ON THE ANALOGUE OF THE $\overline{\partial}$-PROBLEM IN QUATERNIONIC ANALYSIS*

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Abstract. The $\overline{\partial}$-problem in complex analysis has stimulated deep and fruitful investigations. Since two-dimensional complex analysis can be embedded into one-dimensional quaternionic analysis, one expects that the study of the analogous problem ${}^{\psi}\mathcal{D}[u] = \alpha$ in a domain $\Omega \subset \mathbf{H} = \{quaternions\}$, for fixed parameter $\psi \in \mathbf{H}^4$, will be useful and promising. We establish some basic facts concerning operators on quaternionic Hilbert modules, and establish connections among the hyperholomorphic Bergman projector, the so-called T-operator, and a special solution of the ${}^{\psi}\mathcal{D}$ equation.

Key words: $\overline{\partial}$-problem, hyperholomorphic function, Bergman projector

1. Preliminaries

We study here the analogue of the classical $\overline{\partial}$-problem, that is, ${}^{\psi}\mathcal{D}[u] = \alpha$, in one quaternionic variable, where

$$ {}^{\psi}\mathcal{D}[f] := \sum_{k=0}^{3} \psi^k \cdot \frac{\partial f}{\partial x_k} \ , $$

for suitable normalized coefficients ("structural set") $\psi = (\psi^0, \psi^1, \psi^2, \psi^3)$. This continues lines of investigation initiated in [7],[8]. In particular we establish basic properties of various operators related to this problem. Most of the results stated carry over without difficulty to Clifford algebras.

First we fix the basic notation and establish the main definitions related to Hilbert space with quaternionic scalars.

Let $\mathbf{H}$ be the set of quaternions $a = \sum_{k=0}^{3} a_k i_k$, where $a_k \in \mathbf{R}$, i_0 is the unit, and i_1, i_2, i_3 are the imaginary units.

Let $H = H^R$ be a right $\mathbf{H}$-module; that is, there is a multiplication $H \times \mathbf{H} \to H$ which is additive in both factors and for which

$$ (a\lambda)\mu = a(\lambda\mu) $$

* This work was partially supported by CONACYT project 1821-E9211

when $a \in H$ and λ, $\mu \in \mathbf{H}$. A *scalar product* on H^R is a map $\langle\cdot,\cdot\rangle_H : H \times H \to \mathbf{H}$ satisfying the following axioms: for any $a, b, c \in H$ and for any $\lambda \in \mathbf{H}$, we have

$$\begin{aligned}
&\langle a, b\rangle = \overline{\langle b, a\rangle}\ , \\
&\langle a, b\cdot\lambda\rangle = \langle a, b\rangle \cdot \lambda\ , \\
&\langle a + b, c\rangle = \langle a, c\rangle + \langle b, c\rangle\ , \quad \text{and} \\
&\langle a, a\rangle \geq 0 \text{ with } \ \langle a, a\rangle = 0 \text{ only when } \ a = 0\ .
\end{aligned}$$

In a left $\mathbf{H}$-module H^L, the second axiom is replaced by $\langle\lambda\cdot a,\, b\rangle = \lambda\cdot\langle a, b\rangle$. We will sometimes omit the "$\cdot$" indicating multiplication by a quaternion when there is no possibility of confusion.

An $\mathbf{H}$-module $H = H^R$ or H^L endowed with a scalar product is called a *Hilbert (right or left) module* if it is complete with respect to the norm generated by the scalar product $\|a\| := \sqrt{\langle a, a\rangle}$. A functional $\mathcal{X} : a \in H^R \mapsto \mathcal{X}(a) \in \mathbf{H}$ is *linear* when it satisfies $\mathcal{X}(a\cdot\lambda+b) = \mathcal{X}(a)\cdot\lambda+b$; in H^L this takes the form $\mathcal{X}(\lambda\cdot a+b) = \lambda\cdot\mathcal{X}(a)+b$. The Riesz representation theorem holds in a natural way for quaternionic modules.

Let $H = H^R$, and let the linear operator A be defined on $\operatorname{dom} A \subset H$ with $\operatorname{dom} A$ a nonclosed dense subspace. The domain of the adjoint is defined as usual by

$$\operatorname{dom} A^* := \{y \in H \,|\, \text{the functional } \mathcal{X} : x \mapsto \langle y, Ax\rangle \text{ is bounded on } \operatorname{dom} A\} \tag{1}$$

and by means of the Riesz theorem, the operator A^* is defined to satisfy $\langle y, Ax\rangle = \langle A^*y, x\rangle$ for $x \in \operatorname{dom} A, y \in \operatorname{dom} A^*$.

2. Hyperholomorphic Functions and Cauchy Integral-Type Operators

2.1. Hyperholomorphic Functions

Let Ω be a bounded domain in $\mathbf{R}^4$ with its boundary $\partial\Omega$ a compact, sufficiently smooth hypersurface (codimension 1 manifold). We work with $\mathbf{H}$-valued functions $f = \sum_{k=0}^{3} f^k \cdot i_k$ defined in Ω. $L^2(\Omega; \mathbf{H})$ has the structure of an $\mathbf{H}$-bimodule (that is, simultaneously a left and right $\mathbf{H}$-module), and the scalar product

$$\langle f, g\rangle := \int_\Omega \overline{f(t)} \cdot g(t) dx$$

makes $L^2(\Omega; \mathbf{H})$ into a right Hilbert $\mathbf{H}$-module, while $\int_\Omega f(t) \cdot \overline{g(t)dx}$ makes it into a left one. Usually we will take $L^2(\Omega; \mathbf{H})$ to be a right module.

Let $\psi := \{\psi^0, \cdots, \psi^3\} \in \mathbf{H}^4$. The operators ${}^{\psi}\!\mathcal{D}$ and $\mathcal{D}^{\psi}$ are defined on $C^1(\Omega; \mathbf{H})$ by the formulae

$${}^{\psi}\!\mathcal{D}[f] := \sum_{k=0}^{3} \psi^k \cdot \partial_k f\ , \qquad \mathcal{D}^{\psi}[f] := \sum_{k=0}^{3} \partial_k f \cdot \psi^k\ , \tag{2}$$

where $\partial_k = \partial/\partial x_k$. If $\triangle$ denotes the 4-dimensional Laplace operator, then the equality

$${}^{\psi}\!\mathcal{D} \cdot {}^{\overline{\psi}}\!\mathcal{D} = {}^{\overline{\psi}}\!\mathcal{D} \cdot {}^{\psi}\!\mathcal{D} = \mathcal{D}^{\psi} \cdot \mathcal{D}^{\overline{\psi}} = \mathcal{D}^{\overline{\psi}} \cdot \mathcal{D}^{\psi} = \triangle \tag{3}$$

is true if and only if for every j, k we have

$$\langle \vec{\psi^j}, \vec{\psi^k} \rangle_{\mathbf{R}^4} = \delta_{jk} \cdot \lambda \tag{4}$$

for some real constant λ (see [7], [8]). Any ψ which satisfies this will be called a *structural set*. We assume further, without any loss of generality, that $\lambda = 1$. The elements of the sets ${}^{\psi}\mathcal{M}(\Omega) := \ker {}^{\psi}\mathcal{D}$ and $\mathcal{M}^{\psi}(\Omega) := \ker \mathcal{D}^{\psi}$ are called left (respectively, right) ψ-hyperhomomorphic $\mathbf{H}$-valued functions in Ω. These spaces are right (respectively, left) Hilbert $\mathbf{H}$ modules.

Let $d\hat{x}_k$ denote the differential form obtained from $dx_0 \wedge \ldots \wedge dx_3$ with the factor dx_k omitted, and let $\sigma^{(3)}_{\psi,x} := \sum_{k=0}^{3} (-1)^k \ \ \psi^k d\hat{x}_k$. Denote by ds_3 the 3 dimensional volume element of a hypersurface in $\mathbf{R}^4$. Then $|\sigma^{(3)}_{\psi,x}| = ds_3$ and we call $\sigma^{(3)}_{\psi,x}$ the $\mathbf{H}$-valued area form associated to ψ.

Denote by $\theta_4(x) = (1/2\pi)|x|^{-2}$, $x \in \mathbf{R}^4 - \{0\}$, the fundamental solution of the Laplace operator. Denote by $\mathcal{K}_{\psi}$ the (common) fundamental solution of the operators ${}^{\psi}\mathcal{D}$ and $\mathcal{D}^{\psi}$, namely

$$\mathcal{K}_{\psi}(x) := \frac{1}{2\pi^2|x|^4} \sum_{k=0}^{3} \overline{\psi^k} \cdot x^k \ . \tag{5}$$

The following well known statements can be found in many sources (see, for example, [1], [7]). Here $C^2(\overline{\Omega}; \mathbf{H})$ denotes the subclass of functions which can be extended smoothly to an open set containing the closure $\overline{\Omega}$.

Integral form of the quaternionic Stokes formula. Let Ω be a bounded domain in $\mathbf{R}^4$ with a smooth boundary $\partial\Omega$, and let $f, g \in C^1(\overline{\Omega}; \mathbf{H})$. Then

$$\int_{\partial\Omega} g(x)\sigma_{\psi,x} f(x) = \int_{\Omega} (\mathcal{D}^{\psi}[g](x) f(x) + g(x)\, {}^{\psi}\mathcal{D}[f](x))dx. \tag{6}$$

Borel-Pompeiu (Cauchy-Green) quaternionic formula. Let Ω be a bounded domain in $\mathbf{R}^4$ with smooth boundary and $f \in C^1(\overline{\Omega}; \mathbf{H})$. Then

$$f(x) = \int_{\partial\Omega} \mathcal{K}_{\psi}(\tau - x)\sigma_{\psi,\tau} f(x) - \int_{\Omega} \mathcal{K}_{\psi}(\tau - \lambda)\, {}^{\psi}\mathcal{D}[f](\tau)d\tau, \quad x \in \Omega. \tag{7}$$

2.2. Quaternionic Distribution Theory

We now consider the notion of $\mathbf{H}$-valued generalized function. Let $C_0^{\infty}(\Omega; \mathbf{H})$ denote the set of all compactly supported functions in $C^{\infty}(\Omega; \mathbf{H})$. It is clear that $C_0^{\infty}(\Omega; \mathbf{H})$ is an $\mathbf{H}$-bimodule, and we consider it at will as a right or as a left module. Let $f \in L^1_{loc}(\Omega; \mathbf{H})$ be a locally integrable function. Then the mappings

$$f^{(R)} : \varphi \longmapsto \int_{\Omega} f(x) \cdot \varphi(x) dx$$

$$f^{(L)} : \varphi \longmapsto \int_{\Omega} \varphi(x) \cdot f(x) dx$$

are respectively right and left linear, continuous functionals on $C_0^\infty(\Omega;\mathbf{H})$ which are called right and left *regular generalized functions*, or right and left *regular distributions*. They satisfy natural linearity properties, which characterize right and left distributions in general. The definitions of ${}^{\psi}\mathcal{D}$ and $\mathcal{D}^{\psi}$ extend to distributions in the natural way. As is the case for holomorphic functions of a complex variable, distributions which are annihilated by ${}^{\psi}\mathcal{D}$ or $\mathcal{D}^{\psi}$ are regular distributions.

2.3. L^2 Adjoint Operator of ${}^{\psi}\mathcal{D}$

We want an operator acting on L^2, and thus are led to define

$$\operatorname{dom}{}^{\psi}\mathcal{D} := \left\{ f| \;\; f \in L^2 \text{and } \frac{\partial f}{\partial x_k} \in L^2, \text{ all } k \right\}. \tag{8}$$

which is dense in $L^2(\Omega;\mathbf{H})$. The following is easily verified by integration by parts.

Lemma 2.1 *Let $f, h \in C^1(\overline{\Omega};\mathbf{H})$. Then*

$$\langle h, {}^{\psi}\mathcal{D}[f]\rangle = \int_{\partial\Omega} \overline{h(x)} \cdot \sigma_{\psi,x} \cdot f(x) - \langle {}^{\overline{\psi}}\mathcal{D}[h], f\rangle \ .$$

Corollary 2.2 *If $f, h \in C^1(\overline{\Omega};\mathbf{H})$ and also for each $x \in \partial\Omega$ either $f(x) = 0$ or $h(x) = 0$, then $\langle h, {}^{\psi}\mathcal{D}[f]\rangle = -\langle {}^{\overline{\psi}}\mathcal{D}[h], f\rangle$.*

We begin now a more formal investigation of the domains of ${}^{\psi}\mathcal{D}$ and ${}^{\psi}\mathcal{D}^*$. The following is analogous to a result given by Kohn [3] in a complex analysis setting.

Theorem 2.3 *Let $h \in C^1(\overline{\Omega};\mathbf{H})$. Then $h \in \operatorname{dom}{}^{\psi}\mathcal{D}^*$ if and only if $h|\partial\Omega = 0$.*

Proof. First suppose $h \in \operatorname{dom}{}^{\psi}\mathcal{D}^*$. This means that for all $f \in \operatorname{dom}{}^{\psi}\mathcal{D}$, $\langle h, {}^{\psi}\mathcal{D}[f]\rangle = \langle {}^{\psi}\mathcal{D}^*[h], f\rangle$. By Corollary 2.2, for $\varphi \in C_0^1(\Omega;\mathbf{H})$ and $h \in \operatorname{dom}{}^{\psi}\mathcal{D}^*$,

$$\langle h, {}^{\psi}\mathcal{D}[\varphi]\rangle = -\langle {}^{\overline{\psi}}\mathcal{D}[h], \varphi]\rangle, \tag{9}$$

and so, taking into account the fact that $C_0^1(\Omega;\mathbf{H}) \subset \operatorname{dom}{}^{\psi}\mathcal{D}$, we have $\langle {}^{\psi}\mathcal{D}^*[h], \varphi\rangle = -\langle {}^{\overline{\psi}}\mathcal{D}[h], \varphi\rangle$ for all $\varphi \in C_0^1(\overline{\Omega};\mathbf{H})$. Since $C_0^1(\Omega;\mathbf{H})$ is dense in $L^2(\Omega;\mathbf{H})$ we obtain ${}^{\psi}\mathcal{D}^*[h] = -{}^{\overline{\psi}}\mathcal{D}[h]$. Therefore, again from Corollary 2.2 we see

$$\langle {}^{\psi}\mathcal{D}^*[h], \varphi\rangle = -\langle {}^{\overline{\psi}}\mathcal{D}[h], \varphi\rangle \tag{10}$$

for all $\varphi \in C^1(\overline{\Omega};\mathbf{H})$, and from Lemma 2.1 we have

$$\int_{\partial\Omega} \overline{h(x)} \cdot \sigma_{\psi,x} \cdot \varphi(x) = 0 \tag{11}$$

for all $\varphi \in C^1(\overline{\Omega};\mathbf{H})$. It is known that $\sigma_{\psi,x} = n_\psi(x) \cdot ds_3$, where $n_\psi(x) = \sum_{k=0}^{3} n_\psi(x) \cdot \psi^k$ with $n(x) := (n_0(x), \ldots, n_3(x))$ being the outward normal to $\partial\Omega$ at the point x.

Substitute $\varphi(x) := \overline{n_\psi(x)} \cdot h(x)$ in (11) to obtain $\int_{\partial\Omega} |h(x)|^2 \cdot ds = 0$, which implies $h|\partial\Omega = 0$.

For the reverse inclusion, let now $h|\partial\Omega = 0$. Then (11) is true, and thus by Lemma 2.1 the equality (9) holds. Since $C_0^\infty(\Omega; \mathbf{H})$ is dense in $L^2(\Omega; \mathbf{H})$, $\langle h, {}^{\psi}\mathcal{D}[\varphi]\rangle = -\langle {}^{\overline{\psi}}\mathcal{D}[h], \varphi\rangle$ for all $\varphi \in \operatorname{dom} {}^{\psi}\mathcal{D}$. Therefore $h \in \operatorname{dom} {}^{\psi}\mathcal{D}^*$ as claimed.

From (10) we see that $-{}^{\overline{\psi}}\mathcal{D}$, which has the same domain as ${}^{\psi}\mathcal{D}$, is an extension of ${}^{\psi}\mathcal{D}^*$. Theorem 2.3 characterizes the elements of the domain of ${}^{\psi}\mathcal{D}^*$ which are smooth up to the boundary. This is a dense set in the L^2 norm. From a Lemma of Hörmander [5, p. 95] it is not difficult to verify the following.

Theorem 2.4 *Let $\Omega \subset \mathbf{H}$ be a bounded domain with smooth (C^1) boundary. (i) The set $C^\infty(\Omega; \mathbf{H}) \cap \operatorname{dom} {}^{\psi}\mathcal{D}$ is dense in $\operatorname{dom} {}^{\psi}\mathcal{D}$ in the graph norm; (ii) $C^\infty(\Omega; \mathbf{H}) \cap \operatorname{dom} {}^{\psi}\mathcal{D}^*$ is dense in $\operatorname{dom} {}^{\psi}\mathcal{D}^*$ in the graph norm.*

3. The Quaternionic $\overline{\partial}$-Problem

The $\overline{\partial}$-problem $\overline{\partial}u = \alpha$ is of great importance in the theory of one and several complex variables. Here we will examine the corresponding problem for one quaternionic variable, namely

$$ {}^{\psi}\mathcal{D}[u] = \alpha. \tag{12} $$

$\Omega \subset \mathbf{H}$ will denote throughout a bounded domain for which $\partial\Omega$ is a smooth hypersurface. It is convenient to take into account spaces of functions more general than $u \in L^2(\Omega; \mathbf{H})$ since certain operators such as the T-operator do not necessarily conserve L^2. We limit the discussion to the left operator ${}^{\psi}\mathcal{D}$ since the theory for $\mathcal{D}^{\psi}$ is completely analogous.

We begin with the (weakly singular) integral operator ${}^{\psi}T$. In $\mathbf{H}$, as in the case of $\mathbf{C}^1$, no special "compatibility condition" is necessary for the existence of a solution. The solution given by ${}^{\psi}T$ is the one described by the normalization ${}^{\psi}K[u] = 0$. This has certain inconveniences, since ${}^{\psi}K$ is an unbounded operator, and ${}^{\psi}K$ cannot be applied to general L^2 functions, since it is defined in terms of their boundary values. We therefore contrast this approach with a second solution of (12) given by means of the quaternionic Bergman kernel; the characterization for this case is orthogonality to the hyperholomorphic L^2 functions.

3.1. T-Operator Solution

In this subsection we will collect some facts about the K and T operators which are essentially the same as for $\mathbf{C}^1$. They are mainly from Goldschmidt [2] and Iftimie [6] (in the more general context of Clifford algebras).

Recall the integral kernel $\mathcal{K}_\psi(x)$ given by (5), and define

$$ {}^{\psi}K[f](x) = \int_{\partial\Omega} \mathcal{K}_\psi(\tau - x)\sigma_{\psi,\tau} f(\tau) \tag{13} $$

for any f which is integrable on $\partial\Omega$. When $f \in C^0(\overline{\Omega}; \mathbf{H})$ we will write ${}^{\psi}K[f]$ for

${}^{\psi}K[f|\partial\Omega]$. Define further

$$ {}^{\psi}T[\alpha](x) := -\int_{\Omega} \mathcal{K}_{\psi}(\tau - x)\alpha(\tau)d\tau \tag{14} $$

for any $\alpha \in L^2(\Omega; \mathbf{H})$. It is known that then ${}^{\psi}T[\alpha] \in L^2$, and

$$ {}^{\psi}\mathcal{D} \cdot {}^{\psi}T[\alpha] = \alpha \ , \tag{15} $$

so the solution set of (12) is ${}^{\psi}T[\alpha] + {}^{\psi}\mathcal{A}_2(\Omega)$. However, for $f \in L^p(\Omega; \mathbf{H})$ the most we can say in general [2] is that ${}^{\psi}T[f] \in L^1_{\text{loc}}(\Omega; \mathbf{H}) = \mathrm{L}^1(\Omega; \mathbf{H})$. Therefore we may consider generalized derivatives of ${}^{\psi}T[f]$. With this proviso in mind, we have the following [6].

Proposition 3.1 *Let $p \geq 2$. Then ${}^{\psi}\mathcal{D}{}^{\psi}T[f] = f$ for $f \in L^p(\Omega; \mathbf{H})$.*

It follows that for $\alpha \in L^2(\Omega; \mathbf{H})$ we have ${}^{\psi}T[\alpha] \in \operatorname{dom} {}^{\psi}\mathcal{D}$. The following estimates may be obtained by arguments similar to those in [2].

Proposition 3.2 *Let $p > 4$. If $f \in L^p(\Omega; \mathbf{H})$ and $u := {}^{\psi}T[f] \in L^q(\Omega; \mathbf{H})$, with $1 < q < 4$, then*

$$ |u(z)| \leq M_1 \|f\|_{L^p} \ , \qquad z \in \mathbf{C} \ , $$
$$ |u(z_1) - u(z_2)| \leq M_1 \|f\|_{L^p} \cdot |z_1 - z_2|^{(p-4)/p} \ , \quad z_1, z_2 \in \Omega \ , $$

where M_1 is a constant for fixed p and Ω, and M_2 constant for fixed p. Thus ${}^{\psi}T(L^p(\Omega)) \subset C^{0,(p-4)/p}(\Omega; \mathbf{H})$.

From this we deduce that for $f \in L^p(\Omega; \mathbf{H})$, ${}^{\psi}T[f]$ extends continuously to $\partial\Omega$. Therefore it makes sense to speak of ${}^{\psi}K[{}^{\psi}T[f]]$. As a result, The Borel-Pompeiu formula (7), which we will write as

$$ I \ = \ {}^{\psi}K \ - \ {}^{\psi}T{}^{\psi}\mathcal{D} \ , \tag{16} $$

is valid when applied to functions in the range of ${}^{\psi}T$. For $f \in L^p(\Omega; \mathbf{H})$, $p > 4$, we have that $u = {}^{\psi}T[f]$ is the only solution of (12) with ${}^{\psi}K[u] = 0$.

3.2. Bergman Operator Solution

The hyperholomorphic Bergman kernel, introduced in [8], is given by

$$ {}^{\psi}B[f](x) \ := \ \int_{\Omega} {}_{\psi}\mathcal{B}(x, \tau) f(\tau) d\tau \tag{17} $$

where ${}_{\psi}\mathcal{B}(x,\tau) \ := {}^{\psi}\mathcal{D}_{\tau} \cdot \overline{{}^{\psi}\mathcal{D}}_{x}[g](x,\tau) = \overline{{}^{\psi}\mathcal{D}}_{x} \cdot {}^{\psi}\mathcal{D}_{\tau}[g](x,\tau)$, where $g(s,\tau)$ is the classical Green's function for $\Omega \subset \mathbf{R}^4$, and $(x,\tau) \in \overline{\Omega} \setminus$ diagonal . The basic facts pertaining to ${}^{\psi}B$ are as follows [8].

Proposition 3.3 *On any $L^p(\Omega; \mathbf{H})$, ($p > 1$) the relation ${}^{\psi}\mathcal{D}{}^{\psi}B = 0$ holds.*

Proposition 3.4 ${}^{\psi}B : L^2(\Omega; \mathbf{H}) \to {}^{\psi}\mathcal{A}_2(\Omega; \mathbf{H})$ *is a self-adjoint orthogonal projection.*

Proposition 3.5 *For functions in the range of ${}^{\psi}T$, ${}^{\psi}B{}^{\psi}K = {}^{\psi}K$.*

3.3. Relations Among the Operators

Now introduce the operator

$$ {}^{\psi}S \; := \; (I - {}^{\psi}B){}^{\psi}T. \tag{18} $$

which is defined on every $L^p(\Omega; \mathbf{H})$, $p > 1$. In $L^2(\Omega; \mathbf{H})$, clearly ${}^{\psi}S[\alpha]$ is the unique solution of (12) which is orthogonal to ${}^{\psi}\mathcal{A}_2(\Omega; \mathbf{H})$. We relate ${}^{\psi}S$ to ${}^{\psi}\mathcal{D}$, ${}^{\psi}K$, and ${}^{\psi}B$ as follows.

Theorem 3.6 ${}^{\psi}B = I - {}^{\psi}S{}^{\psi}\mathcal{D}$ *on* $\operatorname{dom}{}^{\psi}\mathcal{D}$.

Proof. By (18) and (16), ${}^{\psi}S{}^{\psi}\mathcal{D} = (I - {}^{\psi}B){}^{\psi}T{}^{\psi}\mathcal{D} = (I - {}^{\psi}B)(I - {}^{\psi}K) = I - {}^{\psi}B - {}^{\psi}K + {}^{\psi}B{}^{\psi}K$. By Corollary (3.3) this is equal to $I - {}^{\psi}B$, which proves the result.

Theorem 3.7 ${}^{\psi}K{}^{\psi}B = {}^{\psi}B$ *on the set* $\{f \in L^2(\Omega; \mathbf{H}) |\ {}^{\psi}B[f] \in C^1(\overline{\Omega}; \mathbf{H})\}$.

Proof. Using (16) and the previous result we have

$$ {}^{\psi}K{}^{\psi}B = (I - {}^{\psi}T{}^{\psi}\mathcal{D})(I - {}^{\psi}S{}^{\psi}\mathcal{D}) = I - {}^{\psi}T{}^{\psi}\mathcal{D} - {}^{\psi}S{}^{\psi}\mathcal{D} + {}^{\psi}T{}^{\psi}\mathcal{D}{}^{\psi}S{}^{\psi}\mathcal{D}\ . $$

But ${}^{\psi}\mathcal{D}{}^{\psi}S = {}^{\psi}\mathcal{D}({}^{\psi}T - {}^{\psi}B{}^{\psi}T) = {}^{\psi}\mathcal{D}{}^{\psi}T - D{}^{\psi}B{}^{\psi}T = I$ by Theorem 3.1 and Corollary 3.3. Therefore ${}^{\psi}K{}^{\psi}B = I - {}^{\psi}S{}^{\psi}\mathcal{D} = {}^{\psi}B$ by Theorem 3.6 again.

On $L^2(\Omega; \mathbf{H})$ the operator ${}^{\psi}B$ is bounded. In contrast, ${}^{\psi}K$, with an appropiate natural domain, is not bounded because it annihilates $C_0^{\infty}(\overline{\Omega}; \mathbf{H})$. From the above, it is of interest to contrast the projection $I - {}^{\psi}B$, whose image contains $({}^{\psi}\mathcal{A}_2)^{\perp}$, and $I - {}^{\psi}K$, whose image contains the image of ${}^{\psi}T$. It is not known whether the intersection of these two images is trivial. Similarly, it is not known what the kernel of ${}^{\psi}B - {}^{\psi}K$ looks like.

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