ON THE BERGMANN KERNEL FUNCTION IN THE CLIFFORD ANALYSIS*

MICHAEL V. SHAPIRO and NIKOLAI L. VASILEVSKI
Department of Mathematics, Odessa State University,
Odessa, UKRAINE
and
Departamento de Matemáticas, CINVESTAV del IPN,
México, D.F., MEXICO

Abstract. We define and study the Bergmann kernel function and corresponding integral operator for Clifford algebra–valued hyperholomorphic functions. For any bounded domain Ω with a smooth enough boundary $\Gamma = \partial\Omega$ we have established the following facts:

The formula which gives an independent definition of ${}_\psi\mathcal{B}$ by means of the Green function g of a domain Ω; ${}_\psi\mathcal{B}$ is left-ψ-h.h. with respect to the first variable, is right–anti–ψ–h.h. with respect to the second one, is hermitian conjugate; Bergmannh.h. operator ${}^\psi B$ (i.e. the integral operator with the kernel ${}_\psi\mathcal{B}$) reproduces left–ψ–h.h. functions; the integral representation of ${}_\psi\mathcal{B}$ containing the integral over the boundary with the Clifford Cauchy kernel and the integral over Ω giving a continuous function.

Key words: Bergmann kernel function, reproducing property, integral representations, hyperholomorphic analysis

In the paper we define and study the Bergmann kernel function and corresponding integral operator for Clifford algebra–valued hyperholomorphic functions. The Bergmann function K_Ω of a domain $\Omega \subset \mathbb{C}$ in one-dimensional complex analysis has a lot of equivalent definitions given in different terms (see [B]). From our point of view the most productive definition is the following one:

$$K_\Omega(z, \overline{\zeta}) := -\frac{2}{\pi}\frac{\partial^2 G(z,\zeta)}{\partial z \partial \overline{\zeta}}, \tag{0.1}$$

where G is the Green function of a domain Ω.

This definition allows to obtain a lot of delicate and deep properties of K_Ω.

In the multidimensional complex analysis the simple connection between the Bergmann and Green functions is lost. To study the properties of the Bergmann kernel function one should use more powerful and more complex tools (see, for example, [R], [F1] [F2]).

The deep structural analogy between the one-dimensional complex analysis and the Clifford analysis allows to make use of the equality (0.1) to generalize the notion of the Bergmann kernel function and allows to follow in our investigation the general line of one-dimensional study of it.

* This work was partially supported by CONACYT project 1821-E9211

This work is a natural development of our paper [SV] which treats with the quaternionic case.

We know only two papers more, related to the problem. In [BDS,§29] the Bergmann function in Clifford analysis had been constructed for the unit ball using the general approach of reproducing kernels in a Hilbert space. In [Dz] the author uses the language of $\mathbb{C}^2$-valued vectors and 2×2-matrices with operator entries which is, in fact, a matrix representation of a special case of our quaternionic considerations (see [SV], Section 8).

In our work we study the notion of the Bergmann ψ-hyperholomorphic kernel function ${}_{\psi}\mathcal{B}$.For any bounded domain Ω with a smooth enough boundary $\Gamma = \partial\Omega$ we have established the following facts:

1. The formula which gives an independent definition of ${}_{\psi}\mathcal{B}$ by means of the Green function g of a domain Ω.
2. ${}_{\psi}\mathcal{B}$ is left-ψ-h.h. with respect to the first variable, is right–anti–ψ–h.h. with respect to the second one; is hermitian conjugate.
3. Bergmannh.h. operator ${}^{\psi}B$ (i.e. the integral operator with the kernel ${}_{\psi}\mathcal{B}$) reproduces left–ψ–h.h. functions.
4. The integral representation of ${}_{\psi}\mathcal{B}$ containing the integral over the boundary with the Clifford Cauchy kernel and the integral over Ω giving a continuous function.

In the quaternionic case we have found quite sudden and unpredictable novelty: to investigate ${}_{\psi}\mathcal{B}$ for some fixed class of ψ–h.h. functions we necessarily have to use some other class of φ–h.h. functions. A new effect arises in general situation: it appears that the method of producing of integral representation does not work for m even.

1. Preliminaries

1.1 Let $\mathcal{A}_n^-$, denotes the universal real Clifford algebra with the quadratic form $-\sum\limits_{k=1}^{n} t_k^2$. In particular, this means that if $i_1, \ldots, i_n$ is the basis of n–dimensional space on which the algebra $\mathcal{A}_n^-$ is built and i_0 is the unit of $\mathcal{A}_n^-$, then

$$i_k^2 = -i_0, \quad i_p i_q + i_q i_p = 0$$

for $k \in \{1, \ldots, n\}$, $p \neq q$. All necessary properties of this algebra one can find in [1] for instance. Also from [1] the terms and notations has been taken which are used without explanation. Let $m \in \{1, \ldots, n\}$. We shall consider functions with values in algebra $\mathcal{A}_n^-$ and defined in a domain $\Omega \subset \mathbb{R}^{m+1}$. If $n = 2$, i.e. $\mathcal{A}_2^-$ is the scew-field of the real quaternions, then all reasonings will remain true for $\Omega \subset \mathbb{R}^4$ also.

1.2 We denote by $\mathcal{A}_n^{-,k}$ the set of k–vectors from $\mathcal{A}_n^-$ and let $\psi := \{\psi^0, \psi^1, \ldots, \psi^m\} \subset \mathcal{A}_n^{-,0} \oplus \mathcal{A}_n^{-,1}$. On $C^1(\Omega, \mathcal{A}_n^-)$ we introduce the operator ${}^{\psi}D$ putting

$$ {}^{\psi}D[f] := \sum_{k=0}^{m} \psi^k \frac{\partial f}{\partial x_k}. \tag{1.1}$$

Let the conditions

$$\psi^p \cdot \overline{\psi^q} + \psi^q \cdot \overline{\psi^p} = \langle \psi^p, \psi^q \rangle_{\mathbb{R}^{m+1}} = 2\delta_{p,q}$$

be fulfilled for all $p, q \subset \{0, 1, \ldots, m\}$, where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^{m+1}$, $\delta_{p,q}$ is the Kronecker's delta. Of course this means that ψ is an orthonormalized system in $\mathbb{R}^{n+1} \approx \mathcal{A}_n^{-,0} \oplus \mathcal{A}_n^{-,1}$.

In this case the Laplace operator Δ has the following representation:

$$\Delta = {}^{\psi}D \cdot {}^{\overline{\psi}}D = {}^{\overline{\psi}}D \cdot {}^{\psi}D. \tag{1.2}$$

The operator ${}^{\psi}D$ will be called (left) Cauchy–Riemann operator for $\mathcal{A}_n^{-}$–valued functions, and the elements of the set ${}^{\psi}\mathfrak{M}(\Omega, \mathcal{A}_n^{-}) := ker^{\psi}D$ will be called left–ψ–hyperholomorphic in a domain Ω $\mathcal{A}_n^{-}$–valued functions of $(m+1)$–dimensional variable. Of course this name will be shortened when misunderstanding cannot arise. It follows from (1.2) that if $f \in {}^{\psi}\mathfrak{M}(\Omega)$ then its coordinate functions are harmonic in Ω.

2. Definition of the Hyperholomorphic Bergmann Kernel and Some its Principal Properties

2.1 Let Ω be a bounded multiconnected domain in $\mathbb{R}^{m+1}$ with the smooth boundary $\Gamma = \partial\Omega$. Denote by g the harmonic (or classical) Green function of a domain Ω. This means that g is a function of two ((m+1)-dimensional) variables x and ξ, $(x, \xi) \in \Omega \times \Omega\setminus$ diag, diag $:= \{(x, x) \mid \ x \in \Omega\}$, and it has the following properties:

1. $g(x, \xi) := \theta_{m+1}(\xi - x) + h(x, \xi)$, $\quad (x, \xi) \in \Omega \times \Omega \setminus \text{diag}$, *where*

$$\theta_{m+1}(x) = -\frac{1}{(m-1)\cdot \mid \mathbb{S}^m \mid} \mid x \mid^{-m+1} \quad (m \geq 3),$$

is the fundamental solution of the Laplace equation, $\mid \mathbb{S}^m \mid$ *is the area of the unit sphere in* $\mathbb{R}^{m+1}$, *h is harmonic in* $\Omega \times \Omega$ *and is such that*

$$\begin{aligned} h(x, \xi) &:= -\theta_{m+1}(\xi - x), \qquad (x, \xi) \in \Omega \times \partial\Omega, \\ h(x, \xi) &:= h(\xi, x), \qquad \textit{on} \quad \Omega \times \Omega; \end{aligned}$$

2. g *is harmonic in* $\Omega \times \Omega \setminus \text{diag}$ *and* $g(x, \xi) = g(\xi, x)$ in $\Omega \times \Omega \setminus \text{diag}$;
3. $g(x, \xi) = 0 \quad \forall (x, \xi) \in \Omega \times \Gamma$;
4. $g \in C(\bar{\Omega} \times \bar{\Omega} \setminus \text{diag})$.

It is known (see, for instance, [HK], p.26) that such function exists and is unique.

For example, in the case of the unit ball, i.e. $\Omega = \mathbb{B}(0, 1)$, we have (see, [HK] p.26, [Do], p.14)

$$g(x, \xi) = -\frac{1}{(m-1) \mid \mathbb{S}^m \mid} \left(\frac{1}{|\xi - x|^{m-1}} - \frac{1}{(1 - 2\langle \xi, x\rangle + |\xi|^2 \cdot |x|^2)^{(m-1)/2}} \right),$$

where $\langle \xi, x \rangle = \sum_{k=0}^{m} \xi_k \cdot x_k$.

2.2 For an arbitrary structural set ψ introduce the function ${}_{\psi}\mathcal{B}$:

$${}_{\psi}\mathcal{B}(x, \xi) := D_{\xi}^{\psi} \cdot {}^{\bar{\psi}}D_x[g](x, \xi) = {}^{\bar{\psi}}D_x \cdot {}^{\psi}D_{\xi}[g](x, \xi), \quad (x, \xi) \in \bar{\Omega} \times \bar{\Omega} \setminus \text{diag}. \tag{2.1}$$

We shall call it the Bergmann left-ψ-hyperholomorphic kernel function and shall write the Bergmann h.h. kernel (or simply kern) function. Sometimes we shall use also the term the *Bergmann h.h. kernel.*

In particular, for the unit ball (compare with [BDS,§29]):

$$
{}_{\psi}\mathcal{B}(x,\xi) = \frac{m+1}{\mid \mathbb{S}^m \mid} \cdot \frac{1}{(1-2\langle \xi, x\rangle + |\xi|^2 \cdot |x|^2)^{(m+1)/2}} -
$$

$$
-\frac{2}{\mid \mathbb{S}^m \mid} \cdot \frac{x_{\overline{\psi}} \cdot \xi_{\psi}}{(1-2\langle \xi, x\rangle + |\xi|^2 \cdot |x|^2)^{(m+1)/2}} +
$$

$$
+\frac{m+1}{\mid \mathbb{S}^m \mid} \cdot \frac{(\xi - x\cdot|\xi|^2)_{\overline{\psi}} \cdot (x - \xi\cdot |x|^2)_{\psi}}{(1-2\langle \xi, x\rangle + |\xi|^2 \cdot |x|^2)^{(m+3)/2}},
$$

here for $x \in \mathbb{R}^{m+1}$ $x_{\psi} := \sum_{k=0}^{m} x_k \cdot \psi^k$.

2.3 Remark. *The function ${}_{\psi}\mathcal{B}$ can be continuously extended onto a diagonal of $\Omega \times \Omega$, and, moreover, in the definition of ${}_{\psi}\mathcal{B}$ the function g may be substituted by h.*

Really

$$
\begin{aligned}
{}_{\psi}\mathcal{B} &= D_{\xi}^{\psi} \cdot {}^{\bar{\psi}}D_x[g] = D_{\xi}^{\psi} \cdot {}^{\bar{\psi}}D_x[\theta_{m+1}(\xi - x) + h] \\
&= D_{\xi}^{\psi} \cdot {}^{\bar{\psi}}D_x[h] \quad on \quad \bar{\Omega} \times \Omega \setminus diag.
\end{aligned}
$$

Hence we assume further that ${}_{\psi}\mathcal{B}$ is defined in a whole $\Omega \times \Omega$.

2.4 Theorem. *Bergmann h.h. kern-function has the following properties*:

1. *for any fixed* $\xi \in \Omega$
$$ {}_{\psi}\mathcal{B}(\cdot, \xi) \in {}^{\psi}\mathfrak{M}(\Omega), $$
2. *for any fixed* $x \in \Omega$
$$ {}_{\psi}\mathcal{B}(x, \cdot) \in \mathfrak{M}^{\bar{\psi}}(\Omega), $$
3. ${}_{\psi}\mathcal{B}(x,\xi) = \overline{{}_{\psi}\mathcal{B}(\xi, x)}$.

Proof is the same as in the case of quaternionic functions [SV]. First two assertions are the simple corollaries of the factorization (1.2), of the fact that Δ is a "real" operator (i.e. it maps real-valued functions into real-valued ones) and of Remark 2.3 which allows substitute h instead of g.

And besides we have

$$
\overline{{}_{\psi}\mathcal{B}(\xi, x)} = \overline{D_x^{\psi} \cdot {}^{\bar{\psi}}D_{\xi}[g](\xi, x)} = {}^{\bar{\psi}}D_x \, {}^{\psi}D_{\xi}[g] = {}_{\psi}\mathcal{B}(x,\xi).
$$

2.5 Theorem. (Reproducing property of the Bergmann h.h. kernel). *Let $\Gamma = \partial\Omega$ be a smooth surface and $f \in {}^{\psi}\mathfrak{M}(\Omega, \mathcal{A}_n^-) \cap C(\bar{\Omega}, \mathcal{A}_n^-)$. Then*

$$
{}^{\psi}B[f](x) := \int_{\Omega} {}_{\psi}\mathcal{B}(x,\xi) f(\xi)\, d\xi = f(x), \quad x \in \Omega. \tag{2.2}
$$

The natural name for ${}^{\psi}B$ is the Bergmann h.h. operator.

Proof. For $\varepsilon > 0$ consider

$$A(x,\varepsilon) := \int_{\Omega_{x,\varepsilon}} {}_{\psi}\mathcal{B}(x,\xi) f(\xi)\, d\xi = \int_{\Omega_{x,\varepsilon}} D_{\xi}^{\psi} \cdot {}^{\bar{\psi}}D_x[g] \cdot f(\xi)\, d\xi.$$

Use the Stokes formula [BDS]:

$$\int_{\partial\Omega_{x,\varepsilon}} \left({}^{\psi}D_x[g]\right) \cdot \sigma_{\psi} \cdot f = \int_{\Omega_{x,\varepsilon}} \left(D_{\xi}^{\psi} \cdot {}^{\bar{\psi}}D_x[g] \cdot f + {}^{\bar{\psi}}D_x[g] \cdot {}^{\psi}D_{\xi}[f]\right) d\xi$$

$$= \int_{\Omega_{x,\varepsilon}} D_{\xi}^{\psi} \cdot {}^{\bar{\psi}}D_x[g] \cdot f\, d\xi,$$

because of $f \in {}^{\psi}\mathfrak{M}(\Omega)$. Hence

$$A_{x,\varepsilon} = \int_{\partial\Omega_{x,\varepsilon}} {}^{\bar{\psi}}D[g] \cdot \sigma_{\psi} \cdot f.$$

By the property 3. from 2.1 we have

$${}^{\bar{\psi}}D_x[g] = 0 \qquad \forall \xi \in \Gamma.$$

As $\partial\Omega_{x,\varepsilon} = \Gamma \cup \mathbb{S}^{-}(x,\varepsilon)$ with negatively oriented sphere then

$$A_{x,\varepsilon} = -\int_{\mathbb{S}(x,\varepsilon)} {}^{\bar{\psi}}D[g] \cdot \sigma_{\psi} \cdot f.$$

Calculation

$${}^{\bar{\psi}}D_x[g](x,\xi) = {}^{\bar{\psi}}D_x[\theta_{m+1}(\xi - x)] + {}^{\bar{\psi}}D_x[h](x,\xi)$$

$$= -{}^{\bar{\psi}}D_{\xi}[\theta_{m+1}(\xi - x)] + h_1(x,\xi) = -\mathcal{K}_{\psi}(\xi - x) + h_1(x,\xi)$$

gives

$$A_{x,\varepsilon} = \int_{\mathbb{S}(x,\varepsilon)} \mathcal{K}_{\psi}(\xi - x) \cdot \sigma_{\psi,\xi} \cdot f(\xi) + o(\varepsilon) = f(x) + o(\varepsilon),$$

and tending $\varepsilon \to 0$ we obtain the necessary result.

2.6 Remark. *Theorem* 2.4 *and* 2.5 *justify the name* Bergmann kernel function *for* ${}_{\psi}\mathcal{B}$*, by our opinion.*

3. Integral Representation of the Bergmann Hyperholomorphic Kernel

3.1 Given arbitrary structural sets ψ and φ, introduce the function ${}_{\varphi,\psi}\mathcal{L}$ on $\bar{\Omega} \times \bar{\Omega} \setminus$ diag:

$${}_{\varphi,\psi}\mathcal{L} := {}^{\varphi}D_{\xi} \cdot {}^{\psi}D_x[g](x,\xi) = D_x^{\psi} \cdot {}^{\varphi}D_x[g](x,\xi).$$

It is clear that when $\varphi = \bar{\psi}$ we have

$${}_{\bar{\psi},\psi}\mathcal{L} := {}_{\psi}\bar{\mathcal{B}} \quad \text{or} \quad {}_{\psi}\mathcal{B}(x,\xi) = {}_{\bar{\psi},\psi}\mathcal{L}(\xi,x).$$

From subsection 2.1 it follows that

$$_{\varphi,\psi}\mathcal{L}(x,\xi) = {}_{\varphi,\psi}\Lambda(x,\xi) + {}_{\varphi,\psi}\ell(x,\xi), \tag{3.1}$$

where

$$\begin{aligned} {}_{\varphi,\psi}\Lambda(x,\xi) &= {}^{\varphi}D_{\xi} \cdot {}^{\psi}D_x[\theta_{m+1}(\xi - x)] = D_x^{\psi} \cdot {}^{\varphi}D_{\xi}[\theta_{m+1}(\xi - x)] \\ &= \frac{(m+1) \cdot (\xi - x)_{\varphi} \cdot (\xi - x)_{\psi} - \mid \xi - x \mid^2 \cdot \sum_{k=0}^{m} \varphi^k \cdot \psi^k}{\mid \mathbb{S}^m \mid \cdot \mid \xi - x \mid^{m+3}}, \\ {}_{\varphi,\psi}\ell(x,\xi) &= {}^{\varphi}D_{\xi} \cdot {}^{\psi}D_x[h](x,\xi) = D_x^{\psi} \cdot {}^{\varphi}D_{\xi}[h](x,\xi). \end{aligned}$$

3.2 Lemma. *Let g_o: $\Omega \to \mathbb{R}$ be a function, smooth enough, for which Γ is a level surface. Then for any structural set ψ and for any $\xi \in \Gamma$*

$$\nu_{\psi} \cdot {}^{\bar{\psi}}D[g_0](\xi) = \varepsilon_0 \cdot \mid grad\, g_0(\xi) \mid,$$

and hence left hand side does not depend on a choice of ψ. Here $\varepsilon_0 = \pm 1$, $\vec{\nu} = (\nu_0, \nu_1, \nu_2, \nu_3)$ is a unit outer normal vector to a surface Γ in a point $\xi \in \Gamma$,

$$\nu_{\psi} := \sum_{k=0}^{m} \nu_k \cdot \psi^k.$$

Proof. It is known from the standard calculus that under our assumptions $\vec{\nu}$ is collinear to $\operatorname{grad} g_0(\xi)$, hence

$$\operatorname{grad} g_0(\xi) = \varepsilon_0 |\operatorname{grad} g_0(\xi)| \cdot \nu(\xi).$$

Thus

$$\begin{aligned} \nu_{\psi} \cdot {}^{\bar{\psi}}D[g_0](\xi) &= \nu_{\psi}(\xi) \cdot \varepsilon_0 \cdot |\operatorname{grad} g_0(\xi)| \cdot \nu_{\bar{\psi}}(\xi) \\ &= \varepsilon_0 |\operatorname{grad} g_0(\xi)|. \end{aligned}$$

3.3 Corollary. *Functions $_{\psi}\mathcal{B}$ and $_{\varphi,\psi}\mathcal{L}$ are connected by the formula*

$$\begin{gathered} \nu_{\psi}(\xi) \cdot {}_{\psi}\mathcal{B}(\xi, x) = \nu_{\bar{\varphi}} \cdot {}_{\varphi,\psi}\mathcal{L}(x,\xi), \\ (x,\xi) \in \Omega \times \Gamma. \end{gathered} \tag{3.2}$$

Proof. For $(x,\xi) \in \Omega \times \Gamma$ we have $g(x,\xi) = 0$, hence Γ is a level surface for

$$\tilde{g}_k := \frac{\partial g}{\partial x_k}(x, \cdot), \qquad x \in \Omega,\ k \in \{0, 1, ..., m\}.$$

And so

$$\begin{aligned} \nu_{\psi}(\xi) \cdot {}_{\psi}\mathcal{B}(\xi, x) &= \nu_{\psi}(\xi) \cdot {}^{\bar{\psi}}D_{\xi} \cdot {}^{\psi}D_x[g] \\ &= \sum_{k=0}^{m} \nu_{\psi} \cdot {}^{\bar{\psi}}D_{\xi}[\tilde{g}_k] \cdot \psi^k \\ &= \sum_{k=0}^{m} \nu_{\bar{\varphi}} \cdot {}^{\varphi}D_{\xi}[\tilde{g}_k] \cdot \psi^k \\ &= \nu_{\bar{\varphi}} \cdot {}^{\varphi}D_{\xi} \cdot {}^{\psi}D[g] = \nu_{\bar{\varphi}} \cdot {}_{\varphi\psi}\mathcal{L}(x,\xi). \end{aligned}$$

3.4 In the quaternionic case [SV] the following fact was of principal value: for any structural set ψ there exists a structural set φ such that

$$\sum_{k=0}^{3} \varphi^k \cdot \psi^k = 0.$$

The question arises whether the corresponding equality for an arbitrary m

$$\sum_{k=0}^{m} \varphi^k \cdot \psi^k = 0 \tag{3.3}$$

is true for some φ.

Let first m be odd, i.e. $(m+1)$ is even. Then, given ψ, define φ by the rule

$$\varphi^k := \begin{cases} \overline{\psi^k}, & k \in K_1, \\ -\overline{\psi^k}, & k \in K_2, \end{cases}$$

where $K_1 \cup K_2 = \{0, 1, \ldots, m\}, K_1 \cap K_2 = \emptyset$, and each of K_1, K_2 has $\frac{m+1}{2}$ elements. Clear that for such φ we have (3.3).

Let us show now that for m even the condition (3.3) is contradictive, and hence φ cannot be found. We can write $\psi^k = \psi_0^k + \vec{\psi^k}$, $\varphi^k = \varphi_0^k + \vec{\varphi^k}$, where $\psi_0^k, \varphi_0^k \in \mathcal{A}_n^{-,0}$,$\vec{\psi^k}, \vec{\varphi^k} \in A_n^{-,1}$.

Equating k–vectors ($k \in \{0, 1, 2\}$) in (3.3) we arrive at the system

$$\sum_{k=0}^{m} \varphi_0^k \cdot \psi_0^k - \sum_{k=0}^{m} \langle \vec{\varphi^k}, \vec{\psi^k} \rangle_{\mathbb{R}^n} = 0, \tag{3.4}$$

$$\sum_{k=0}^{m} (\varphi_0^k \cdot \vec{\psi^k} + \psi_0^k \cdot \vec{\varphi^k}) = 0,$$

$$\sum_{k=0}^{m} \vec{\varphi^k} \wedge \vec{\psi^k} = 0, \tag{3.5}$$

which is equivalent to (3.3).

Denote by $\widetilde{\psi}$ a fixed basis in $\mathbb{R}^{n+1} \approx \mathcal{A}_n^{-,0} \oplus \mathcal{A}_n^{-,1}$ containing the set $\overline{\psi}$ (if $m = n$ then $\widetilde{\psi} := \overline{\psi}$).

To simplify calculations we assume that $\psi^0 := i_0$, then $\psi^k \in \mathcal{A}_n^{-,1}$ for $k \neq 0$, and $\widetilde{\psi}$ has a form

$$\widetilde{\psi} = \{i_0, -\psi^1, \ldots, -\psi^m, \widetilde{\psi}^{m+1}, \ldots, \widetilde{\psi}^n\}.$$

We may assume also that φ is given just in this basis: $\varphi^k = \sum\limits_{j=0}^{n} \varphi_j^k \cdot \widetilde{\psi}^j$.

Condition (3.4) gives now

$$\sum_{k=0}^{m} \varphi_k^k = 0, \tag{3.6}$$

and (3.5) gives

$$\sum_{1\le k<j\le m} (\varphi_j^k - \varphi_k^j)\psi^k \wedge \psi^j = 0,$$

$$\varphi_j^k = 0 \quad \forall j > m.$$

So in fact $\varphi^k = \sum_{j=0}^{m} \varphi_j^k \cdot \overline{\psi}^j$. Introduce $(m+1)\times(m+1)$ matrix

$$\phi := \begin{pmatrix} \varphi^0 \\ \dots \\ \varphi^m \end{pmatrix}.$$

It is orthogonal by the definition of a structural set. And (3.6) means that ϕ has zero trace which is impossible.

3.5 Theorem. (Integral representation of the Bergmann hyperholomorphic kernel function). *Let m be odd and let ψ and φ be two structural sets with the only condition*

$$\sum_{k=0}^{m} \varphi^k \cdot \psi^k = 0,$$

then the following equality holds

$${}_{\psi}\mathcal{B}(x,\xi) = \int_{\Gamma} \mathcal{K}_{\psi}(\tau - x)\cdot \sigma_{\bar{\varphi},\tau} \cdot {}_{\varphi,\psi}\Lambda(\tau-\xi) + \int_{\Omega} \overline{{}_{\varphi,\psi}\ell(x,\tau)} \cdot {}_{\varphi,\psi}\ell(\xi,\tau)\, d\tau,$$

for $\forall (x,\xi)\in\Omega\times\Omega$.

Proof. For any fixed $\zeta\in\Omega$ the function ${}_{\psi}\mathcal{B}(\cdot,\zeta)$ is left-ψ-hyperholomorphic. Hence from the Cauchy formula we have

$${}_{\psi}\mathcal{B}(x,\zeta) = \int_{\Gamma} \mathcal{K}_{\psi}(\xi - x)\cdot \sigma_{\psi,\xi} \cdot {}_{\psi}\mathcal{B}(\xi,\zeta), \quad (x,\zeta)\in\Omega\times\Gamma. \tag{3.7}$$

Multiplying (3.2) by ds and substituting into (3.7) we obtain

$${}_{\psi}\mathcal{B}(x,\zeta) = \int_{\Gamma} \mathcal{K}_{\psi}(\xi - x)\cdot \sigma_{\bar{\varphi},\xi} \cdot \mathcal{L}_{\varphi,\psi}(\zeta,\xi).$$

Using the representation (3.1) we arrive at

$$\begin{aligned} {}_{\psi}\mathcal{B}(x,\zeta) = &\int_{\Gamma} \mathcal{K}_{\psi}(\xi - x)\cdot \sigma_{\bar{\varphi},\xi} \cdot {}_{\varphi,\psi}\Lambda(\xi-\zeta) \\ &+ \int_{\Gamma} \mathcal{K}_{\psi}(\xi - x)\cdot \sigma_{\bar{\varphi},\xi} \cdot {}_{\varphi,\psi}\ell(\zeta,\xi). \end{aligned} \tag{3.8}$$

The general Stokes formula gives now

$$\int_{\Gamma} \mathcal{K}_{\psi}(\xi - x)\cdot \sigma_{\bar{\varphi},\xi} \cdot {}_{\varphi,\psi}\ell(\zeta,\xi) = \int_{\mathbb{S}(x,\varepsilon)} \mathcal{K}_{\psi}(\xi - x)\cdot \sigma_{\bar{\varphi},\xi} \cdot {}_{\varphi,\psi}\ell(\zeta,\xi)$$

$$+\int_{\Omega_{x,\varepsilon}} D_\xi^{\bar\varphi}\left[\mathcal{K}_\psi(\xi-x)\right]\cdot {}_{\varphi,\psi}\ell(\zeta,\xi)\,d\xi+\int_{\Omega_{x,\varepsilon}}\mathcal{K}_\psi(\xi-x)\cdot {}^{\bar\varphi}D_\xi[{}_{\varphi,\psi}\ell](\zeta,\xi)\,d\xi. \quad (3.9)$$

The definition of ${}_{\varphi,\psi}\ell$ implies ${}^{\bar\varphi}D_\xi[{}_{\varphi,\psi}\ell]=0$. Calculate:

$$N_{x,\varepsilon}:=\int_{\mathbb{S}(x,\varepsilon)}\mathcal{K}_\psi(\xi-x)\cdot\sigma_{\bar\varphi,\xi}\cdot {}_{\varphi,\psi}\ell(\zeta,\xi)$$

$$=\frac{1}{(m-1)\mid\mathbb{S}^m\mid\varepsilon^{m+1}}\int_{\mathbb{S}(x,\varepsilon)}\sum_{k=0}^m\bar\psi_k(\xi_k-x_k)\cdot\sigma_{\bar\varphi,\xi}\cdot {}_{\varphi,\psi}\ell(\zeta,\xi).$$

Using

$$\sum_{k=0}^m\varphi^k\cdot\psi^k=0$$

we have

$$D_\xi^{\bar\varphi}\left(\sum_{k=0}^m\overline{\psi^k}(\xi_k-x_k)\right)=\sum_{k=0}^m\overline{\psi^k}\cdot\overline{\varphi^k}=0,$$

and consequently Cauchy integral theorem means that $N_{x,\varepsilon}=0\ \forall x,\varepsilon$. Thus the equality (3.9) turns into

$$\int_\Gamma\mathcal{K}_\psi(\xi-x)\cdot\sigma_{\bar\varphi,\xi}\cdot {}_{\varphi,\psi}\ell(\zeta,\xi)=\int_{\Omega_{x,\varepsilon}}D_\xi^{\bar\varphi}\left[\mathcal{K}_\psi(\xi-x)\right]\cdot {}_{\varphi,\psi}\ell(\zeta,\xi)\,d\xi. \quad (3.10)$$

Using the Stokes formula and the equality ${}^{\bar\varphi}D[{}_{\varphi\psi}\ell]=0$ we have for $(x,\zeta)\in\Omega\times\Omega$

$$\int_{\Omega_{x,\varepsilon}}{}_{\varphi,\psi}\overline{\mathcal{L}(x,\xi)}\cdot {}_{\varphi,\psi}\ell(\zeta,\xi)\,d\xi=\int_{\Omega_{x,\varepsilon}}D_\xi^{\bar\varphi}\cdot D_\xi^{\bar\psi}[g](x,\xi)\cdot {}_{\varphi,\psi}\ell(\zeta,\xi)\,d\xi$$

$$=\int_{\Gamma\cup\mathbb{S}(x,\varepsilon)}D_x^{\bar\psi}[g](x,\xi)\cdot\sigma_{\bar\varphi,\xi}\cdot {}_{\varphi,\psi}\ell(\zeta,\xi). \quad (3.11)$$

Substitute $g=\theta_{m+1}(\xi-x)+h$ into (3.11) and use $N_{x,\varepsilon}=0$:

$$\int_{\Omega_{x,\varepsilon}}D_\xi^{\bar\varphi}\,{}^{\bar\psi}D_x\left[\theta_{m+1}(\xi-x)\right]\cdot {}_{\varphi,\psi}\ell(\eta,\xi)\,d\xi+\int_{\Omega_{x,\varepsilon}}D_\xi^{\bar\varphi}\cdot {}^{\psi}D_x[h]\,{}_{\varphi,\psi}\ell(\eta,\xi)\,d\xi$$

$$=-\int_{\partial\mathbb{B}(x,\varepsilon)}\mathcal{K}_\psi(\xi-x)\cdot\sigma_{\bar\varphi,\xi}\cdot {}_{\varphi,\psi}\ell(\eta,\xi)+\alpha(\varepsilon)=\alpha(\varepsilon),$$

where

$$\alpha(\varepsilon)=-\int_{\partial\mathbb{B}_(x,\varepsilon)}D_x^{\psi}[h](x,\xi)\cdot\sigma_{\bar\varphi,\xi}\cdot {}_{\varphi,\psi}\ell(\eta,\xi),$$

or

$$\int_{\Omega_{x,\varepsilon}}D_\xi^{\bar\varphi}\left[\mathcal{K}_\psi(\xi-x)\right]\cdot {}_{\varphi,\psi}\ell(\zeta,\xi)\,d\xi=\int_{\Omega_{x,\varepsilon}}{}_{\varphi,\psi}\overline{\ell(x,\xi)}\cdot {}_{\varphi,\psi}\ell(\zeta,\xi)\,d\xi+o(\varepsilon). \quad (3.12)$$

From (3.10), (3.12) under $\varepsilon\to0$ we obtain

$$\int_\Gamma\mathcal{K}_\psi(\xi-x)\cdot\sigma_{\bar\varphi,\zeta}\cdot {}_{\varphi,\psi}\ell(\zeta,\xi)=\int_\Omega{}_{\varphi,\psi}\overline{\ell(x,\xi)}{}_{\varphi,\psi}\ell(\zeta,\xi)\,d\xi. \quad (3.13)$$

Finally, comparing (3.8) and (3.13), we finish the proof.

3.6 Corollary. *Under the assumptions of Theorem 3.5 we have also*

$$ {}_{\psi}\mathcal{B}(x,\xi) = \int_{\Gamma} {}_{\varphi,\psi}\overline{\Lambda(\tau - x)} \cdot \sigma_{\varphi,\tau} \cdot \mathcal{K}_{\bar{\psi}}(\tau - \xi) + \int_{\Omega} {}_{\varphi,\psi}\overline{\ell(x,\tau)} \cdot {}_{\varphi,\psi}\ell(\xi,\tau)\, d\tau. $$

It follows from the Hermitian simmetry of function ${}_{\psi}\mathcal{B}$.

References

[B] S. Bergmann. *The Kernel Function and Conformal Mappings.* AMS, Providence, R. I. 1970.

[BDS] F. Brackx, R. Delanghe, F. Sommen. *Clifford Analysis.* Boston, Pitman, 1982.

[Do] J.L. Doob. *Classical Potential Theory and Its Probabilistic Counterpart.* Springer: New York Berlin Heidelberg, 1984.

[Dz] A. Dzhuraev. *On kernel matrices and holomorphic vectors.* Complex Variables, v. 16, 1991, p. 43–57.

[F1] C. Fefferman. *The Bergman kernel and biholomorphic mappings of pseudoconvex domains.* Inventiones Math., v. 26, 1974, p. 1–65.

[F2] C. Fefferman. *Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains.* Annals of Math., v. 103, 1976, p. 395–416.

[HK] W.K. Hayman, P.B. Kennedy. *Subharmonic Functions.* Academic Press; London New York San Francisco, 1976.

[R] R.M. Range. *Holomorphic Functions and Integral Representations in Several Complex Variables.* Springer: New York Berlin Heidelberg, 1986.

[SV] M.V. Shapiro, N.L. Vasilevski. *On the Bergman kernel function in hypercomplex analysis.* Reporte Interno No. 115, Departamento de Matemáticas, CINVESTAV del I.P.N., Mexico City, 1993, 35p.