $(\in L_\infty(\Pi))$ the characteristic function of the set Π_k $k = \overline{1,n}$.

Introduce the algebra

$$\mathcal{A}_n = \{a_1\chi_{\Pi_1} + \ldots + a_n\chi_{\Pi_n} \; : \; a_k \in \mathbb{C}, \quad k = \overline{1,n}\} \cong \mathbb{C}^n.$$

We will study the algebras $\mathcal{T}(\mathcal{A}_n)$ and $\mathcal{R}(\mathcal{A}_n, B_\Pi)$.

Theorem 5.1 *The algebra $\mathcal{T}(\mathcal{A}_n)$ is isomorphic and isometric to the algebra $C(\Delta)$, where*

$$\Delta = \Delta(Y_1, ..., Y_n) = \operatorname{clos}\{\, (\gamma_{\chi_1}(x), ..., \gamma_{\chi_n}(x)) : x \in \mathbb{R}_+ \,\} \quad \subset \quad \Delta_{n-1}, \tag{5.1}$$

and $\gamma_{\chi_k}(x)$, $k = \overline{1,n}$, are given by

$$\gamma_{\chi_k}(x) = \int_{\mathbb{R}_+} \chi_k(\frac{\eta}{2x})\, |\ell_0(\eta)|^2\, d\eta = \int_{2x\cdot Y_k} |\ell_0(\eta)|^2\, d\eta.$$

The isomorphism

$$\tau_n : \mathcal{T}(\mathcal{A}_n) \longrightarrow C(\Delta)$$

is generated by the following mapping of generators T_a, $a = a_1\chi_{\Pi_1} + \ldots + a_n\chi_{\Pi_n} \in \mathcal{A}_n$, of the algebra $\mathcal{T}(\mathcal{A}_n)$

$$\tau_n : T_a \longmapsto a_1t_1 + \ldots + a_nt_n, \quad t = (t_1, ..., t_n) \in \Delta.$$

Proof. Follows directly from Theorem 3.1. $\square$

The algebra $\mathcal{R}(\mathcal{A}_n, B_\Pi)$ can be defined by another set of generators. Namely, introduce the orthogonal projections

$$P = B_\Pi, \quad \text{and} \quad Q_k = \chi_{\Pi_k}I, \quad k = \overline{1,n}, \tag{5.2}$$

acting on $L_2(\Pi)$. These projections are obviously all-but-one, and the algebra $\mathcal{R}(\mathcal{A}_n, B_\Pi) = \mathcal{C}^*(P, Q_1, ..., Q_n)$ is generated by them. Thus we have the following theorem.

Theorem 5.2 *The C^*-algebra $\mathcal{R}(\mathcal{A}_n, B_\Pi)$ is isomorphic and isometric to a subalgebra of the algebra $\mathfrak{S}(\Delta) \oplus \mathbb{C}^n$, where $\Delta = \Delta(Y_1, ..., Y_n)$ is given by (5.1). The isomorphic imbedding*

$$\nu : \mathcal{R}(\mathcal{A}_n, B_\Pi) \longrightarrow \mathfrak{S}(\Delta) \oplus \mathbb{C}^n$$

is generated by the following mapping of the generators of the algebra $\mathcal{R}(\mathcal{A}_n, B_\Pi)$:

$$\begin{aligned} \nu : B_\Pi \quad &\longmapsto \quad (p(t), (0, 0, \ldots, 0)), \\ \nu : \chi_{\Pi_k}I \quad &\longmapsto \quad (q_k(t), (0, \ldots, 0, \underset{k-place}{1}, 0, \ldots, 0)), \quad k = \overline{1,n}, \end{aligned}$$

where $p(t)$ and $q_k(t)$ are defined in (2.5) and (2.6), $t = (t_1, \ldots, t_n) \in \Delta = \Delta(Y_1, ..., Y_n)$.

PROOF. The all-but-one projections (5.2) are obviously satisfy (2.3). The analysis of the projections $UB_{\Pi}U^{-1} = \chi_+(x)I \otimes P_0$ and $U\chi_{\Pi_k}U^{-1} = \chi_{\Pi_k}(\frac{y}{2|x|})I$, $k = \overline{1,n}$, shows that the property (2.4) does not hold for the projections (5.2) for all $k = \overline{1,n}$. Thus we are in the situation of Theorem 2.6 with $m = n$. The joint spectrum of the operators $C_k = PQ_kP = T_{\chi_{\Pi_k}}$, $k = \overline{1,n}$, is obviously equal to $\Delta = \Delta(Y_1, ..., Y_n)$. $\square$

Remark 5.3 It is easy to construct the measurable sets Y_k, $k = \overline{1,n}$, for which the corresponding set $\Delta = \Delta(Y_1, ..., Y_n)$ belongs to the interior of the simplex Δ_{n-1}. In such a case the algebra $\mathfrak{S}(\Delta)$ in Theorem 5.2 is just the algebra $C(\Delta, \mathrm{Mat}_n(\mathbb{C}))$.

6 General case

Denote by L_∞^0 the subalgebra of L_∞, with consists of all functions having zero limit at the the point $0 \in R_+$. And let $\mathcal{A}_\infty^0 = \mathbb{C} \otimes L_\infty^0 \subset \mathcal{A}_\infty$,

$$\widetilde{\mathcal{A}}_\infty^0 = U_0^{-1}\mathcal{A}_\infty^0 U_0 = \{a(\omega(z)) : a \in \mathcal{A}_\infty^0\} \subset \widetilde{\mathcal{A}}_\infty.$$

Besides the algebra $\widetilde{\mathcal{A}}_\infty^0$ we will consider its rotated variants. To do this introduce the unitary (rotation) operator on $L_2(\mathbb{D})$: given $t_1, t_2 \in \partial\,\mathbb{D}$, let

$$(U_{(t_1,t_2)}\varphi)(z) = \varphi(\frac{t_2}{t_1}z),$$

then, of course, $U_{(t_1,t_2)}^{-1} = U_{(t_2,t_1)}$. Denote now $\widetilde{\mathcal{A}}_\infty^0(i) = \widetilde{\mathcal{A}}_\infty^0$, and for each $t \in \partial\,\mathbb{D}$ introduce

$$\widetilde{\mathcal{A}}_\infty^0(t) = U_{(i,t)}^{-1}\widetilde{\mathcal{A}}_\infty^0(i)U_{(i,t)} = \{a(\frac{i}{t}z) : a \in \widetilde{\mathcal{A}}_\infty^0(i)\}.$$

All the functions from $\widetilde{\mathcal{A}}_\infty^0(t)$ are continuous at each point of $\partial\,\mathbb{D} \setminus \{t\}$, and their restrictions onto $\partial\,\mathbb{D} \setminus \{t\}$ are identically zero.

Lemma 6.1 *Given two different points $t_1, t_2 \in \partial\,\mathbb{D}$, the algebras $\mathcal{R}(\mathbb{C} + \widetilde{\mathcal{A}}_\infty^0(t_1), B_{\mathbb{D}})$ and $\mathcal{R}(\mathbb{C} + \widetilde{\mathcal{A}}_\infty^0(t_2), B_{\mathbb{D}})$, as well as the algebras $\mathcal{T}(\mathbb{C} + \widetilde{\mathcal{A}}_\infty^0(t_1))$ and $\mathcal{T}(\mathbb{C} + \widetilde{\mathcal{A}}_\infty^0(t_2))$, are unitary equivalent.*

For all functions $a_k \in \widetilde{\mathcal{A}}_\infty^0(t_k)$, $k = 1, 2$, the operators $T_{a_1 \cdot a_2}$ and $T_{a_1}T_{a_2}$ are compact.

PROOF. The first statement follows from

$$\widetilde{\mathcal{A}}_\infty^0(t_2) = U_{(t_1,t_2)}^{-1}\widetilde{\mathcal{A}}_\infty^0(t_1)U_{(t_1,t_2)}, \qquad B_{\mathbb{D}} = U_{(t_1,t_2)}^{-1}B_{\mathbb{D}}U_{(t_1,t_2)}.$$

The operator $T_{a_1 \cdot a_2}$ is compact by Theorem 4.4. Let now $c_{(t_1,t_2)}(z)$ be a continuous on $\overline{\mathbb{D}}$ function, equals to 0 at a small neighborhood of the point t_1, and equals to 1 at a small neighborhood of the point t_2. Then the compactness of the operator $T_{a_1}T_{a_2}$ follows from

$$\begin{aligned} T_{a_1}T_{a_2} &= B_{\mathbb{D}}a_1B_{\mathbb{D}}(c_{(t_1,t_2)} + 1 - c_{(t_1,t_2)})B_{\mathbb{D}}a_2B_{\mathbb{D}} \\ &= B_{\mathbb{D}}a_1c_{(t_1,t_2)}B_{\mathbb{D}}a_2B_{\mathbb{D}} + B_{\mathbb{D}}a_1B_{\mathbb{D}}(1 - c_{(t_1,t_2)})a_2B_{\mathbb{D}} + K \\ &= T_{a_1c_{(t_1,t_2)}} \cdot T_{a_2} + T_{a_1} \cdot T_{(1-c_{(t_1,t_2)})a_2} + K, \end{aligned}$$

where K is compact, and the compactness of the operators $T_{a_1 c_{(t_1,t_2)}}$ and $T_{(1-c_{(t_1,t_2)})a_2}$. $\square$

Theorem 6.2 *For each finite (ordered) set $\Lambda = \langle n_0, n_1, ..., n_m\rangle$, where $1 = n_0 < n_1 < ... < n_m \leq \infty$, and $n_k \in \mathbb{N} \cup \{\infty\}$, there is an algebra $\mathcal{A}_\Lambda$, the subalgebra of $L_\infty(\mathbb{D})$, such that the symbol (Calkin) algebra $\operatorname{Sym}\mathcal{T}(\mathcal{A}_\Lambda)$ of the algebra $\mathcal{T}(\mathcal{A}_\Lambda)$ is commutative, while the symbol algebra $\operatorname{Sym}\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ of the algebra $\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ has irreducible representations exactly of dimensions n_0, n_1, ..., n_m.*

Proof. Fix a set $\Lambda = \langle n_0, n_1, ..., n_m\rangle$, and denote by t_1, t_2, ... t_m any m different points on the unit circle $\partial\mathbb{D}$. Now for each k ($k = \overline{1,m}$, if $n_m < \infty$, and $k = \overline{1, m-1}$, if $n_m = \infty$) select the subalgebra $\widetilde{\mathcal{A}}^0_{n_k}(t_k)$ of the algebra $\widetilde{\mathcal{A}}^0_\infty(t_k)$ as follows.

Consider a special case of the algebra $\mathcal{A}_n$ of Section 5, the algebra $\mathcal{A}_{n_k}$, generated by sets $Y_{k,1}$, ..., Y_{k,n_k}, such that $Y_{k,n_k} = [0, \eta_k]$, for some positive η_k, and that the corresponding set (5.1) intersects the boundary of the simplex Δ_{n_k-1} only by the point $(0, ..., 0, 1)$. The algebra $\mathcal{A}^0_{n_k} = \mathcal{A}^0_\infty \cap \mathcal{A}_{n_k} \cong \mathbb{C}^{n_k-1}$ is generated by the characteristic functions $\chi_{\Pi_{k,j}}$, where $\Pi_{k,j} = \mathbb{R} + iY_{k,j}$, $j = \overline{1, n_k - 1}$, and, of course, $\mathbb{C} + \mathcal{A}^0_{n_k} = \mathcal{A}_{n_k}$. Set now

$$\widetilde{\mathcal{A}}^0_{n_k}(t_k) = U^{-1}_{(i,t_k)} U_0^{-1} \mathcal{A}^0_{n_k} U_0 U_{(i,t_k)} = \{a(\omega(\frac{i}{t_k} z)) : a \in \mathcal{A}^0_{n_k}\} \subset \mathcal{A}^0_\infty(t_k).$$

If $n_m = \infty$, set $\widetilde{\mathcal{A}}^0_{n_k}(t_m) = \widetilde{\mathcal{A}}^0_\infty(t_k)$. Finally define the algebra $\mathcal{A}_\Lambda$ as

$$\mathcal{A}_\Lambda = C(\overline{\mathbb{D}}) + \widetilde{\mathcal{A}}^0_{n_1}(t_1) + ... + \widetilde{\mathcal{A}}^0_{n_m}(t_m).$$

Now the commutativity of the symbol algebra $\operatorname{Sym}\mathcal{T}(\mathcal{A}_\Lambda) = \mathcal{T}(\mathcal{A}_\Lambda)/\mathcal{K}$ follows from the one for $\operatorname{Sym}\mathcal{T}(C(\overline{\mathbb{D}}))$, Theorem 3.1 and Lemma 6.1.

To analyze the symbol algebra $\operatorname{Sym}\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ of the algebra $\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ we will use the standard local principle (see, for example, [3], [9]). The algebra $C(\overline{\mathbb{D}})$ is obviously a central commutative subalgebra of $\widehat{\mathcal{R}} = \operatorname{Sym}\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$. For each $z_0 \in \overline{\mathbb{D}}$ denote by J_{z_0} the maximal ideal of $C(\overline{\mathbb{D}})$ corresponding to z_0, and by $J(z_0)$ the closed two-sided ideal of the algebra $\widehat{\mathcal{R}}$, generated by J_{z_0}. Then the local algebra at the point z_0 is defined as $\mathcal{R}(z_0) = \widehat{\mathcal{R}}/J(z_0)$, the natural projection

$$\pi_{z_0} : \mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D}) \longrightarrow \operatorname{Sym}\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D}) \longrightarrow \mathcal{R}(z_0)$$

identifies locally equivalent at the point z_0 elements of the algebra $\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$, and all the irreducible representations of the algebra $\operatorname{Sym}\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ are the collection of irreducible representations of the local algebras $\mathcal{R}(z_0)$, $z_0 \in \overline{\mathbb{D}}$.

For each $z_0 \in \mathbb{D} = \operatorname{Int}\overline{\mathbb{D}}$ the Bergman projection $B_\mathbb{D}$ is locally equivalent to 0. Thus the local algebra $\mathcal{R}(z_0)$ is commutative, and all the irreducible representations, with correspond to points of $\mathbb{D}$, are just one-dimensional representations of the algebra $\mathcal{A}_\Lambda|_\mathbb{D}$.

Recall that each function $a(z)$, which is continuous at some point z_0, is locally equivalent at this point to its value $a(z_0)$, and that all the functions from $\widetilde{\mathcal{A}}^0_{n_k}(t_k)$ are continuous in each point of $\partial\mathbb{D} \setminus \{t_k\}$, and are identically zero in $\partial\mathbb{D} \setminus \{t_k\}$.

At the points $z_0 \in \partial\mathbb{D} \setminus \{t_1, ..., t_m\}$ functions $a(z) \in C(\overline{\mathbb{D}})$ are locally equivalent to $a(z_0) \in \mathbb{C}$, while all functions from each $\widetilde{\mathcal{A}}^0_{n_k}(t_k)$ are locally equivalent to 0. The local algebra $\mathcal{R}(z_0)$ in this case is generated by $\mathbb{C}$ and the projection $\pi_{z_0}(B_{\mathbb{D}})$, and thus is commutative and isomorphic to $\mathbb{C}^2$.

Let finally $z_0 = t_k$ for some $k = \overline{1,m}$. Again, functions $a(z) \in C(\overline{\mathbb{D}})$ are locally equivalent to $a(z_0) \in \mathbb{C}$, but now functions from only $\widetilde{\mathcal{A}}^0_{n_j}(t_j)$, with $j \neq k$, are locally equivalent to 0. Thus the local algebra $\mathcal{R}(t_k)$ is generated by images, under the mapping π_{t_k}, of $B_{\mathbb{D}}$ and functions from $\mathbb{C} + \widetilde{\mathcal{A}}^0_{n_k}(t_k)$.

Further, the algebra $\mathcal{R}(\mathbb{C}+\widetilde{\mathcal{A}}^0_{n_k}(t_k), B_{\mathbb{D}})$ is unitary equivalent to $\mathcal{R}(\mathcal{A}_{n_k}, B_{\Pi})$, which does not contain any non-zero compact operator. And thus its image on the symbol algebra (which can be described via continuous functions on $\overline{\mathbb{D}}$, valued in local algebras $\mathcal{R}(z_o)$ for each z_0) is isomorphic to the initial algebra $\mathcal{R}(\mathbb{C} + \widetilde{\mathcal{A}}^0_{n_k}(t_k), B_{\mathbb{D}})$. But $\pi_{z_0}(B_{\mathbb{D}}) = 0$ for each $z_0 \in \mathbb{D}$, and $\pi_{z_0}(\widetilde{\mathcal{A}}^0_{n_k}(t_k)) = 0$ for each $z_0 \in \partial\mathbb{D} \setminus \{t_k\}$. Thus

$$\mathcal{R}(t_k) \cong \mathcal{R}(\mathbb{C} + \widetilde{\mathcal{A}}^0_{n_k}(t_k), B_{\mathbb{D}}) \cong \mathcal{R}(\mathcal{A}_{n_k}, B_{\Pi}),$$

and by Theorem 3.5 (for $n_k = \infty$), or Theorem 5.2 (for $n_k < \infty$) the algebra $\mathcal{R}(t_k)$ has irreducible representations of only dimensions n_k and 1. $\square$

Remark 6.3 Given Λ, the constructed algebra $\mathcal{A}_\Lambda$ is not the unique one. Proving the theorem we rather demonstrate key steps to construct such algebras. It is unknown, if there exist algebras $\mathcal{A}_\Lambda$ having fundamentally another structure.

Functions a from the algebra $\mathcal{A}_\Lambda = C(\overline{\mathbb{D}}) + \widetilde{\mathcal{A}}^0_{n_1}(t_1) + ... + \widetilde{\mathcal{A}}^0_{n_m}(t_m)$ are obviously continuous in each point of $\partial\mathbb{D} \setminus \{t_1, ..., t_m\}$, and have well-defined limit values at the points t_k, $k = \overline{1,m}$, over the unit circle $\partial\mathbb{D}$:

$$a(t_k) = \lim_{\partial\mathbb{D} \ni t \to t_k} a(t). \tag{6.1}$$

Routine verification shows that for each $a \in \mathcal{A}_\Lambda$ and each $k = \overline{1,m}$ there is the unique function $\tilde{a}_k \in \widetilde{\mathcal{A}}^0_{n_k}(t_k)$, such that

$$\lim_{\mathbb{D} \ni z \to t_k} (a(z) - a(t_k) - \tilde{a}_k(z)) = 0.$$

For the functions $\tilde{a}_k(z) \in \widetilde{\mathcal{A}}^0_{n_k}(t_k)$ we will use their "origins" $a_k \in \mathcal{A}^0_{n_k} \subset \mathbb{C} \otimes L^0_\infty(\mathbb{R}_+)$, i.e., the functions

$$a_k(\omega) = \tilde{a}_k(\frac{t_k}{i} z(\omega)), \quad \omega \in \Pi. \tag{6.2}$$

Denote by Γ the disjoint union of the unit circle $\partial\mathbb{D}$ and m copies of $(0, +\infty] = (0, +\infty]_k$, $k = \overline{1,m}$, with the points $t_k \in \partial\mathbb{D}$ and $+\infty \in (0, +\infty]_k$ identified, for each $k = \overline{1,m}$.

Corollary 6.4 *The symbol algebra* $\mathrm{Sym}\,\mathcal{T}(\mathcal{A}_\Lambda)$ *is isomorphic and isometric to a subalgebra of* $C_b(\Gamma)$*. The homomorphism*

$$\mathrm{sym} : \mathcal{T}(\mathcal{A}_\Lambda) \longrightarrow \mathrm{Sym}\,\mathcal{T}(\mathcal{A}_\Lambda) \subset C_b(\Gamma)$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}_\Lambda)$

$$\mathrm{sym}\,: T_a \longmapsto \begin{cases} a(t), & \text{if } \ t \in \partial\mathbb{D} \setminus \{t_1, ..., t_m\} \\ a(t_k), & \text{if } \ t = t_k \\ \gamma_{a_k}(x), & \text{if } \ x \in (0, +\infty)_k \end{cases},$$

where $a \in \mathcal{A}_\Lambda$*, and for the function* a_k*, defined in (6.2), the function* $\gamma_{a_k}(x)$ *is given by (3.1).*

Corollary 6.5 *The essential spectrum of each Toeplitz operator* T_a*,* $a \in \mathcal{A}_\Lambda$*, is connected and coincides with the closure of the range of its symbol*

$$\text{ess-sp}\, T_a = \mathrm{clos}\, \{(\mathrm{sym}\, T_a)(u) : u \in \Gamma\}.$$

To describe the symbol algebra of the algebra $\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ introduce some notations. For each $a \in \mathcal{A}_\Lambda$ its restriction on $\partial\mathbb{D} \setminus \{t_1, ..., t_m\}$ is continuous. Defining values of a at the points t_k by (6.1), we obtain a continuous on $\partial\mathbb{D}$ function, which we will denote by $a|_{\partial\mathbb{D}}$. For each k with $n_k < \infty$, the function (6.2) has obviously the form

$$a_k = \alpha'_{k,1}\chi_{\Pi_{k,1}} + ... + \alpha'_{k,n_k-1}\chi_{\Pi_{k,n_k-1}},$$

for some $\alpha'_{k,j} \in \mathbb{C}$, and let $\alpha_{k,j} = a(t_k) + \alpha'_{k,j}$, $j = \overline{1, n_k - 1}$. For such k denote by $\Delta_k = \Delta_k(Y_{k,1}, ..., Y_{k,n_k})$ corresponding set (5.1).

Assume as well that $n_m = \infty$ in the set $\Lambda = \langle n_0, n_1, ..., n_m\rangle$. For the case $n_m < \infty$, one needs to omite all, which is connected with index m, considering the set $\Lambda = \langle n_0, n_1, ..., n_{m-1}\rangle$ in what follows.

Finally, denote by $\widetilde{\Gamma}$ the following disjoint union

$$\widetilde{\Gamma} = \{t_1, ..., t_m\} \sqcup \partial\mathbb{D} \sqcup \overline{\mathbb{D}}.$$

Corollary 6.6 *The symbol algebra* $\mathrm{Sym}\,\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ *of the algebra* $\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$ *is isomorphic and isometric to a subalgebra of*

$$\mathfrak{A} = \bigoplus_{k=1}^{m-1} C(\Delta_k, \mathrm{Mat}_{n_k}(\mathbb{C})) \oplus \mathcal{R}(\mathbb{C} + \mathcal{A}^0_{n_m}, B_\Pi) \oplus C(\partial\mathbb{D}) \oplus \mathcal{A}_\Lambda.$$

The homomorphism

$$\mathrm{sym}\,: \mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D}) \longrightarrow \mathrm{Sym}\,\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D}) \subset \mathfrak{A}$$

is generated by the following mapping of generators of the algebra $\mathcal{R}(\mathcal{A}_\Lambda, B_\mathbb{D})$

$$\mathrm{sym}\,: aI \quad\longmapsto \begin{cases} \mathrm{diag}(\alpha_{k,1}, ..., \alpha_{k,n_k-1}, a(t_k)) \in C(\Delta_k, \mathrm{Mat}_{n_k}(\mathbb{C})), & \text{for } \ k \neq m \\ (a(t_m) + a_m)I \in \mathcal{R}(\mathbb{C} + \mathcal{A}^0_{n_m}, B_\Pi), & \text{for } \ k = m \\ a|_{\partial\mathbb{D}}(t) \in C(\partial\mathbb{D}), & \text{on } \ \partial\mathbb{D} \\ a \in \mathcal{A}_\Lambda, & \text{on } \ \overline{\mathbb{D}} \end{cases},$$

$$\mathrm{sym}\,: B_\mathbb{D} \quad\longmapsto \begin{cases} p(t) \in C(\Delta_k, \mathrm{Mat}_{n_k}(\mathbb{C})), & \text{for } \ k \neq m \\ B_\Pi \in \mathcal{R}(\mathbb{C} + \mathcal{A}^0_{n_m}, B_\Pi), & \text{for } \ k = m \\ 1 \in C(\partial\mathbb{D}), & \text{on } \ \partial\mathbb{D} \\ 0 \in \mathcal{A}_\Lambda, & \text{on } \ \overline{\mathbb{D}} \end{cases},$$

where $a \in \mathcal{A}_\Lambda$*, and* $p(t) = (\sqrt{t_i}\sqrt{t_j})_{i,j=1}^{n_k}$ *with* $t = (t_1, ..., t_{n_k}) \in \Delta_k$*.*

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