The hyperholomorphic Bergman projector and its properties

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Abstract

The hyperholomorphic Bergman kernel function and the corresponding Bergman projector (integral operator with the Bergman kernel) are studied. This work is a continuation of a series of papers devoted to the hyperholomorphic Bergman function in Clifford and quaternionic analysis.

0 Introduction

This article is a continuation of the series of works [SV1], [SV2], [VS] where the Bergman kernel function for quaternionic and Clifford analysis were studied. This study was based on the intimate connection between the Bergman function and the Green function of the domain.

In the quaternionic case [SV1] it has been found that to study the Bergman function ${}_{\psi}\mathcal{B}$ for some fixed class of ψ–hyperholomorphic functions, it is necessary to use another class of φ–hyperholomorphic functions. A new phenomenon (for details see [SV2]) arises in the case of general Clifford analysis: it turns out that a very important method of producing integral representations for ${}_{\psi}\mathcal{B}$ does not work for m even, where $m+1$ denotes the number of independent variables. Hence deeper and more impressive properties can be established only in the case of m odd.

The first section gives the necessary preliminaries, including the description of the main results of previous works.

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Section 2 deals with the well–known T–operator and with the singular integral operator ${}^{\varphi,\psi}S$ over a domain in $\Omega \subset \mathbb{R}^{m+1}$. The properties established here allows us, in Section 3, to represent the Bergman hyperholomorphic projector ${}^{\varphi}B$ (i.e., the integral operator with the kernel ${}_{\varphi}\mathcal{B}$) as an element of the algebra generated by classical Mikhlin–Calderon–Zygmund operator over the domain Ω.

In Section 4 we describe the structure of two operator algebras. The first one is generated by ${}^{\psi}B$ and multiplication by continuous function operators, the second algebra is generated by the Toeplitz–Bergman operators with right continuous presymbols. Fredholm criteria are given as well.

1 Preliminaries

1.1 Let $\mathcal{A}_n^-$ denote the universal real Clifford algebra with the quadratic form $-\sum\limits_{k=1}^{n} t_k^2$. In particular, this means that if $i_1, \ldots, i_n$ is the basis of the n–dimensional space on which the algebra $\mathcal{A}_n^-$ is built and i_0 is the unit of $\mathcal{A}_n^-$, then

$$i_k^2 = -i_0,\ \ i_p i_q + i_q i_p = 0$$

for $k \in \{1, \ldots, n\}, p \neq q$. One can find all necessary properties of this algebra as well as commonly used notations in [BDS], for instance. Let $m \in \{1, \ldots, n\}$. We shall consider functions with values in the algebra $\mathcal{A}_n^-$ and defined in a domain $\Omega \subset \mathbb{R}^{m+1}$.

1.2 We denote by $\mathcal{A}_n^{-,k}$ the set of k–vectors from $\mathcal{A}_n^-$ and let $\psi := \{\psi^0, \psi^1, \ldots, \psi^m\} \subset \mathcal{A}_n^{-,0} \oplus \mathcal{A}_n^{-,1}$, and $\overline{\psi} := \{\overline{\psi^0}, \overline{\psi^1}, \ldots, \overline{\psi^m}\}$. On $\mathcal{C}^1(\Omega, \mathcal{A}_n^-)$ we introduce the operator ${}^{\psi}D$ putting

$${}^{\psi}D[f] := \sum_{k=0}^{m} \psi^k \frac{\partial f}{\partial x_k} \quad D[f]^{\psi} := \sum_{k=0}^{m} \frac{\partial f}{\partial x_k} \psi^k. \tag{1.1}$$

Let the conditions

$$\psi^p \cdot \overline{\psi^q} + \psi^q \cdot \overline{\psi^p} = 2\langle \psi^p, \psi^q \rangle_{\mathbb{R}^{m+1}} = 2\delta_{p,q}$$

be fulfilled for all $p, q \in \{0, 1, \ldots, m\}$, where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^{m+1}$ and $\delta_{p,q}$ is the Kronecker's delta. Of course this means that ψ is an orthonormalized system in $\mathbb{R}^{n+1} \approx \mathcal{A}_n^{-,0} \oplus \mathcal{A}_n^{-,1}$.

Now the Laplace operator Δ has the following representation:

$$\Delta = {}^{\psi}D \cdot {}^{\overline{\psi}}D = {}^{\overline{\psi}}D \cdot {}^{\psi}D. \tag{1.2}$$

The operator ${}^{\psi}D$ will be called the (left) Cauchy–Riemann operator for $\mathcal{A}_n^-$–valued functions, and the elements of the set ${}^{\psi}\mathfrak{M}(\Omega, \mathcal{A}_n^-) := \ker {}^{\psi}D$ will be called left–ψ–hyperholomorphic (in a domain Ω) $\mathcal{A}_n^-$–valued functions of $(m+1)$–dimensional variable. This name may be shortened when misunderstandings cannot arise. It follows from (1.2) that if $f \in {}^{\psi}\mathfrak{M}(\Omega)$, then its coordinate functions are harmonic in Ω.

1.3 Let Ω be a bounded domain in $\mathbb{R}^{m+1}$ with smooth boundary $\Gamma = \partial\Omega$. Denote by g the harmonic (or classical) Green's function of Ω. The exact definition and the properties of this function can be found in [HK], p. 26, for instance. It is known, in particular, that such a function exists, is unique and has the form:

$$g(x, \xi) := \theta_{m+1}(\xi - x) + h(x, \xi), \quad (x, \xi) \in \Omega \times \Omega \setminus \mathrm{diag}\ ,$$

where

$$\theta_{m+1}(x) = -\frac{1}{(m-1) \cdot |\mathbb{S}^m|} |x|^{-m+1} (m \geq 3),$$

is the fundamental solution of the Laplace equation, $|\mathbb{S}^m|$ is the area of the unit sphere in $\mathbb{R}^{m+1}$, h is harmonic in $\Omega \times \Omega$ and is such that

$$\begin{aligned} h(x, \xi) &:= -\theta_{m+1}(\xi - x), \quad (x, \xi) \in \Omega \times \partial\Omega, \\ h(x, \xi) &:= h(\xi, x), \quad \text{on} \quad \Omega \times \Omega; \end{aligned}$$

diag$:= \{(x, x) | x \in \Omega\}$.

For an arbitrary structural set ψ the Bergman left–ψ–hyperholomorphic kernel function (or Bergman h.h. kernel function or simply kern–function) ${}_{\psi}\mathcal{B}$ has been defined in [SV2], (see also [SV1]), by the formula

$${}_{\psi}\mathcal{B}(x, \xi) := D_{\xi}^{\psi} \cdot {}^{\overline{\psi}}D_x[g](x, \xi) = {}^{\overline{\psi}}D_x \cdot {}^{\psi}D_{\xi}[g](x, \xi), \quad (x, \xi) \in \overline{\Omega} \times \overline{\Omega} \setminus \ \mathrm{diag}\ .$$

The following properties have been established in [SV1] and [SV2]:

1. The function ${}_{\psi}\mathcal{B}$ can be continuously extended onto the diagonal of $\Omega \times \Omega$;
2. for any fixed $\xi \in \Omega$ $\qquad {}_{\psi}\mathcal{B}(\cdot, \xi) \in {}^{\psi}\mathfrak{M}(\Omega)$,
3. for any fixed $x \in \Omega$ $\qquad {}_{\psi}\mathcal{B}(x, \cdot) \in \mathfrak{M}^{\overline{\psi}}(\Omega)$,
4. ${}_{\psi}\mathcal{B}(x, \xi) = \overline{{}_{\psi}\mathcal{B}(\xi, x)}$.

5. (Reproducing property of the Bergman h.h. kernel). Let $\Gamma = \partial\Omega$ be a smooth surface and $f \in {}^{\psi}\mathfrak{M}(\Omega, A_n^-) \cap \mathcal{C}(\overline{\Omega}, A_n^-)$. Then
$$ {}^{\psi}B[f](x) := \int_{\Omega} {}_{\psi}\mathcal{B}(x,\xi) f(\xi) d\xi = f(x), \quad x \in \Omega. $$
The natural name for ${}^{\psi}B$ is the Bergman h.h. operator.

6. (Integral representation of the Bergman hyperholomorphic kernel function). Let m be odd and let ψ and φ be two structural sets with the only condition
$$ \sum_{k=0}^{m} \varphi^k \cdot \psi^k = 0, \tag{1.3} $$
then the equality
$$ {}_{\psi}\mathcal{B}(x,\xi) = \int_{\Gamma} \mathcal{K}_{\psi}(\tau - x) \cdot \sigma_{\overline{\varphi},\tau} \cdot {}_{\varphi,\psi}\Lambda(\tau,\xi) + \int_{\Omega} {}_{\varphi,\psi}\overline{\ell(x,\tau)} \cdot {}_{\varphi,\psi}\ell(\xi,\tau) d\tau, \tag{1.3$'$} $$
holds for $\forall (x,\xi) \in \Omega \times \Omega$.
Here $\mathcal{K}_{\psi}$ denotes the Cauchy kernel of the Clifford analysis:
$$ \mathcal{K}_{\psi}(\tau - x) := \frac{1}{|\mathbb{S}^m| \cdot |\tau - x|^{m+1}} \cdot \sum_{k=0}^{m} \overline{\psi^k} \cdot x_k, \tag{1.4} $$
${}_{\varphi,\psi}\Lambda$ and ${}_{\varphi,\psi}\ell$ are of the form
$$ \begin{aligned} {}_{\varphi,\psi}\Lambda(x,\xi) &= {}^{\varphi}D_{\xi} \cdot {}^{\psi}D_x[\theta_{m+1}(\xi - x)] = D_x^{\psi} \cdot {}^{\varphi}D_{\xi}[\theta_{m+1}(\xi - x)] \\ &= \frac{(m+1)\cdot(\xi - x)_{\varphi} \cdot (\xi - x)_{\psi} - |\xi - x|^2 \cdot \sum_{k=0}^{m} \varphi^k \cdot \psi^k}{|\mathbb{S}^m| \cdot |\xi - x|^{m+3}}, \\ {}_{\varphi,\psi}\ell(x,\xi) &= {}^{\varphi}D_{\xi} \cdot {}^{\psi}D_x[h](x,\xi) = D_x^{\psi} \cdot {}^{\varphi}D_{\xi}[h](x,\xi), \end{aligned} $$
$$ x_{\varphi} = \sum_{0}^{m} x_k \varphi_k, $$
$$ \sigma_{\overline{\psi},\tau} := \sum_{k=0}^{m} (-1)^k \overline{\psi^k} d\hat{\tau}_k, $$
and $d\hat{\tau}_k$ denotes the differential form $d\tau_0 \wedge d\tau_1 \wedge d\tau_2 \wedge \ldots \wedge d\tau_m$ with the omitted multiple $d\tau_k$.

7. Under the same assumptions we have also
$$ {}_{\psi}\mathcal{B}(x,\xi) = \int_{\Gamma} {}_{\varphi,\psi}\overline{\Lambda(\tau,x)} \cdot \sigma_{\varphi,\tau} \cdot \mathcal{K}_{\overline{\psi}}(\tau - \xi) + \int_{\Omega} {}_{\varphi,\psi}\overline{\ell(x,\tau)} \cdot {}_{\varphi,\psi}\ell(\xi,\tau) d\tau. $$
This follows from the Hermitian symmetry of the function ${}_{\psi}\mathcal{B}$.

2 The T–operator and the singular integral operator over a domain.

2.1 The well known T–operator (or the Vekua operator) is defined by the equality

$$^{\psi}T[f](x) := -\int_{\Omega} \mathcal{K}_{\psi}(\tau - x) \cdot f(\tau) \cdot d\tau$$

with $\mathcal{K}_{\psi}$ as in (1.4).

For an arbitrary pair of structural sets φ and ψ introduce the operator $^{\varphi,\psi}S$:

$$\begin{aligned} ^{\varphi,\psi}S[f](x) : &= \int_{\Omega} \overline{_{\varphi,\psi}\Lambda(\tau, x)} \cdot f(\tau) \cdot d\tau = \\ &= \int_{\Omega} \frac{(m+1) \cdot (\tau - x)_{\overline{\psi}} \cdot (\tau - x)_{\overline{\varphi}} - |\tau - x|^2 \cdot \sum_{k=0}^{m} \overline{\psi^k} \cdot \overline{\varphi^k}}{|\mathbb{S}^m| \cdot |\tau - x|^{m+1}} \cdot f(\tau) \cdot d\tau. \end{aligned}$$

It is clear that both $^{\psi}T$ and $^{\varphi,\psi}S$ are linear bounded operators on the right $\mathcal{A}_n^-$–modules $L_p(\Omega, \mathcal{A}_n^-)$, $p > 1$, and the Hölder function space $\mathcal{C}^{0,\mu}(\Omega, \mathcal{A}_{\mu}^-)$, $0 < \mu < 1$.

2.2 Theorem. *In the spaces $L_p(\Omega, \mathcal{A}_n^-)$, $p > 1$, and $\mathcal{C}^{0,\mu}(\Omega, \mathcal{A}_n^-)$, $0 < \mu < 1$, the composition $^{\psi}D \cdot {}^{\psi}T$ is well–defined and in these spaces*

$$^{\psi}D \cdot {}^{\psi}T = I, \tag{2.1}$$

where I is the identity operator.

Proof. Assume that f belongs to $\mathcal{C}^2(\overline{\Omega}, \mathcal{A}_n^-)$ which is dense in L_p or $\mathcal{C}^{0,\mu}$. For $\tau \neq x$ we have

$$d_{\xi}(\theta_{m+1}(\tau - x) \cdot \sigma_{\overline{\psi},\tau} \cdot f(\tau)) = (\mathcal{K}_{\psi}(\tau - x) \cdot f(\tau) + \theta_{m+1}(\tau - x) \cdot {}^{\overline{\psi}}D[f](\tau))d\tau, \tag{2.2}$$

or in the integral form

$$\int_{\partial\Omega} \theta_{m+1}(\tau - x) \cdot \sigma_{\overline{\psi},\tau} \cdot f(\tau) = \int_{\Omega} \mathcal{K}_{\psi}(\tau - x) \cdot f(\tau) d\tau + \int_{\Omega} \theta_{m+1}(\tau - x) \cdot {}^{\overline{\psi}}D[f](\tau) d\tau, \tag{2.3}$$

which implies

$$^{\psi}T[f](x) = \int_{\Omega} \theta_{m+1}(\tau - x) \cdot {}^{\overline{\psi}}D[f](\tau) d\tau - \int_{\partial\Omega} \theta_{m+1}(\tau - x) \cdot \sigma_{\overline{\psi},\tau} f(\tau) d\tau. \tag{2.4}$$

Applying the operator ${}^{\psi}D$ and using the Borel–Pompeiu formula we obtain:

$$\begin{aligned}
{}^{\psi}D \cdot {}^{\psi}T[f](x) &= \int_{\Omega} {}^{\psi}D_x[\theta_{m+1}(\tau - x)] \cdot {}^{\psi}D[f](\tau)d\tau \\
&\quad - \int_{\partial\Omega} {}^{\psi}D_x[\theta_{m+1}(\tau - x)] \cdot \sigma_{\overline{\psi},\tau} \cdot f(\tau) \\
&= - \int_{\Omega} \mathcal{K}_{\overline{\psi}}(\tau - x) \cdot {}^{\overline{\psi}}D[f](\tau)d\tau \\
&\quad + \int_{\partial\Omega} \mathcal{K}_{\overline{\psi}}(\tau - x) \cdot \sigma_{\overline{\psi},\tau} \cdot f(\tau) \\
&= f(x).
\end{aligned}$$

Thus for $f \in \mathcal{C}^2(\Omega, \mathcal{A}_n^-)$ we have

$$ {}^{\psi}D \cdot {}^{\psi}T[f] = f. \tag{2.5}$$

The equality (2.1) follows from (2.5), the continuity of the identity operator and the density of $\mathcal{C}^2$ in L_p and $\mathcal{C}^{0,\mu}$.

2.3 Theorem. *Let m be odd and let two structural sets φ and ψ satisfy (1.3). Then the operator ${}^{\psi}D \cdot {}^{\psi}T$ is well–defined in both $L_p(\Omega, \mathcal{A}_n^-), p > 1$ and $\mathcal{C}^{0,\mu}(\Omega, \mathcal{A}_n^-)$ and*

$$ {}^{\varphi}D \cdot {}^{\overline{\psi}}T = {}^{\varphi,\psi}S. $$

The proof is similar to the previous one. First we check this equality on functions from an appropriate dense subset and then extend it by continuity.

3 Bergman hyperholomorphic spaces and Bergman hyperholomorphic projector

3.1 Consider any smooth function f in $\overline{\Omega}$ and apply the Bergman h.h. operator ${}^{\psi}B$ to it. Using the formula in property 7 from 1.3 we have

$$\begin{aligned}
{}^{\psi}B[f](x) &= \int_{\Omega} {}_{\psi}B(x, \xi)f(\xi)d\xi = \\
&= \int_{\Omega} \Big(\int_{\Gamma} {}_{\varphi,\psi}\overline{\Lambda(\tau, x)} \cdot \sigma_{\varphi,\tau} \cdot \mathcal{K}_{\overline{\psi}}(\tau - \xi) \\
&\quad + \int_{\Omega} {}_{\varphi,\psi}\overline{\ell(x, \tau)} \cdot {}_{\varphi,\psi}\ell(\xi, \tau)d\tau \Big) f(\xi)d\xi \\
&= {}^{\overline{\psi}}D_x \left(\int_{\Gamma} \mathcal{K}_{\varphi}(\tau - x) \cdot \sigma_{\varphi,\tau} \cdot {}^{\overline{\psi}}T[f](\tau) \right) + {}_{\varphi,\psi}C[f](x),
\end{aligned} \tag{3.1}$$

where ${}_{\varphi,\psi}C$ is the operator with the continuous kernel

$$\int_\Omega {}_{\varphi,\psi}\overline{\ell(x,\tau)} \cdot {}_{\varphi,\psi}\ell(\xi,\tau)d\tau.$$

Now apply to (3.1) the Borel–Pompeiu formula and Theorems 2.2 and 2.3:

$$\begin{aligned}
{}^{\psi}B[f](x) &= {}^{\overline{\psi}}D\left({}^{\overline{\psi}}T[f](\tau) + \int_\Omega \mathcal{K}_\varphi(\tau - x)\cdot {}^{\varphi}D_\tau\left[{}^{\overline{\psi}}T[f]\right](\tau)d\tau\right) + {}_{\varphi,\psi}C[f](x) \\
&= f(x) - ({}^{\overline{\psi}}D_x \cdot {}^{\psi}T)\cdot({}^{\varphi}D_\tau \cdot {}^{\overline{\varphi}}T)[f](x) +_{\varphi,\psi} C[f](x) \qquad (3.2) \\
&= f(x) - {}^{\overline{\psi}\cdot\overline{\varphi}}S \cdot {}^{\varphi,\psi}S[f](x) + {}_{\varphi,\psi}C[f](x).
\end{aligned}$$

Thus finally we have that the following operator equality holds on the smooth functions in $\overline{\Omega}$:

$$ {}^{\psi}B = I - {}^{\overline{\psi},\overline{\varphi}}S \cdot {}^{\varphi,\psi}S + {}_{\varphi,\psi}C. \qquad (3.3)$$

The right–hand side is well–defined both in $L_p(\Omega,\mathcal{A}_n^-)$ and in $\mathcal{C}^{0,\mu}(\overline{\Omega},\mathcal{A}_n^-)$, being linear and bounded. Thus the following theorem has been proved:

3.2 Theorem. *Let m be odd. The Bergman (left) hyperholomorphic operator acts, linearly and boundedly, on $L_p(\Omega,\mathcal{A}_n^-)$, $p>1$, and on $\mathcal{C}^{0,\mu}(\overline{\Omega},\mathcal{A}_n^-)$, $0<\mu<1$; moreover the following equality holds:*

$$ {}^{\psi}B = I - {}^{\overline{\psi},\overline{\varphi}}S \cdot {}^{\varphi,\psi}S + {}_{\varphi,\psi}C, \qquad (3.4)$$

where φ,ψ are two structural $(m+1)$–tuples with the property (1.3), and ${}_{\varphi,\psi}C$ is a compact operator.

3.3 Definition. *Given a structural set $\psi, p>1$ and a domain Ω, we denote by ${}_{\psi}\mathcal{A}^p(\Omega,\mathcal{A}_n^-)$ the subset of $L_p(\Omega,\mathcal{A}_n^-)$ consisting of all left–ψ–hyperholomorphic functions:*

$${}_{\psi}\mathcal{A}^p(\Omega,\mathcal{A}_n^-) := L_p(\Omega,\mathcal{A}_n^-) \cap {}^{\psi}\mathfrak{M}(\Omega,\mathcal{A}_n^-).$$

It is clear that ${}_{\psi}\mathcal{A}^p$ is a Banach submodule of the right $\mathcal{A}_n^-$–module L_p, and that ${}_{\psi}\mathcal{A}^2$ is a Hilbert submodule of the Hilbert $\mathcal{A}_n^-$–module L_2. We shall call ${}_{\psi}\mathcal{A}^p(\Omega,\mathcal{A}_n^-)$ the Bergman h.h. space.

3.4 Theorem. *Let m be odd. For any Ω, ψ and p under the previous restrictions we have*

$$ {}^{\psi}B(L_p(\Omega,\mathcal{A}_n^-)) = {}_{\psi}\mathcal{A}^p(\Omega,\mathcal{A}_n^-) \qquad (3.5)$$

and
$$ {}^{\psi}B^2 = {}^{\psi}B. \tag{3.6}$$
Hence, in the other words, the operator ${}^{\psi}B$ *is a projector of* $L_p(\Omega, \mathcal{A}_n^-)$ onto ${}_{\psi}\mathcal{A}^p(\Omega, \mathcal{A}_n^-)$.

Proof. We know that ${}_{\psi}\mathcal{B}$ has singularities only on the boundary. So if $x \in \Omega$ and $f \in L_p(\Omega, \mathcal{A}_n^-)$ then
$$ {}^{\psi}D \cdot {}^{\psi}B[f](x) = 0. $$
Taking also into account Theorem 3.2, we obtain that
$$ {}^{\psi}B[f] \in {}^{\psi}\mathfrak{M}(\Omega, \mathcal{A}_n^-) \cap L_p(\Omega, \mathcal{A}_n^-), $$
i.e.
$$ {}^{\psi}B(L_p) \subset {}_{\psi}\mathcal{A}^p. \tag{3.7}$$
The boundedness of ${}^{\psi}B$ together with the density of ${}^{\psi}\mathfrak{M}(\Omega, \mathcal{A}_n^-) \cap \mathcal{C}(\overline{\Omega}, \mathcal{A}_n^-)$ in ${}_{\psi}\mathcal{A}^p$ and the reproducing property, imply that
$$ f = {}^{\psi}B[f] \tag{3.8}$$
for each $f \in {}_{\psi}\mathcal{A}^p$. Hence ${}_{\psi}\mathcal{A}^p \subset {}^{\psi}B(L_p)$, and (3.5) is true. Taking $g \in L_p$ we get a function $f := {}^{\psi}B[g]$ which belongs to ${}_{\psi}\mathcal{A}^p$. Substitution of this functions into (3.8) gives (3.6).

3.5 Theorem. *The Bergman h.h. operator is self–adjoint in* $L_2(\Omega, \mathcal{A}_n^-)$.

Proof. By a simple calculation, using the fact that kernel ${}_{\psi}\mathcal{B}$ is hermitian:
$$\begin{aligned} \langle f, {}^{\psi}B[g]\rangle &= \int_\Omega \overline{f(t)} \left(\int_\Omega {}_{\psi}\mathcal{B}(t,\xi) \cdot g(\xi) d\xi \right) dt \\ &= \int_\Omega \overline{\left(\int_\Omega \overline{{}_{\psi}\mathcal{B}(t,\xi)} f(t) dt \right)} \cdot g(\xi) d\xi \\ &= \int_\Omega \overline{\left(\int_\Omega {}_{\psi}\mathcal{B}(t,\xi) f(t) dt \right)} \cdot f(\xi) d\xi \\ &= \langle {}^{\psi}B[f], g\rangle. \end{aligned}$$

3.6 Corollary *The Bergman h.h. operator is an orthogonal projector in* $L_2(\Omega, \mathcal{A}_n^-)$.

3.7 Remark In [GS, Ch.3] for the case of hyperholomorphic functions $f : \Omega \subset \mathbb{R}^3 \cong \mathbb{H}$ the orthogonal decomposition of $L_2(\Omega, \mathbb{H})$ has been constructed. The first summand is the set of hyperholomorphic functions, the second one is described in the terms of some Sobolev space and the Cauchy-Riemann operator. Thus, in fact,the authors give the description of the Bergmann projector in this situation.

4 Some algebras generated by the Bergman projector.

4.1 Let Ω be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. On the (real) spaces $L_p(\Omega, \mathcal{A}_n^-)$, $1 < p < \infty$, for a given function $a \in \mathcal{C}(\overline{\Omega}, \mathcal{A}_n^-)$ introduce the operators $m_r(a)$ and $m_\ell(a)$ as follows

$$\begin{aligned} m_r(a) : f \in L_p(\Omega, \mathcal{A}_n^-) &\longmapsto f \cdot a \in L_p(\Omega, \mathcal{A}_n^-), \\ m_\ell(a) : f \in L_p(\Omega, \mathcal{A}_n^-) &\longmapsto a \cdot f \in L_p(\Omega, \mathcal{A}_n^-). \end{aligned}$$

These operators are obviously bounded.

The decomposition (3.4) and the properties of the Mikhlin–Calderon–Zygmund operators ${}^{\varphi,\psi}S$ and ${}^{\overline{\varphi},\overline{\psi}}S$ imply:

4.2 Lemma. *For all $a \in \mathcal{C}(\overline{\Omega}, \mathcal{A}_n^-)$ the commutator $[{}^{\psi}B, m_r(a)] := {}^{\psi}B \cdot m_r(a) - m_r(a) \cdot {}^{\psi}B$ is compact.*

4.3 Remark. At the same time, for an arbitrary $a \in \mathcal{C}(\overline{\Omega}, \mathcal{A}_n^-)$, the commutator $[{}^{\psi}B, m_\ell(a)]$ is not in general compact.

4.4 Denote by $\mathcal{R}$ the algebra generated by all operators of the form

$$A = m_r(a) + m_r(b)\, {}^{\psi}B + K, \tag{4.1}$$

acting on $L_p(\Omega, \mathcal{A}_n^-)$, where a, $b \in \mathcal{C}(\overline{\Omega}, \mathcal{A}_n^-)$ and K is a compact operator.

Theorem 3.4 and Lemma 4.2 show that each operator of the algebra $\mathcal{R}$ has the form (4.1) for some a, b and K.

Let us restrict ourselves now onto the case of the space L_2. A standard use of the local principle [D], [S], together with the properties of the Bergman kernel (as it follows from (1.3′) the Bergman kernel has singularities only at the points $x = \xi \in \Gamma$), leads to

4.5 Theorem. *An operator A of the form (4.1) is Fredholm if and only if*

- $a(x)$ *is invertible in* $\mathcal{A}_n^-$ *for all* $x \in \overline{\Omega}$
- $a(x) + b(x)$ *is invertible in* $\mathcal{A}_n^-$ *for all* $x \in \Gamma$.

4.6 For a function $a \in \mathcal{C}(\overline{\Omega}, \mathcal{A}_n^-)$ denote by T_a the Toeplitz–Bergman operator with the right presymbol a:

$$T_a = {}^{\psi}Bm_r(a)|_{\psi\mathcal{A}^2(\Omega,\mathcal{A}_n^-)} : f \in {}_{\psi}\mathcal{A}^2 \longmapsto {}^{\psi}B(f \cdot a) \in {}_{\psi}\mathcal{A}^2.$$

Let $\mathcal{TB}(\mathcal{C}(\overline{\Omega}, \mathcal{A}_n^-))$ be the algebra generated by all Toeplitz–Bergman operators T_a with continuous right presymbols acting on the Bergman space ${}_{\psi}\mathcal{A}^2(\Omega, \mathcal{A}_n^-)$.

As a corollary of 4.4 and Theorem 4.5 we have immediately

4.7 Theorem. *An arbitrary operator* $T \in \mathcal{TB}(\mathcal{C}(\overline{\Omega}, \mathcal{A}_n^-))$ *has the form*

$$T = T_a + K,$$

where $a \in \mathcal{C}(\overline{\Omega}, \overline{\mathcal{A}}_n)$ *and* K *is a compact operator. The operator* T *is Fredholm if and only if* $a(x)$ *is invertible in* $\mathcal{A}_n^-$ *for all* $x \in \Gamma$.

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