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Group Convolution Operators on Standard CR–manifolds. I. Structural Properties.

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Right, left and two–sided group convolution operators for a class of step two nilpotent Lie groups are considered. To understand the structure of these operators the direct integral construction and symplectic changes of variables are used.

As the application to the complex analysis the simple isomorphic description of the Hardy space and of the Szegö projector associated with the type II Siegel domains is given.

Introduction

In the paper we deal with the convolution operators on $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{C}^n$, endowed with the following group operation

$$(t_1, z_1) \cdot (t_2, z_2) = (t_1 + t_2 + 2\,\mathrm{Im}\; F(z_1, z_2), \quad z_1 + z_2), \tag{0.1}$$

where $(t_k, z_k) \in \mathbb{R}^m \times \mathbb{C}^n$, $k = 1, 2$, and $F(z_1, z_2)$ is some Hermitian form defined on $\mathbb{C}^n$ and valued in $\mathbb{C}^m$.

Let us denote by $\mathbb{H}_F$ the space $\mathbb{R}^m \times \mathbb{C}^n$ with the operation (0.1). It is known (see, for instance, [20, 28]) that $\mathbb{H}_F$ is a step two nilpotent Lie group and its (two-sided) Haar measure coincides with the Lebesque measure on $\mathbb{R}^N$. For $m = 1$ and $F(z_1, z_2) = z_1 \cdot \overline{z}_2$ $\mathbb{H}_F$ is just the Heisenberg group.

Objects of this kind appear naturally in complex analysis. Let us consider the so–called standard CR–manifold M which is selected in $\mathbb{C}^{m+n}$ by the rule

$$M = \{(w, z) \in \mathbb{C}^m \times \mathbb{C}^n : \;\; \mathrm{Im}\; w - F(z, z) = 0\}. \tag{0.2}$$

The mapping

$$\kappa_0 : \quad (t, z) \in \mathbb{R}^m \times \mathbb{C}^n \longmapsto (t + iF(z, z),\, z) \in M$$

identifies $\mathbb{R}^m \times \mathbb{C}^n$ with the CR-manifold M. If, in addition, the form F is nondegenerate then the manifold (0.2) coincides with the distinguished boundary of the following type II Siegel domain (see [9, 21])

$$D = \{(w, z) \in \mathbb{C}^m \times \mathbb{C}^n : \;\mathrm{Im}\; w - F(z, z) \in V\},$$

where a convex pointed cone $V(\subset \mathbb{R}^m)$ is the cone of values of the form F, *i.e.*

$$\overline{V} = \{t \in \mathbb{R}^m : \ \ t = F(z,z) \ \text{ for some } z \in \mathbb{C}^n\}.$$

The group of the biholomorphic automorphisms of the domain D [9, 21] of the form

$$\omega_{(a,c)} : \ \ (w,z) \longmapsto (w + a + 2iF(z,c) + iF(c,c), \ \ z + c),$$

for all $(a,c) \in \mathbb{R}^m \times \mathbb{C}^n$, preserves the distinguished boundary M and allots it with the Lie groups structure. If we transfer this structure from M to $\mathbb{R}^m \times \mathbb{C}^n$ with the help of the mapping κ_0 we obtain the operation (0.1) exactly.

The noncommutativity of the operation (0.1) induces (in contrast to the standard summation operation in $\mathbb{R}^N$) the existence of two shift operators: the right and the left shift operators are defined on $L_2(\mathbb{H}_F)$ by the rule

$$(\pi_r(g)\varphi)(h) = \varphi(hg), \tag{0.3}$$

$$(\pi_l(g)\varphi)(h) = \varphi(g^{-1}h). \tag{0.4}$$

Integrating these shift operators with some kernel functions we obtain the right and the left group convolution operators

$$B = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{H}_F} b(g)\, \pi_r(g)\, dg, \tag{0.5}$$

$$C = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{H}_F} c(g)\, \pi_l(g)\, dg. \tag{0.6}$$

The investigation of group convolution operators on $\mathbb{H}_F$ and pseudodifferential operators connected with them has sufficiently long history by this time. We mention here the works [1, 6-8, 10, 13, 14, 16, 18-20, 22, 27, 28] only, which have extensive bibliography.

As a rule the study of the convolution operators has been based on the use of the group Fourier transformation and that is why it has necessarily exploited the description of the irreducible representations the group $\mathbb{H}_F$.

Note, that the using of the Fourier transformation allows to consider the one ("right" or "left") case of shift and of convolution operators only.

The method of the investigation in this paper is based on another idea and allows to study the operators (0.5) and (0.6) simultaneously. Moreover it turns out to be useful to consider the two–sided convolution operator

$$K = (2\pi)^{-N} \int_{\mathbb{H}_F \times \mathbb{H}_F} k(g_1, g_2)\, \pi_r(g_1)\, \pi_l\,(g_2)\, dg_1\, dg_2. \tag{0.7}$$

The operators (0.3) and (0.4) are mutually commuting thus they may be written in (0.7) in an arbitrary order.

The main idea of the investigation as follows. The operators (0.5), (0.6) and (0.7) commute with the (left) shift operator $\pi_l(t,0)$ on elements $g = (t,0)$ of the center of $\mathbb{H}_F$. Thus they commute also with all elements of the von Neumann algebra $\mathfrak{A}$ generated by these shift operators π_l. Thus these convolution operators belong to the commutant

$$\mathfrak{A}' = F^{-1} L_\infty(\mathbb{R}^n) F \otimes \mathfrak{B}(L_2(\mathbb{C}^n))$$

of the algebra $\mathfrak{A}$. Here $\mathfrak{B}(X)$ is an algebra of all bounded linear operators acting on a Hilbert space X. Thus all of them have the form

$$(F^{-1} \otimes I)(I \otimes K(\lambda))(F \otimes I) \tag{0.8}$$

for some essentially bounded measurable operator–function

$$K(\cdot): \ \mathbb{R}^m \longmapsto \mathfrak{B}(L_2(\mathbb{C}^n)).$$

And in fact we come to the investigation of operators $K(\lambda)$ acting on the space $L_2(\mathbb{C}^n)$. Selecting then the appropriate symplectic change of variables in $T^*(\mathbb{C}^n)$ it is possible to reduce the operators $K(\lambda)$ to a simple canonical form.

The paper is organized in the following way: Section 1 contains all (mainly known) facts being necessary later on and describing the structure of the standard CR-manifolds.

In Section 2 the two–sided group convolution operators (0.7) with L_1-kernels are considered. The representation (0.8) is justified with the help of the direct integral construction. The formulas for the Weyl symbols of the operators $K(\lambda)$ and K are given.

Section 3 contains the reduction of $K(\lambda)$ to some simple canonical form.

In Section 4 the boundedness on $L_2(\mathbb{H}_F)$ of the group convolution operators is proved. The results of this Section extend the known ones (see, for example, [2, 13, 14]) onto the case of the noncontinuous Fourier image of the convolution operator kernel with respect to the “real” variable $\lambda \in \mathbb{R}^m$.

Section 5 contains some applications to the complex analysis. We give here the simple and more evident than, for example, in [9, 17, 20] isomorphic description of the Hardy space and of the Szegö projector associated with the type II Siegel domains.

Some results of this paper were announced in [30-32].

1 Standard CR–manifolds

1.1

Let us denote by $M(\subset \mathbb{C}^N)$ the standard Cauchy–Riemann manifold *) of the following form

$$M = \{(w, z) \in \mathbb{C}^m \times \mathbb{C}^n : \ \mathrm{Im}\, w - F(z, z) = 0\}, \tag{1.1}$$

where $N = m + n$, F is a $\mathbb{C}^m$–valued Hermitian form in $\mathbb{C}^n$.

The notation

$$\tilde{z} = (w, z) = (t, s, x, y)$$

will be used for the points of the space $\mathbb{C}^N = \mathbb{C}^m \times \mathbb{C}^n = \mathbb{R}^{2m} \times \mathbb{R}^{2n}$.

Denote by $T(\mathbb{C}^N)$ and by $T^*(\mathbb{C}^N)$ the tangent and the cotangent bundles over $\mathbb{C}^N$. As usual we take the system

$$\left\{\frac{\partial}{\partial t_v}, \frac{\partial}{\partial s_v}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\right\} \quad (\{dt_\nu, ds_\nu, dx_j, dy_j\}),\ \nu = \overline{1,m},\ j = \overline{1,n},$$

*) For references see, for instance, [29].

for the orthonormalized basis in fibres of the bundle $T(\mathbb{C}^N)$ $(T^*(\mathbb{C}^N))$. The existence of the complex structure in fibres of $T(\mathbb{C}^N)$ is displayed in the existence of an automorphism

$$J: \;\; T(\mathbb{C}^N) \longrightarrow T(\mathbb{C}^N), \quad J^2 = -id,$$

which continues the operator of multiplication by i onto fibres of $T(\mathbb{C}^n)$.

The operator J acts on the basis vectors by the rule

$$J\left(\frac{\partial}{\partial s}\right) = -\frac{\partial}{\partial t}, \;\; J\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial s}, \;\; J\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}, \;\; J\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}.$$

Then the adjoint operator J^* acts on the dual basis by the rule

$$J^*(ds) = dt, \quad J^*(dt) = -ds, \quad J^*(dy) = dx, \quad J^*(dx) = -dy.$$

The imbedding of the manifold M into $\mathbb{C}^N$ generates an injection

$$\text{in: } T(M) \longrightarrow T(\mathbb{C}^N)|_M.$$

We will identify $T(M)$ with its image in $T(\mathbb{C}^N)|_M$. The adjoint morphism

$$\text{in}^* \;:\; T(\mathbb{C}^N)|_M \longrightarrow T^*(M)$$

is a surjection. Its kernel coincides with the conormal bundle $N^*(M)$ of the manifold M. Thus a vector field X of the bundle $T(\mathbb{C}^N)|_M$ belongs to $\Gamma(T(M))$ if and only if $\Theta(X) = 0$ for all forms $\Theta \in \Gamma(N^*(M))$.

There exists a maximal subbundle of $T(M)$ among all subbundles which are invariant under the action of the operator J. This subbundle is called a complex tangent bundle of M and is denoted by $T^{\mathbb{C}}(M)$. We have

$$T^{\mathbb{C}}(M) = T(M) \bigcap JT(M) = \{\bigcap \ker \Theta : \quad \Theta \in \Gamma(N^*(M) + J^*N^*(M))\}.$$

Recall that each smooth section X of the bundle $T(M)$ generates an operator (which we will denote by the same letter) acting on the space $C^\infty(M)$ by the rule: $(Xf)(\tilde{z})$ is the derivative of $f \in C^\infty(M)$ along the direction $X(\tilde{z})$, $\tilde{z} \in M$.

The operator J has a complex spectrum (sp $J = \{\pm i\}$), therefore it is appropriate to pass to the complexification $T^{\mathbb{C}}(M) \otimes_{\mathbb{R}} \mathbb{C}$ of the bundle $T^{\mathbb{C}}(M)$. The bundle $T^{\mathbb{C}}(M) \otimes_{\mathbb{R}} \mathbb{C}$ is decomposed into the direct sum $T^{(1,0)} \oplus T^{(0,1)}$ of its “holomorphic” and “antiholomorphic parts, where

$$T^{(1,0)}(M) = \{X - iJX : \quad X \in T^{\mathbb{C}}(M)\}, \; T^{(0,1)}(M) = \{X + iJX : \quad X \in T^{\mathbb{C}}(M)\}. \quad (1.2)$$

In this context the operator J acts invariantly on the space of sections of each subbundle (1.2) and is reduced there to the multiplication (by i and $-i$ respectively) operator.

Finally a function $f \in C^\infty(M)$ is called a CR–function if $Xf = 0$ for all $X \in \Gamma(T^{(0,1)}(M))$.

1.2

The geometry of the standard CR–manifold M (1.1) is closely connected with the vector Hermitian form F.

The set

$$V(F) = \{t \in \mathbb{R}^m : \ t = F(z,z), \ \ z \in \mathbb{C}^n\}$$

is a cone, obviously. Denote by $E(F)$ the vector subspace in $\mathbb{R}^m$ generated by the cone $V(F)$ (we will write E and V later on).

The decomposition $\mathbb{R}^m = E \oplus E^\perp$ generates the representation of M in the direct product form $M = M_0 \times M_1$, where

$$\begin{aligned} M_0 &= \{w_0 \in E^\perp \otimes_{\mathbb{R}} \mathbb{C} : \ \text{Im } w_0 = 0\}, \\ M_1 &= \{(w_1, z) \in (E \otimes_{\mathbb{R}} \mathbb{C}) \times \mathbb{C}^n : \ \text{Im } w_1 - F(z,z) = 0\}. \end{aligned}$$

The manifold M_1 is a standard CR–manifold with smaller dimension (if $M_0 \neq \{0\}$), and M_0 is a completely real manifold. Further we will assume that coordinates in $\mathbb{C}^m$ are compatible with the decomposition

$$\mathbb{C}^m = \mathbb{R}^m \otimes_{\mathbb{R}} \mathbb{C} = (E \otimes_{\mathbb{R}} \mathbb{C}) \oplus (E^\perp \otimes_{\mathbb{R}} \mathbb{C}).$$

Let $(F_1, F_2, \ldots, F_m)$ be the coordinate notation of the vector form F. Here $F_\nu, \ \ \nu = \overline{1,m}$, are the scalar Hermitian forms in $\mathbb{C}^n$ and thus they have the form

$$F_\nu(z_1, z_2) = z_1' F_\nu \overline{z}_2, \ \ \nu = \overline{1,m}. \tag{1.3}$$

As usual a form and a corresponding matrix are denoted by the same letter.

From (1.3) we have

$$\frac{\partial F_\nu}{\partial z_j} = e_j' F_\nu \overline{z} = F_\nu(e_j, z) = \overline{F_\nu(z, e_j)}, \ \frac{\partial F_\nu}{\partial \overline{z}_j} = z' F_\nu e_j = F_\nu(z, e_j), \ \frac{\partial^2 F_\nu(z,z)}{\partial z_j \partial \overline{z}_k} = F_\nu(e_j, e_k),$$

where $e_j = (0, \ldots, 0, \ 1_{\underline{j^{th}} \text{ place}}, \ 0, \ldots, 0) \in \mathbb{R}^n$.

Then

$$\frac{\partial F_\nu(z,z)}{\partial x_j} = \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \overline{z}_j}\right) F_\nu(z,z) = 2\,\text{Re } F_\nu(z, e_j), \tag{1.4}$$

$$\frac{\partial F_\nu(z,z)}{\partial y_j} = \frac{1}{i}\left(\frac{\partial}{\partial \overline{z}_j} - \frac{\partial}{\partial z_j}\right) F_\nu(z,z) = 2\,\text{Im } F_\nu(z, e_j). \tag{1.5}$$

Now let $\lambda \in \mathbb{R}^m$. Denote by $L(\lambda)$ the following Hermitian form

$$L(\lambda)(z_1, z_2) = \lambda \cdot F(z_1, z_2) = \sum_{\nu=1}^{m} \lambda_\nu \cdot F_\nu(z_1, z_2). \tag{1.6}$$

It can be shown that $L(\lambda)$ coincides with the Levi form of the manifold M along the direction λ. This form will play an important role further.

1.3

Rewrite the definition of the manifold M in the form

$$M = \{\tilde{z} \in \mathbb{C}^N; \quad r(\tilde{z}) = 0\},$$

where $r(\tilde{z}) = (r_1, \dots, r_m)$ and $r_\nu = \mathrm{Im}\ w_\nu - F_\nu(z, z), \quad \nu = \overline{1, m}.$

The conormal bundle $N^*(M)$ of the manifold M is generated by the vector form dr. We have

$$dr_\nu = \sum_{k=1}^{m} \left(\frac{\partial r_\nu}{\partial t_k} dt_k + \frac{\partial r_\nu}{\partial s_k} ds_k \right) + \sum_{j=1}^{n} \left(\frac{\partial r_\nu}{\partial x_j} dx_j + \frac{\partial r_\nu}{\partial y_j} dy_j \right).$$

It is easy to see that $\frac{\partial r_\nu}{\partial t_k} = 0$ and $\frac{\partial r_\nu}{\partial s_k} = \delta_{\nu,k}$, where $\delta_{\nu,k}$ is the Kronecker delta.

According to (1.4) and (1.5) we have

$$dr_\nu = ds_\nu - 2 \sum_{j=1}^{n} \{\mathrm{Re}\ F_v(z, e_j)\, dx_j + \mathrm{Im}\ F_\nu(z, e_j)\, dy_j\} = ds_\nu - 2\, \mathrm{Re}\ F_\nu(z, dz).$$

Thus

$$dr = ds + 2\, \mathrm{Re}\ F(z, dz). \tag{1.7}$$

The equality $J^* dz = i\, dz$ implies

$$J^* dr = dt - 2\, \mathrm{Re}\ F(z, idz) = dt + 2\, \mathrm{Re}\ iF(z, dz) = dt - 2\, \mathrm{Im}\ F(z, dz). \tag{1.8}$$

The formulas (1.7) and (1.8) show that

$$N^*_{\tilde{z}}(M) \bigcap J^* N^*_{\tilde{z}}(M) = \{0\}.$$

Thus the vector space $N^*_{\tilde{z}}(M) + J^* N^*_{\tilde{z}}(M)$ is a $2m$–dimensional one and is generated by the vectors $dr_v(\tilde{z})$ and $J^* dr_\nu(\tilde{z}), \quad v = \overline{1, m}.$

Set now

$$\mathfrak{X}^0_j = \frac{1}{2} \frac{\partial}{\partial x_j} + \sum_{v=1}^{m} (\mathrm{Im}\ F_\nu(z, e_j)\, \frac{\partial}{\partial z_\nu} + \ \mathrm{Re}\ F_\nu\, (z, e_j) \frac{\partial}{\partial s_\nu}), \ \ j = \overline{1, n}. \tag{1.9}$$

It is obvious that vector fields $\mathfrak{X}^0_j$ are the smooth sections of the bundle $T(\mathbb{C}^N)|_M$ and the system $\{\mathfrak{X}^0_j\}_{j=1}^n$ is linearly independent in each point $\tilde{z} \in M$.

Lemma 1.1 *The following inclusions*

$$\mathfrak{X}^0_j \in \Gamma(T^{\mathbb{C}}(M)), \ \ j = \overline{1, n},$$

hold.

The proof of the lemma is reduced to the verification of the following equalities:

$$dr(\mathfrak{X}^0_j) = 0 \ \text{ and } \ \ J^* dr(\mathfrak{X}^0_j) = 0. \square$$

Let $\mathfrak{X}^0 = (\mathfrak{X}^0_1, \mathfrak{X}^0_2, \dots, \mathfrak{X}^0_n)$, then (1.9) shows that

$$\mathfrak{X}^0 = \frac{1}{2} \frac{\partial}{\partial x} - 2\, \mathrm{Im}\ L(\frac{\partial}{\partial w}) \overline{z}.$$

Introduce

$$\Upsilon^0 = J\mathfrak{X}^0 = \frac{1}{2}\frac{\partial}{\partial y} - 2\,\mathrm{Im}\ L(i\frac{\partial}{\partial w})\overline{z} \ = \ \frac{1}{2}\frac{\partial}{\partial y} - 2\,\mathrm{Re}\ L(\frac{\partial}{\partial w})\overline{z}.$$

Lemma 1.1 implies that the system of the vector fields $\{\mathfrak{X}^0_j(\tilde{z}), \Upsilon^0_j(\tilde{z})\}^n_{j=1}$ is a basis in $T^{\mathbb{C}}_{\tilde{z}}(M)$.

1.4

Let us consider the tangential Cauchy–Riemann equation on M. It is know that a function $f \in C^\infty(M)$ is a CR–function if it satisfies the tangential Cauchy–Riemann equation on M, *i.e.*

$$(X + iJX)f = 0 \tag{1.10}$$

for all $X \in \Gamma(T^{\mathbb{C}}(M))$.

It is obvious that (1.10) must be verified for $X = \mathfrak{X}^0_j,\ \ j = \overline{1, n}$, only. Let

$$\overline{Z} = \mathfrak{X}^0 + iJ\mathfrak{X}^0 = \mathfrak{X}^0 + i\Upsilon^0,$$

then

$$\begin{aligned}\overline{Z} \ &= \ \frac{\partial}{\partial\overline{z}} - 2\ \mathrm{Im}\ L(\frac{\partial}{\partial w})\overline{z} - 2\ i\,\mathrm{Re}\ L(\frac{\partial}{\partial w})\overline{z} = \\ &= \ \frac{\partial}{\partial\overline{z}} - 2i\,\overline{L}(\frac{\partial}{\partial\overline{w}})z = \frac{\partial}{\partial\overline{z}} - 2i(\frac{\partial}{\partial\overline{w}}\cdot\overline{F})z.\end{aligned}$$

The similar calculations give

$$Z = \mathfrak{X}^0 - i\Upsilon^0 = \frac{\partial}{\partial z} + 2iL(\frac{\partial}{\partial w})\overline{z}.$$

The following theorem holds.

Theorem 1.2 *A function $f \in C^\infty(M)$ is a CR–function on M if and only if*

$$\frac{\partial f}{\partial\overline{z}} - 2i\overline{L}(\frac{\partial f}{\partial\overline{w}})z = 0.$$

1.5

The mapping $\kappa_0 : \mathbb{R}^m \times \mathbb{C}^n \longrightarrow M$ defined by the rule

$$\kappa_0(t, z) = (t + iF(z, z),\ z)$$

identifies the space $\mathbb{R}^m \times \mathbb{C}^n$ with M. Then in usual way each bundle ξ on M generates the induced bundle $\kappa^*_0(\xi)$ on $\mathbb{R}^m \times \mathbb{C}^n$. Thus all bundles on M mentioned above can be transferred onto $\mathbb{R}^m \times \mathbb{C}^n$. Besides each function f on M generates the function $f_0 = f \circ \kappa_0$ on $\mathbb{R}^m \times \mathbb{C}^n$

$$f_0(t, z) = f(t + iF(z, z),\ z), \quad (t, z) \in \mathbb{R}^m \times \mathbb{C}^n.$$

It easy to see that

$$\begin{aligned}\frac{\partial f_0}{\partial x} &= \frac{\partial f}{\partial x} + \sum_{\nu=1}^{m} \frac{\partial f}{\partial s_\nu} \cdot \frac{\partial F_\nu(z,z)}{\partial x} = \frac{\partial f}{\partial x} + 2\ \mathrm{Re}\ L(\frac{\partial f}{\partial s})\overline{z},\\ \frac{\partial f_0}{\partial y} &= \frac{\partial f}{\partial y} + \sum_{\nu=1}^{m} \frac{\partial f}{\partial s_0} \cdot \frac{\partial F_\nu(z,z)}{\partial y} = \frac{\partial f}{\partial y} - 2\ \mathrm{Im}\ L(\frac{\partial f}{\partial s})\overline{z}\end{aligned}$$

and that

$$\frac{\partial f_0}{\partial t} = \frac{\partial f}{\partial t}.$$

Thus

$$\kappa_0^*(\mathfrak{X}^0) = \kappa_0^* \left(\frac{1}{2}\frac{\partial}{\partial x} - \mathrm{Im}\ L(\frac{\partial}{\partial t})\overline{z} + \mathrm{Re}\ L(\frac{\partial}{\partial s}\overline{z}) \right) = \frac{1}{2}\frac{\partial}{\partial x} - \ \mathrm{Im}\ L(\frac{\partial}{\partial t})\overline{z}.$$

The similar calculation gives

$$\kappa_0^*(\Upsilon^0) = \frac{1}{2}\frac{\partial}{\partial y} - \ \mathrm{Re}\ L(\frac{\partial}{\partial t})\overline{z}.$$

We will preserve the notations $\mathfrak{X}^0, \Upsilon^0, \overline{Z}$ and Z for their images under the mapping κ_0^*. Thus (on $\mathbb{R}^m \times \mathbb{C}^n$):

$$\mathfrak{X}^0 = \frac{1}{2}\frac{\partial}{\partial x} - \ \mathrm{Im}\ L(\frac{\partial}{\partial t})\overline{z}, \tag{1.11}$$

$$\begin{aligned}\Upsilon^0 &= \frac{1}{2}\frac{\partial}{\partial y} - \mathrm{Re}\ L(\frac{\partial}{\partial t})\overline{z}, &(1.12)\\ \overline{Z} &= \frac{\partial}{\partial \overline{z}} - i\overline{L(\frac{\partial}{\partial t})}z, &(1.13)\\ Z &= \frac{\partial}{\partial z} + iL(\frac{\partial}{\partial t})\overline{z}. &(1.14)\end{aligned}$$

1.6

Let us consider now the case when M coincides with the distinguished boundary of a type II Siegel domain. Recall that the Siegel domain of the type II has the following form

$$D = \{(w,z) \in \mathbb{C}^{m+n} : \ \ \mathrm{Im}\ w - F(z,z) \in V\},$$

where V is a convex pointed cone in $\mathbb{R}^m$, and a nondegenerate Hermitian form F has an additional property: for all $z \in \mathbb{C}^n$ $F(z,z) \in \overline{V}$.

The distinguished boundary of the domain D is the following CR–manifold

$$B = \{(w,z) \in \mathbb{C}^{m+n} : \mathrm{Im}\ w - F(z,z) = 0\}.$$

Thus B is a special case of the standard CR–manifolds mentioned above. It is known (see, for instance, [13, §7]) that a function f defined on the distinguished boundary admits an analytic extension into the domain D if only it f is a CR–function (*i.e.* satisfies the tangential Cauchy–Riemann equations).

Identify B with $\mathbb{R}^m\times\mathbb{C}^n$ with the help of the mapping $\kappa_0:(t,z)\longmapsto(t+iF(z,z),z)$. Thus the tangential Cauchy–Riemann equations have the form

$$\overline{Z}f_0=\left(\frac{\partial}{\partial\overline{z}}-i\overline{L(\frac{\partial}{\partial t})z}\right)f_0=0, \tag{1.15}$$

where $f_0(t,z)=f(t+iF(z,z),\ z)$.

Consider the space $L_2(\mathbb{R}^m\times\mathbb{C}^n)$ and its subspace $H^2(\mathbb{R}^m\times\mathbb{C}^n)$ which is a closure in $L_2(\mathbb{R}^m\times\mathbb{C}^n)$ of a linear vector space of the functions satisfying the equations (1.15). The operator $F_I=F_t\otimes I$ of the partial Fourier transformation with respect to the variable t gives an isometric isomorphism of the space $L_2(\mathbb{R}^m\times\mathbb{C}^n)$ under which the space $H^2(\mathbb{R}^m\times\mathbb{C}^n)$ turns into a definite space H_1^2.

The space H_1^2 is a closure in $L_2(\mathbb{R}^m\times\mathbb{C}^n)$ of a linear vector space of the functions $\widetilde{f}(\lambda,z)$ satisfying the following equations

$$F_I\overline{Z}F_I^{-1}\widetilde{f}=\left(\frac{\partial}{\partial\overline{z}}+\overline{L(\lambda)z}\right)\widetilde{f}=0.$$

Introduce the space $L_2(\mathbb{R}^m\times\mathbb{C}^n,d\mu_n)$ with the measure

$$d\mu_n=e^{-2\lambda\cdot F(z,z)}\,d\,\mathrm{Re}\ z\,d\,\mathrm{Im}\ z\,d\lambda,$$

and the isometric isomorphism

$$\nu_1:\ \ L_2(\mathbb{R}^m\times\mathbb{C}^n)\longrightarrow L_2(\mathbb{R}^m\times\mathbb{C}^n,d\mu_n)$$

defined by the rule

$$\nu_1(\widetilde{f})=\widetilde{f}e^{\lambda\cdot F(z,z)}=\widetilde{f}e^{z'L(\lambda)\overline{z}}=\varphi(\lambda,z).$$

Using the equality $z'L(\lambda)\overline{z}=(\overline{L(\lambda)}z)\cdot\overline{z}$ we get

$$\frac{\partial\varphi}{\partial\overline{z}}=\frac{\partial\widetilde{f}}{\partial\overline{z}}e^{z'L(\lambda)\overline{z}}+\widetilde{f}\,\overline{L(\lambda)}\cdot ze^{z'L(\lambda)\overline{z}}=e^{z'L(\lambda)\overline{z}}\left(\frac{\partial}{\partial\overline{z}}+\overline{L(\lambda)}z\right)\widetilde{f}.$$

Thus $\widetilde{f}\in H_1^2$ if and only if $\widetilde{f}\in L_2(\mathbb{R}^m\times\mathbb{C}^n)$ and the function $\varphi(\lambda,z)=\widetilde{f}e^{\lambda\cdot F(z,z)}$ is an entire function with respect to z for all $\lambda\in\mathbb{R}^m$. In particular it implies

$$\mathrm{supp}\ \widetilde{f}(\lambda,z)\subset\{(\lambda,z):\ \ z\in\mathbb{C}^n,\ \lambda\in V^*\}.$$

And we have

Theorem 1.3 (S. Gindikin [9], A. Koranyi, E. Stein [17]). *The operator $v_1\circ F_I$ gives an isometric isomorphism of the space $H^2(\mathbb{R}^m\times\mathbb{C}^n)$ onto the space $L_2^a(V^*\times\mathbb{C}^n)$ of all the functions $\varphi(\lambda,t)$ being entire with respect to z and with the norm*

$$\|\varphi\|^2=\int_{V^*}\int_{\mathbb{C}^n}|\varphi(\lambda,z)|^2d\mu_n.$$

A function $f(w,z)\in H^2(\mathbb{R}^m\times\mathbb{C}^n)$ is reestablished from $\varphi(\lambda,z)$ by the rule

$$f(w,z) = (2\pi)^{-\frac{m}{2}} \int_{V^*} e^{i\lambda\cdot w} \varphi(\lambda, z)\, d\lambda.$$

Really we have

$$\begin{aligned}
\left(F_I^{-1} \circ \nu_1^{-1}\right) \varphi &= (2\pi)^{-\frac{m}{2}} \int_{V^*} e^{i\lambda\cdot t} e^{-\lambda\cdot F(z,z)} \varphi(\lambda, z)\, d\lambda = \\
&= (2n)^{-\frac{m}{2}} \int_{V^*} e^{i\lambda\cdot(t+iF(z,z))} \varphi(\lambda, z)\, d\lambda = \\
&= (2\pi)^{-\frac{m}{2}} \int_{V^*} e^{i\lambda\cdot w} \varphi(\lambda, z)\, d\lambda.
\end{aligned}$$

□

Corollary 1.4 *Let $f(w,z) \in H^2(\mathbb{R}^m \times \mathbb{C}^n)$. Then for all $s \in V$ we have*

$$\int_B |f(t+is+iF(z,z),z)|^2 db \leq \|f\|^2_{L_2(B)}.$$

In fact

$$f(t+is+iF(z,z),z) = (2\pi)^{-\frac{m}{2}} \int_{V^*} e^{i\lambda\cdot t - \lambda\cdot s} e^{-\lambda\cdot F(z,z)}\, \varphi(\lambda, z)\, d\,\lambda = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{i\lambda\cdot t-\lambda\cdot s} \widetilde{f}(\lambda, z)\, d\,\lambda.$$

For all $s \in V$ (and $\lambda \in V^*$) we have $\lambda \cdot s \geq 0$ and $e^{-\lambda s} \leq 1$, thus

$$\|f(t+is+iF(z,z),z)\|^2 = \|e^{-\lambda s} \widetilde{f}(\lambda,z)\|^2 \leq |\widetilde{f}(\lambda,z)\|^2 = \|f\|^2.$$

□

2 Group convolution operators with L_1–kernels

2.1

It is known (see, for instance, [29]) that standard CR–manifolds (1.1) are homogeneous. The group of affine transformations of the form

$$\omega_{(a,b)} : (w,z) \longmapsto (w + a + 2iF(z,b) + iF(b,b),\ z+b),$$

where $(a,b) \in \mathbb{R}^m \times \mathbb{C}^n$, acts transitively on the manifold M

$$M = \{(w,z) \in \mathbb{C}^m \times \mathbb{C}^n : \ \ \mathrm{Im}\ w - F(z,z) = 0\}.$$

The mapping $\kappa_0 : (t,z) \longmapsto (t+iF(z,z),z)$, which identifies $\mathbb{R}^m \times \mathbb{C}^n$ with M, allows to transfer the described group of transformations onto $\mathbb{R}^m \times \mathbb{C}^n$. The calculations show that

$$\omega_{(a,b)}(t,z) = (t + a - 2\,\mathrm{Im}\ F(z,b),\ z+b), \quad (a,b) \in \mathbb{R}^m \times \mathbb{C}^n.$$

The identification of points $(t,z) \in \mathbb{R}^m \times \mathbb{C}^n$ with transformations $\omega_{(t,z)}$ provides the space $\mathbb{R}^m \times \mathbb{C}^n$ with the Lie group structure with the operation

$$(t_1,z_1)\cdot(t_2,z_2) = (t_1 + t_2 + 2\,\mathrm{Im}\ F(z_1,z_2),\ z_1+z_2). \tag{2.1}$$

Denote this group by $\mathbb{H}_F$. Note that in the case $m = 1$ and when the scalar Hermitian form F is defined by the unit $n \times n$ matrix e_n, the group $\mathbb{H}_F = \mathbb{H}_{e_n}$ coincides with the n–dimensional Heisenberg group $\mathbb{H}_n$.

It is known that the group $\mathbb{H}_F$ is an unimodular nilpotent Lie group of step two and its (two–sided) Haar measure coincides with the Lebesgue measure in $\mathbb{R}^m \times \mathbb{C}^n = \mathbb{R}^N$, $N = m + 2n$.

Introduce the group convolution operators on $\mathbb{H}_F$. Let π_r and π_l be the right and the left regular representations of the group $\mathbb{H}_F$ on $L_2(\mathbb{H}_F)$, respectively. The operators $\pi_r(g)$ and $\pi_l(g)$, for $g \in \mathbb{H}_F$, have the following form

$$\begin{aligned} (\pi_r(g)f)(h) &= f(hg), \\ (\pi_l(g)f)(h) &= f(g^{-1}h). \end{aligned}$$

As usual the representations π_r and π_l are extended up to the representations of the group algebra of $\mathbb{H}_F$. In particular the right and the left group convolution operators which correspond to the functions $b, c \in L_1(\mathbb{H}_F)$ have the following form

$$B = \pi_r(b) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{H}_F} b(g)\, \pi_r(g)\, dg, \tag{2.2}$$

$$C = \pi_l(c) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{H}_F} c(g)\, \pi_l(g)\, dg. \tag{2.3}$$

Parallel with "one–sided" group convolution operators introduce the "two–sided" group convolution operator, *i.e.* for a function $k \in L_1(\mathbb{H}_F \times \mathbb{H}_F)$ define

$$K = (2\pi)^{-N} \int_{\mathbb{H}_F \times \mathbb{H}_F} k(g_1, g_2)\, \pi_r(g_1)\, \pi_l(g_2)\, dg_1\, dg_2. \tag{2.4}$$

Note, that the operators $\pi_r(g_1)$ and $\pi_l(g_2)$ are mutually commuting, thus they may be written in (2.4) in an arbitrary order. Note also, that putting in (2.4), for instance, $k(g_1, g_2) = (2\pi)^{-\frac{m}{2}} b(g_1) \cdot \delta(g_2)$, where δ is the Dirac δ–function, we obtain (2.2).

The operators of the right and of the left regular representations admit simple decompositions. In fact, let $g = (\tau, w) = (\tau, u, v)$, where $\tau \in \mathbb{R}^m$, $w \in \mathbb{C}^n$, $u, v \in \mathbb{R}^n$. Introduce the operators

$$\begin{aligned} (\pi'_g \varphi)(t, z) &= \varphi(t + 2\, \mathrm{Im}\, F(z, w),\, z), \\ (\pi''_g \varphi)(t, z) &= \varphi(t + \tau,\, z + w). \end{aligned}$$

Then we have

$$\pi_r(g) = \pi'_g \cdot \pi''_g, \quad \pi_l(g) = \pi'_g \cdot \pi''_{-g}, \tag{2.5}$$

and the operators π'_g and π''_g are mutually commuting.

Now introduce the (unbounded on L_2) operators

$$T = \frac{1}{i} \cdot \frac{\partial}{\partial t}, \quad D_x = \frac{1}{i} \cdot \frac{\partial}{\partial x}, \quad D_y = \frac{1}{i} \cdot \frac{\partial}{\partial y}.$$

Then it is obvious that

$$\begin{aligned} \pi'_g &= e^{2i\,\mathrm{Im}\ F(z,w)\cdot T}, \\ \pi''_g &= e^{i(\tau\cdot T+u\cdot D_x+v\cdot D_y)}, \\ \pi''_{-g} &= e^{-i(\tau\cdot T+u\cdot D_x+v\cdot D_y)}. \end{aligned} \tag{2.6}$$

The formulas (2.6) together with (2.5) allow to represent the right and the left group shift operators in the terms of the exponential of vector fields.

Remind (see, for instance, [10]) that the Campbell–Hausdorff formula establishes in the algebra of the formal power series of two non–commutative variables X and Y the following equality:

$$e^X\cdot e^Y = e^{A(X,Y)}, \tag{2.7}$$

where $A(X,Y) = X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[X,Y]]+\ldots$.

Thus taking into account the directly verifiable equality

$$[2i\,\mathrm{Im}\ F(z,w)\cdot T,\ i(\tau\cdot T+u\cdot D_x+v\cdot D_y)] = 0$$

we have

$$\pi_r(g) = e^{i[(\tau+2\,\mathrm{Im}\ F(z,w))\cdot T+u\cdot D_x+v\cdot D_y]}, \tag{2.8}$$

$$\pi_l(g) = e^{-i[(\tau-2\,\mathrm{Im}\ F(z,w))\cdot T+u\cdot D_x+v\cdot D_y]}, \tag{2.9}$$

where $w=u+iv\in\mathbb{C}^n$, $\tau\in\mathbb{R}^m$, $g=(\tau,w)$.

Let us transform the representations (2.8) and (2.9). From

$$\mathrm{Im}\ F(z,w) = \mathrm{Im}\ (F(z,u)-i\,F(z,v)) = \mathrm{Im}\ F(z,u) - \mathrm{Re}\ F(z,v)$$

it follows that

$$\begin{aligned} 2\,\mathrm{Im}\ F(z,w)\cdot T &= \frac{2}{i}\cdot\mathrm{Im}\ z' L(\frac{\partial}{\partial t})u - \frac{2}{i}\cdot\mathrm{Re}\ z' L(\frac{\partial}{\partial t})v = \\ &= \frac{2}{i}u\cdot\mathrm{Im}\ \left(L(\frac{\partial}{\partial t})' z\right) - \frac{2}{i}v\cdot\mathrm{Re}\ \left(L(\frac{\partial}{\partial t})' z\right) = \\ &= -\frac{2}{i}u\cdot\mathrm{Im}\ \left(L(\frac{\partial}{\partial t})\overline{z}\right) - \frac{2}{i}v\cdot\mathrm{Re}\ \left(L(\frac{\partial}{\partial t})\overline{z}\right), \end{aligned}$$

here, as in Section 1, $L(\lambda)=\lambda\cdot F$.

Let us introduce now vector fields

$$\begin{aligned} \mathfrak{X}'_r &= \frac{1}{i}\left(\frac{1}{2}\cdot\frac{\partial}{\partial x} - \mathrm{Im}\ \left(L(\frac{\partial}{\partial t})\overline{z}\right)\right), \\ \Upsilon'_r &= \frac{1}{i}\left(\frac{1}{2}\cdot\frac{\partial}{\partial y} - \mathrm{Re}\ \left(L(\frac{\partial}{\partial y})\overline{z}\right)\right), \\ \mathfrak{X}'_l &= \frac{1}{i}\left(\frac{1}{2}\cdot\frac{\partial}{\partial x} + \mathrm{Im}\ \left(L(\frac{\partial}{\partial t})\overline{z}\right)\right), \\ \Upsilon'_l &= \frac{1}{i}\left(\frac{1}{2}\frac{\partial}{\partial y} + \mathrm{Re}\ \left(L(\frac{\partial}{\partial t})\overline{z}\right)\right). \end{aligned}$$

Then it is easy to see that the expressions (2.8) and (2.9) obtain the following form

$$\pi_r(g) = e^{i(\tau\cdot T+2u\cdot\mathfrak{X}_r^{'}+2v\cdot\Upsilon_r^{'})}, \tag{2.10}$$

$$\pi_l(g) = e^{-i(\tau\cdot T+2u\cdot\mathfrak{X}_l^{'}+2v\cdot\Upsilon_l^{'})}. \tag{2.11}$$

Remark 2.1 *The vector fields $\mathfrak{X}_r^{'}$ and $\Upsilon_r^{'}$ up to the constant $\frac{1}{i}$ coincide with the basis $\{\mathfrak{X}^0, \Upsilon^0\}$ of the complex cotangent bundle of a manifold M (see* (1.11), (1,12) *).*

Let us substitute now (2.10) and (2.11) in (2.4), with the help of Campbell– Hausdorff formula we obtain

$$\begin{aligned} K &= (2\pi)^{-N}\int_{\mathbb{R}^{2N}} k(\tau_1,u_1,v_1,\tau_2,u_2,v_2)\cdot \\ &\cdot\ e^{i[(\tau_1-\tau_2)\cdot T+2u_1\cdot\mathfrak{X}_r^{'}-2u_2\cdot\mathfrak{X}_l^{'}+2v_1\cdot\Upsilon_r^{'}-2v_2\cdot\Upsilon_l^{'}]}d\tau_1\,du_1\,dv_1\,d\tau_2\,du_2\,dv_2. \end{aligned} \tag{2.12}$$

2.2

Let us consider now one general construction based on the notion of the direct integral (for references see, for instance, [23, 25]).

Let $\mathbb{R}^N = \mathbb{R}^m\times\mathbb{R}^l$, denote by (t,x) and (λ,ξ) the so–called direct and dual coordinates in $\mathbb{R}^N$. For $t_0\in\mathbb{R}^m$ introduce a shift operator

$$(\tau_{t_0}\varphi)(t) = \varphi(t-to),$$

acting on the space $L_2(\mathbb{R}^m)$. On the space $L_2(\mathbb{R}^N) = L_2(\mathbb{R}^m)\otimes L_2(\mathbb{R}^l)$ let us introduce an operator

$$V_{t_0} = \tau_{t_0}\otimes I.$$

Lemma 2.2. *Let $K\in\mathfrak{B}(L_2(\mathbb{R}^N))$ be an operator commuting with all the operators V_{t_0}, for $t_0\in\mathbb{R}^m$. Then:*

1. $K\in(F^{-1}\otimes I)(L_\infty(\mathbb{R}^m)\otimes\mathfrak{B}(L_2(\mathbb{R}^l)))(F\otimes I)$;
2. *there exists an essentially bounded measurable operator–function*
   $K(\cdot):\ \ \lambda\in\mathbb{R}^m\ \longmapsto K(\lambda)\in\mathfrak{B}(L_2(\mathbb{R}^l))$,
   which is determined by the operator K with respect to a zero–measure set and which determines the following operator acting on the space $L_2(\mathbb{R}^N) = L_2(\mathbb{R}^m)\otimes L_2(\mathbb{R}^l)$:
   $\widetilde{K}f(\lambda,x) = (I\otimes K(\lambda))f(\lambda,x)$;
3. $K = (F^{-1}\otimes I)\widetilde{K}(F\otimes I)$;
4. $\|K\| = ess - sup_{\lambda\in\mathbb{R}^m}\|K(\lambda)\|$.

Really if the operator K commutes with all operators of the form V_{t_0} then it commutes with all elements of the von Neumann algebra $\mathfrak{A}$ generated by them. And thus $K\in\mathfrak{A}^{'}$. Hörmander results [11] imply

$$\mathfrak{A} = F^{-1}L_\infty(\mathbb{R}^m)F\otimes\mathbb{C}\cdot I.$$

Then (see [5; A. 18], [21; 3.2])

$$\mathfrak{A}' = F^{-1}L_\infty(\mathbb{R}^m)F \otimes \mathfrak{B}(L_2(\mathbb{R}^l)) = (F^{-1}\otimes I)(L_\infty(\mathbb{R}^m)\otimes \mathfrak{B}(L_2(\mathbb{R}^l)))(F\otimes I).$$

Thus the first part of the lemma has been proved. The other statements of the lemma are obtained by the application to the operator

$$\widetilde{K} = (F\otimes I)K(F^{-1}\otimes I)$$

of the results from [25; Ch. IV, §7]. □

Remark 2.3 *The values of an operator–function $K(\lambda)$ almost everywhere in $\mathbb{R}^m$ are obtained by applying the partial Fourier transformation with respect to t $(t\to\lambda)$ and by fixing then the value of λ* i.e.

$$I\otimes K(\lambda) = ((F\otimes I)K(F^{-1}\otimes I))(\lambda).$$

Remark 2.4 *Let $\mathcal{Z}\subset \mathbb{H}_F$ be the following set:*

$$\mathcal{Z} = \{(t,z)\in \mathbb{H}_F :\quad z=0\}.$$

It is easy to see that $\mathcal{Z}$ coincides with the center of the group $\mathbb{H}_F$. Thus the operators of the form $\pi_l(g_0)$, $g_0\in\mathcal{Z}$, commute with both the right and the left shift operators. And then they commute with bounded on $L_2(\mathbb{H}_F) = L_2(\mathbb{R}^N)$ right, left and two–sided convolution operators.

is easy to see also that for $g_0 = (t_0, 0)\in\mathcal{Z}$ we have

$$(\pi_l(g_0)f)(t,z) = f(t-t_0,\ z) = V_{t_0}f,$$

here $\mathbb{H}_F = \mathbb{R}^N = \mathbb{R}^m\times\mathbb{C}^n = \mathbb{R}^m\times\mathbb{R}^{2n}$. Thus bounded on the space $L_2(\mathbb{H}_F) = L_2(\mathbb{R}^m)\otimes L_2(\mathbb{R}^{2n})$ the right, the left and the two–sided convolution operators satisfy the conditions of Lemma 2.2.

Let us describe now the operators $K(\lambda)$ for the two–sided convolution operator (2.4) or (2.12).

Introduce the operators

$$\begin{aligned}
\mathfrak{X}_r &= F_I\mathfrak{X}_r'F_I^{-1} = \frac{1}{2}D_x - \text{Im}\ (L(\lambda)\overline{z}),\\
\Upsilon_r &= F_I\Upsilon_r'F_I^{-1} = \frac{1}{2}D_y - \text{Re}\ (L(\lambda)\overline{z}),\\
\mathfrak{X}_l &= F_I\mathfrak{X}_l'F_I^{-1} = \frac{1}{2}D_x + \text{Im}\ (L(\lambda)\overline{z}),\\
\Upsilon_l &= F_I\Upsilon_l'F_I^{-1} = \frac{1}{2}D_y + \text{Re}\ (L(\lambda)\overline{z}),
\end{aligned}$$

where $F_I = F\otimes I$ is a partial Fourier transformation with respect to t. And let $\mathfrak{X}_r(\lambda)$ $(\mathfrak{X}_l(\lambda)$, $\Upsilon_r(\lambda)$, $\Upsilon_l(\lambda)$, respectively) be the (unbounded) operator on $L_2(\mathbb{R}^{2n})$ obtained from $\mathfrak{X}_r(\mathfrak{X}_l$, Υ_r, $\Upsilon_l)$ by fixing a value $\lambda\in\mathbb{R}^m$.

If there will be no any confusion we will write just $\mathfrak{X}_r$ (instead of $\mathfrak{X}_r(\lambda)$). From (2.12) we have

$$K(\lambda) = (2\pi)^{-N} \int_{\mathbb{R}^{2N}} k(\tau_1, u_1, v_1;\ \tau_2, u_2, v_2) \cdot$$
$$\cdot e^{i[(\tau_1-\tau_2)\cdot\lambda + 2u_1\cdot\mathfrak{X}_r - 2u_2\cdot\mathfrak{X}_l + 2v_1\cdot\Upsilon_r - 2v_2\cdot\Upsilon_l]} d\tau_1\, du_1\, dv_1\, d\tau_2\, du_2\, dv_2,$$

or after the applying of the Fourier transformation

$$K(\lambda) = (2\pi)^{-2n} \int_{\mathbb{R}^{4n}} \hat{k}_I(-\lambda, u_1, v_1;\ \lambda, u_2,\ v_2) \cdot e^{2i(u_1\cdot\mathfrak{X}_r - u_2\cdot\mathfrak{X}_l + v_1\cdot\Upsilon_r - v_2\cdot\Upsilon_l)} du_1\, dv_1\, du_2\, dv_2, \tag{2.13}$$

where $\hat{k}_I$ is the Fourier transform of the function k with respect to the variables τ_1 and τ_2. Let us recall that the operator $K(\lambda)$ acts on the space $L_2(\mathbb{R}^{2n})$.

We will calculate now the Weyl symbol (see, for instance, [12, 24]) of the operator $K(\lambda)$.

Let us introduce the $2n \times 2n$ matrix

$$\Lambda(\lambda) = \begin{pmatrix} \mathrm{Im}\ L(\lambda) & -\mathrm{Re}\ L(\lambda) \\ \mathrm{Re}\ L(\lambda) & \mathrm{Im}\ L(\lambda) \end{pmatrix}.$$

Note that the matrix $\Lambda(\lambda)$ is skew–symmetric and homogeneous of the first order with respect to λ, in particular,

$$\Lambda(\lambda) = |\lambda| \cdot \Lambda\left(\frac{\lambda}{|\lambda|}\right).$$

Let us introduce the following vectors in $\mathbb{R}^{2n}$

$$\widetilde{x} = (x, y), \quad \widetilde{\xi} = (\xi, \eta), \quad p = (u_1, v_1), \quad q = (u_2, v_2),$$

and let

$$\widetilde{\mathfrak{X}}_- = \widetilde{\mathfrak{X}}_-(\lambda) = (\mathfrak{X}_r(\lambda), \Upsilon_r(\lambda)),$$
$$\widetilde{\mathfrak{X}}_+ = \widetilde{\mathfrak{X}}_+(\lambda) = (\mathfrak{X}_l(\lambda), \Upsilon_l(\lambda)).$$

The direct verification shows that

$$\widetilde{\mathfrak{X}}_\pm = \frac{1}{2} D_{\widetilde{x}} \pm \Lambda(\lambda)\, \widetilde{x} \tag{2.14}$$

and that

$$2i(u_1\cdot\mathfrak{X}_r - u_2\cdot\mathfrak{X}_l + v_1\cdot\Upsilon_r - v_2\cdot\Upsilon_l) = i((p-q)\cdot D_{\widetilde{x}} - 2(p+q)\cdot\Lambda\widetilde{x}).$$

Then the operator (2.13) takes the form

$$K(\lambda) = (2\pi)^{-2n} \int_{\mathbb{R}^{4n}} \hat{k}_I(-\lambda, p, \lambda, q) e^{-2i(p+q)\cdot\Lambda\widetilde{x} + i(p-q)\cdot D\widetilde{x}} dp\, dq. \tag{2.15}$$

With the help of skew–symmetry of $\Lambda(\lambda)$ let us calculate the commutator

$$[-2i(p+q)\cdot\Lambda\widetilde{x},\ i(p-q)\cdot D\widetilde{x}] = -2[(p-q)\cdot D_{\widetilde{x}},\ \Lambda'(p+q)\cdot\widetilde{x}] = i2(p-q)\cdot\Lambda'(p+q) = -4ip\cdot\Lambda q.$$

From the Campbell–Hausdorff formula and with regard to the scalarness of the commutator we have

$$\left(e^{-2i(p+q)\cdot\Lambda\widetilde{x}+i(p-q)\cdot D_{\widetilde{x}}}\varphi\right)(\widetilde{x}) =$$

$$= \left(e^{-2i(p+q)\cdot\Lambda\widetilde{x}}\cdot e^{i(p-q)\cdot D_{\widetilde{x}}}\cdot e^{2ip\cdot\Lambda q}\cdot\varphi\right)(\widetilde{x}) = e^{-2i(p+q)\cdot\Lambda\widetilde{x}+2ip\cdot\Lambda q}\varphi(\widetilde{x}+p-q). \tag{2.16}$$

From the obvious equality

$$(2\pi)^n\int_{\mathbb{R}^{2n}} e^{i\widetilde{x}\cdot\widetilde{\xi}}\hat{\varphi}(\widetilde{\xi}-\widetilde{h})\,d\widetilde{\xi} = e^{i\widetilde{x}\cdot\widetilde{h}}\varphi(\widetilde{x}),$$

where $\hat{\varphi}$ is a Fourier transform of φ, it follows that

$$\begin{aligned}(2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(\widetilde{x}+p-q)\cdot\widetilde{\xi}}\hat{\varphi}(\widetilde{\xi}-\Lambda(p+q))\,d\widetilde{\xi} &= e^{i\widetilde{x}\cdot\Lambda(p+q)+i(p-q)\Lambda(p+q)}\varphi(\widetilde{x}+p-q) = \\ &= e^{-i(p+q)\cdot\Lambda\widetilde{x}+2ip\cdot\Lambda q}\varphi(\widetilde{x}+p-q).\end{aligned} \tag{2.17}$$

Substituting (2.17) into (2.16) we obtain

$$\begin{aligned} &e^{-2i(p+q)\cdot\Lambda\widetilde{x}+2ip\cdot\Lambda q}\varphi(\widetilde{x}+p-q) = \\ &= e^{-i(p+q)\cdot\Lambda\widetilde{x}}\cdot(2\pi)^{-n}\int_{\mathbb{R}^{4n}} e^{-i(\widetilde{x}+p-q)\cdot\widetilde{\xi}}\hat{\varphi}(\widetilde{\xi}-\Lambda(p+q))\,d\widetilde{\xi} = \\ &= e^{i(p+q)\cdot\Lambda\widetilde{x}}(2\pi)^{-2n}\int_{\mathbb{R}^{4n}} e^{i(\widetilde{x}+p-q)\cdot\widetilde{\xi}}\cdot e^{-\widetilde{y}\cdot(\widetilde{\xi}-\Lambda(p+q))}\varphi(\widetilde{y})\,d\widetilde{y}\,d\widetilde{\xi} = \\ &= (2\pi)^{-2n}\int_{\mathbb{R}^{4n}} e^{-ip\cdot(-\widetilde{\xi}+\Lambda(\widetilde{x}+\widetilde{y}))-iq\cdot(\widetilde{\xi}+\Lambda(\widetilde{x}+\widetilde{y}))}e^{i(\widetilde{x}-\widetilde{y})\cdot\widetilde{\xi}}\varphi(\widetilde{y})\,d\widetilde{y}\,d\widetilde{\xi}.\end{aligned}$$

Finally substituting this formula in (2.16) and computing the Fourier transformation with respect to p and q we obtain

$$(K(\lambda)\varphi)(\widetilde{x}) = (2\pi)^{-2n}\int_{\mathbb{R}^{4n}}\hat{k}(-\lambda,-\widetilde{\xi}+2\Lambda\left(\frac{\widetilde{x}+\widetilde{y}}{2}\right),\ \lambda,\ \widetilde{\xi}+2\Lambda\left(\frac{\widetilde{x}+\widetilde{y}}{2}\right))\cdot e^{i(\widetilde{x}-\widetilde{y})\cdot\widetilde{\xi}}\varphi(\widetilde{y})\,d\widetilde{y}\,d\widetilde{\xi}. \tag{2.18}$$

Thus $K(\lambda)$ is a pseudodifferential operator with the following Weyl symbol

$$\omega_{K(\lambda)}(\widetilde{x},\widetilde{\xi}), = \hat{k}(-\lambda,-\widetilde{\xi}+2\Lambda(\lambda)\widetilde{x};\ \lambda,\ \widetilde{\xi}+2\Lambda(\lambda)\widetilde{x}), \tag{2.19}$$

where $\hat{k}$ is the Fourier transform of the kernel k of the two–sided convolution operator (2.4).

The Weyl symbols of the operators $\widetilde{\mathfrak{X}}_\pm = \widetilde{\mathfrak{X}}_\pm(\lambda)$ (see (2.14)) have the form

$$\omega_{\widetilde{\mathfrak{X}}_\pm}(\widetilde{x},\widetilde{\xi}) = \frac{1}{2}\widetilde{\xi}\pm\Lambda(\lambda)\widetilde{x}.$$

Thus the following theorem holds.

Theorem 2.5 *The Weyl symbol of the operator* (2.13) *has the form*

$$\omega_{K(\lambda)}(\widetilde{x},\widetilde{\xi}) = \hat{k}(-\lambda,-2\omega_{\widetilde{\mathfrak{X}}_-};\ \lambda,\ 2\omega_{\widetilde{\mathfrak{X}}_+}), \tag{2.20}$$

where

$$\omega_{\widetilde{X\pm}}(\widetilde{x},\widetilde{\xi}) = -\frac{1}{2}\widetilde{\xi} \pm \Lambda(\lambda)\widetilde{x}$$

are the Weyl symbols of the operators

$$\widetilde{\mathfrak{X}}_- = (\mathfrak{X}_r, \Upsilon_r), \quad \widetilde{\mathfrak{X}}_+ = (\mathfrak{X}_l, \Upsilon_l).$$

Corollary 2.6 *The Weyl symbols of the operators $B(\lambda)$ and $C(\lambda)$ (corresponding to the right* (2.2) *and the left* (2.3) *convolution operators on $\mathbb{H}_F$) have the forms*

$$\begin{aligned}\omega_{B(\lambda)}(\widetilde{x},\widetilde{\xi}) &= \hat{b}(-\lambda,\ -2\omega_{\widetilde{\mathfrak{X}}_-}),\\ \omega_{C(\lambda)}(\widetilde{x},\widetilde{\xi}) &= \hat{c}(\lambda,\ 2\omega_{\widetilde{\mathfrak{X}}_+}).\end{aligned}$$

Or returning to original coordinates

$$\begin{aligned}\omega_{B(\lambda)}(x,y,\,\xi,\eta) &= \hat{b}(-\lambda,\ -\xi+\ 2\mathrm{Im}\ (L(\lambda)\overline{z}),\ -\eta+2\mathrm{Re}\ (L(\lambda)\overline{z})),\\ \omega_{C(\lambda)}(x,y,\,\xi,\eta) &= \hat{c}(\lambda,\ \xi+\ 2\mathrm{Im}\ (L(\lambda)\overline{z}),\ \eta+2\mathrm{Re}\ (L(\lambda)\overline{z})),\end{aligned}$$

where $z = x + iy$.

Note that Lemma 2.2 and the formula (2.18) allow to obtain the representation of the initial two-sided convolution operator (2.4). We have

$$(Kf)(t,\widetilde{x}) = (2\pi)^{-N}\int_{\mathbb{R}^{2N}} e^{i(\widetilde{x}-\widetilde{y})\cdot\widetilde{\xi}}\cdot e^{i(t-\tau)\cdot\lambda}\cdot\omega_{K(\lambda)}(\frac{1}{2}(\widetilde{x}+\widetilde{y}),\ \widetilde{\xi})f(\tau,\widetilde{y})\,d\tau\,d\widetilde{y}\,d\lambda\,d\widetilde{\xi}.$$

Thus

Corollary 2.7 *In the coordinates $(t,\widetilde{x}) = (t,x,y)$ and $(\lambda,\widetilde{\xi}) = (\lambda,\xi,\eta)$ the two–sided convolution operator K is a pseudodifferential operator on $L_2(\mathbb{R}^N)$ with the following Weyl symbol*

$$\omega_K(t,\widetilde{x},\lambda,\widetilde{\xi}) = \omega_{K(\lambda)}(\widetilde{x},\widetilde{\xi}) =$$

$$\begin{aligned}&= \hat{k}(-\lambda,-\widetilde{\xi}+2\Lambda(\lambda)\widetilde{x},\ \lambda,\ \widetilde{\xi}+2\Lambda(\lambda)\widetilde{x}) =\\ &= \hat{k}(-\lambda,-\xi+2\,\mathrm{Im}\ (L(\lambda)\overline{z}),-\eta+2\,\mathrm{Re}\ (L(\lambda)\overline{z}),\ \lambda,\ \xi+2\,\mathrm{Im}\ (L(\lambda)\overline{z}),\eta+\\ &+\ 2\,\mathrm{Re}\ (L(\lambda)\overline{z})).\end{aligned}$$

Corollary 2.8 *The Weyl symbols of the right* (2.2) *and the left* (2.3) *convolution operators have the following form*

$$\begin{aligned}\omega_B &= \omega_{B(\lambda)}(\widetilde{x},\widetilde{\xi}) = \hat{b}(-\lambda,\ -\widetilde{\xi}+2\Lambda(\lambda)\widetilde{x}) =\\ &= \hat{b}(-\lambda,-\xi+2\,\mathrm{Im}\ (L(\lambda)\overline{z}),-\eta+2\mathrm{Re}\ (L(\lambda)\overline{z})),\\ \omega_C &= \omega_{C(\lambda)}(\widetilde{x},\widetilde{\xi}) = \hat{c}(\lambda,\ \widetilde{\xi}+2\Lambda(\lambda)\widetilde{x}) =\\ &= \hat{c}(\lambda,\ \xi+2\,\mathrm{Im}\ (L(\lambda)\overline{z}),\ \eta+2\,\mathrm{Re}\ (L(\lambda)\overline{z})).\end{aligned}$$

3 The structure of operators $K(\lambda)$

3.1

Let a point $\lambda \in \mathbb{R}^m \setminus \{0\}$ be fixed. And let

$$(K(\lambda)\varphi)(\widetilde{x}) = (2\pi)^{-2n} \int_{\mathbb{R}^{4n}} e^{i(\widetilde{x}-\widetilde{y})\cdot\widetilde{\xi}} \omega_{K(\lambda)}(\frac{1}{2}(\widetilde{x}+\widetilde{y}),\, \widetilde{\xi})\, \varphi(\widetilde{y})\, d\widetilde{y}\, d\widetilde{\xi} \tag{3.1}$$

be a bounded on $L_2(\mathbb{R}^{2n}) = L_2(\mathbb{C}^n)$ operator with the Weyl symbol (see (2.19))

$$\omega_{K(\lambda)}(\widetilde{x},\widetilde{\xi}) = \hat{k}(-\lambda,\, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x};\, \lambda,\, \widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}), \tag{3.2}$$

or (in the complex form)

$$\omega_{K(\lambda)}(z,\zeta) = \hat{k}(-\lambda,\, 2i(\frac{1}{2}i\zeta + \overline{L(\lambda)}z);\, \lambda,\, 2i(-\frac{1}{2}i\zeta + \overline{L(\lambda)}z)),$$

where $z = x+iy$, $\zeta = \xi + i\eta \in \mathbb{C}^n$ for $\widetilde{x} = (x,y)$, $\widetilde{\xi} = (\xi,\eta) \in \mathbb{R}^{2n}$.

Let us introduce (unbounded on $L_2(\mathbb{C}^n)$) operators $Z_r = Z_r(\lambda) = \frac{\partial}{\partial z} + \overline{L(\lambda)}z$ and $Z_l = Z_l(\lambda) = -\frac{\partial}{\partial z} + \overline{L(\lambda)}z$, then we have

$$\begin{aligned} \omega_{Z_r} &= \omega_{Z_r(\lambda)}(z,\zeta) = \frac{1}{2}i\zeta + \overline{L(\lambda)}z, \\ \omega_{Z_l} &= \omega_{Z_l(\lambda)}(z,\zeta) = -\frac{1}{2}i\zeta + \overline{L(\lambda)}z. \end{aligned}$$

And thus

$$\omega_{K(\lambda)}(z,\zeta) = \hat{k}(-\lambda,\, 2i\,\omega_{Z_r(\lambda)}(z,\zeta);\, \lambda,\, 2i\,\omega_{Z_l(\lambda)}(z,\zeta)). \tag{3.3}$$

Let $U = U_1 + iU_2$ be a unitary matrix. In the space $T^*(\mathbb{C}^n) = T^*(\mathbb{R}^{2n})$ let us carry out the following symplectic change of variables

$$x = U_1'x_1 - U_2'y_1, \quad y = U_2'x_1 + U_1'y_1,$$

$$\xi = U_1'\xi_1 - U_2'\eta_1, \quad \eta = U_2'\xi_1 + U_1'\eta_1,$$

or in the complex form of notation

$$z = U'z_1, \quad \zeta = U'\zeta_1.$$

The operator $(\nu_1\varphi)(z_1) = \varphi(U'z_1) : \ L_2(\mathbb{C}^n) \to L_2(\mathbb{C}^n)$ realizes this change of variables. It is easy to see that

$$\nu_1 \circ Z_r \circ \nu_1^{-1} = U'Z_r^1, \quad \nu_1 \circ Z_l \circ \nu_1^{-1} = U'Z_l^1,$$

where

$$\begin{aligned} Z_r^1 &= \frac{\partial}{\partial \overline{z}_1} + \overline{L_1(\lambda)}z_1, \\ Z_l^1 &= -\frac{\partial}{\partial \overline{z}_1} + \overline{L_1(\lambda)}z_1, \\ L_1(\lambda) &= U\,L(\lambda)\,U^*. \end{aligned}$$

Really, let us calculate on the level of Weyl symbols

$$\begin{aligned}
\omega_{Z_r} &= \frac{1}{2}i\zeta + \overline{L(\lambda)}z = \frac{1}{2}iU'\zeta_1 + \overline{L(\lambda)}U'z_1 = \\
&= U'\left(\frac{1}{2}i\zeta_1 + \overline{U\,L(\lambda)\,U}^{*}z_1\right) = U'\left(\frac{1}{2}i\zeta_1 + \overline{U\,L(\lambda)\,U^{*}z_1}\right) = \\
&= \omega_{\nu_1\circ Z_r\circ\nu_1^{-1}} = U'\omega_{Z_r^1}.
\end{aligned}$$

The similar calculation can be done for ω_{Z_l}. Then the Weyl symbol of the operator

$$K_1(\lambda) = \nu_1 \circ K(\lambda) \circ \nu_1^{-1}$$

is calculated by the formula

$$\omega_{K_1(\lambda)}(z_1,\zeta_1) = \hat{k}(-\lambda,\, 2iU'\omega_{Z_r^1}(z_1,\zeta_1);\, \lambda,\, 2iU'\omega_{Z_l^1}(z_1,\zeta_1)). \tag{3.4}$$

The Hermitian matrix $L(\lambda)$ is homogeneous of the first order, thus

$$L(\lambda) = |\lambda| \cdot L\left(\frac{\lambda}{|\lambda|}\right).$$

The matrix $L(\lambda)$ can be reduced with the help of some unitary matrix $U = U\left(\frac{\lambda}{|\lambda|}\right)$ to a diagonal form

$$U(\frac{\lambda}{|\lambda|})\, L(\frac{\lambda}{|\lambda|})\, U(\frac{\lambda}{|\lambda|})^* = E(\frac{\lambda}{|\lambda|}) = E = \mathrm{diag}\,\left(\varepsilon_1^2, \ldots, \varepsilon_{p,}^2 - \varepsilon_{p+1}^2, \ldots, -\varepsilon_{n-k,}^2 \underbrace{0, \ldots, 0}_{k}\right).$$

For this matrix $U = U(\frac{\lambda}{|\lambda|})$ we have

$$Z_r^1 = \frac{\partial}{\partial \overline{z}_1} + |\lambda| E(\frac{\lambda}{|\lambda|})z_1, \quad Z_l^1 = -\frac{\partial}{\partial \overline{z}_1} + |\lambda| E(\frac{\lambda}{|\lambda|})z_1,$$

and thus (3.4) takes the form

$$\omega_{K_1(\lambda)}(z_1,\zeta_1) = \hat{k}(-\lambda,\, 2iU'(\frac{1}{2}i\zeta_1 + |\lambda|E(\frac{\lambda}{|\lambda|})z_1);\, \lambda,\, 2iU'(-\frac{1}{2}i\zeta_1 + |\lambda|E(\frac{\lambda}{|\lambda|})z_1)).$$

Let us introduce the following symplectic in $T^*(\mathbb{C}^n) = T^*(\mathbb{R}^{2n})$ change of variables:
for $j = \overline{1,p}$

$$\begin{aligned}
x_j &= -\frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\eta_{1,j} + \sqrt{|\lambda|}\varepsilon_j x_{1,j}, \quad y_j = \frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\eta_{1,j} + \sqrt{|\lambda|}\varepsilon_j x_{1,j}, \\
\xi_j &= \frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\xi_{1,j} + \sqrt{|\lambda|}\varepsilon_j y_{1,j}, \quad \eta_j = \frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\xi_{1,j} - \sqrt{|\lambda|}\varepsilon_j y_{1,j},
\end{aligned}$$

for $j=\overline{p+1,n-k}$

$$\begin{aligned}
x_j &= -\frac{1}{2\sqrt{|\lambda|\varepsilon_j}}\eta_{1,j}-\sqrt{|\lambda|\varepsilon_j}x_{1,j},\ \ y_j=\frac{1}{2\sqrt{|\lambda|\varepsilon_j}}\eta_{1,j}-\sqrt{|\lambda|\varepsilon_j}x_{1,j},\\
\xi_j &= -\frac{1}{2\sqrt{|\lambda|\varepsilon_j}}\xi_{1,j}+\sqrt{|\lambda|\varepsilon_j}y_{1,j},\ \ \eta_j=-\frac{1}{2\sqrt{|\lambda|\varepsilon_j}}\xi_{1,j}-\sqrt{|\lambda|\varepsilon_j}y_{1,j},
\end{aligned}$$

for $j=\overline{n-k+1,n}$

$$x_j=\frac{1}{2\sqrt{|\lambda|}}\xi_{1,j},\ \ y_j=\frac{1}{2\sqrt{|\lambda|}}\eta_{1,j},$$

$$\xi_j=-2\sqrt{|\lambda|}x_{1,j},\quad \eta_j=-2\sqrt{|\lambda|}y_{1,j}.$$

The formulas of inverse change have the form:
for $j=\overline{1,p}$

$$\begin{aligned}
x_{1,j} &= \frac{1}{2\sqrt{|\lambda|\varepsilon_j}}(x_i+y_i),\ \ y_{1,j}=\frac{1}{2\sqrt{|\lambda|\varepsilon_j}}(\xi_j-\eta_j),\\
\xi_{1,j} &= \sqrt{|\lambda|\varepsilon_j}(\xi_j+\eta_j),\ \ \eta_{1,j}=\sqrt{|\lambda|\varepsilon_j}(-x_j+y_i),
\end{aligned}$$

for $j=\overline{p+1,n-k}$

$$\begin{aligned}
x_{1,j} &= -\frac{1}{2\sqrt{|\lambda|\varepsilon_j}}(x_j+y_i),\ \ y_{1,j}=\frac{1}{2\sqrt{|\lambda|\varepsilon_j}}(\xi_j-\eta_j),\\
\xi_{1,j} &= -\sqrt{|\lambda|\varepsilon_j}(\xi_j+\eta_j),\ \ \eta_{1,j}=\sqrt{|\lambda|\varepsilon_j}(-x_i+y_i),
\end{aligned}$$

for $j=\overline{n-k+1,n}$

$$x_{1,j}=-\frac{1}{2\sqrt{|\lambda|}}\xi_i,\ \ y_{1,j}=-\frac{1}{2\sqrt{|\lambda|}}\eta_i,$$

$$\xi_{1,j}=2\sqrt{|\lambda|}x_j,\quad \eta_{1,j}=2\sqrt{|\lambda|}y_j.$$

Let ν_2 be an (unitary) operator which realizes the above mentioned symplectic change of variables. The direct calculations show that the components of the operators

$$Z_r^{(2)}=\nu_2\circ Z_r^1\circ\nu_2^{-1},\ Z_l^{(2)}=\nu_2\circ Z_l^1\circ\nu_2^{-1}$$

and their Weyl symbols have the following form:

$$Z_{r,j}^{(2)}=\begin{cases}
\sqrt{|\lambda|}\varepsilon_j\left(\frac{\partial}{\partial x_j}+x_i\right), & j=\overline{1,p}\\
\sqrt{|\lambda|}\varepsilon_j,\left(-\frac{\partial}{\partial x_j}+x_j\right), & j=\overline{p+1,n-k}\\
\sqrt{|\lambda|}i(x_j+iy_j), & j=\overline{n-k+1,n}
\end{cases},$$

$$
Z^{(2)}_{l,j} = \begin{cases} \sqrt{|\lambda|}\varepsilon_j\left(-\frac{\partial}{\partial y_j}+y_j\right), & j = \overline{1,p} \\ \sqrt{|\lambda|}\varepsilon_j\left(\frac{\partial}{\partial y_j}+y_i\right), & j = \overline{p+1,n-k} \\ -\sqrt{|\lambda|}i(x_j+iy_j), & j = \overline{n-k+1,n} \end{cases},
$$

$$
\omega_{Z^{(2)}_{r,j}} = \begin{cases} \sqrt{|\lambda|}\varepsilon_j(i\xi_j+x_j), & j = \overline{1,p} \\ \sqrt{|\lambda|}\varepsilon_j(-i\xi_j+x_j), & j = \overline{p+1,n-k} \\ \sqrt{|\lambda|}i(x_j+iy_j), & j = \overline{n-k+1,n} \end{cases},
$$

$$
\omega_{Z^{(2)}_{l,j}} = \begin{cases} \sqrt{|\lambda|}\varepsilon_j(-i\eta_j+y_j), & j = \overline{1,p} \\ \sqrt{|\lambda|}\varepsilon_j(i\eta_j+y_j), & j = \overline{p+1,n-k} \\ -\sqrt{|\lambda|}i(x_i+iy_j), & j = \overline{n-k+1,n} \end{cases}.
$$

Let $z = x+iy \in \mathbb{C}^n$, introduce $x = (x', x''),\ \ y = (y', y'') \in \mathbb{R}^{n-k}\times\mathbb{R}^k$. Let us denote by $\xi', \eta' \in \mathbb{R}^{n-k}$ the dual coordinates for x' and y' respectively. The space $L_2(\mathbb{C}^n) = L_2(\mathbb{R}^{2n})$ can be represented as the following tensor product

$$
L_2(\mathbb{C}^n) = L_2(\mathbb{R}^{n-k}) \otimes L_2(\mathbb{R}^{2k}) \otimes L_2(\mathbb{R}^{n-k}).
$$

In this context a point

$$
(x', x'', y'', y') \in \mathbb{R}^{n-k}\times\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{R}^{n-k} = \mathbb{R}^{n-k}\times\mathbb{R}^{2k}\times\mathbb{R}^{n-k}
$$

corresponds to the point $z = x+iy \in \mathbb{C}^n$.

Let us introduce further n–dimensional complex vectors $\hat{z}_r$ and $\hat{z}_l$ with the following coordinates

$$
\hat{z}_{r,j} = \begin{cases} x_j + i\xi_j = x'_j + i\xi'_j, & j = \overline{1,p} \\ x_j - i\xi_j = x'_j - i\xi'_j, & j = \overline{p+1,n-k}, \\ x_j + iy_j = x''_{j-(n-k)} + iy''_{j-(n-k)}, & j = \overline{n-k+1,n} \end{cases} \tag{3.5}
$$

$$
\hat{z}_{l,j} = \begin{cases} y_i - i\eta_j = y'_j - i\eta'_j, & j = \overline{1,p} \\ y_i + i\eta_j = y'_j + i\eta'_j, & j = \overline{p+1,n-k}. \\ x_j + iy_j = x''_{j-(n-k)} + iy''_{j-(n-k)}, & j = \overline{n-k+1,n} \end{cases} \tag{3.6}
$$

Finally let us introduce $n\times n$ complex matrices

$$
\hat{U}_r = \hat{U}_r\left(\frac{\lambda}{|\lambda|}\right) = 2i\,U'\mathrm{diag}\,(\varepsilon_1,\ldots,\ \varepsilon_{n-k},\ i,\ldots,i) \tag{3.7}
$$

$$
\hat{U}_l = \hat{U}_l\left(\frac{\lambda}{|\lambda|}\right) = 2i\,U'\mathrm{diag}\,(\varepsilon_1,\ldots,\ \varepsilon_{n-k},\ -i,\ldots,-i). \tag{3.8}
$$

It is easy to see that in these notations the Weyl symbol of the operator $K_2(\lambda) = \nu_2 K_1(\lambda)\nu_2^{-1}$ has the following form

$$
\omega_{K_2(\lambda)} = \hat{k}\left(-\lambda, \sqrt{|\lambda|}\hat{U}_r\left(\frac{\lambda}{|\lambda|}\right)\hat{z}_r,\ \lambda,\ \sqrt{|\lambda|}\hat{U}_l\left(\frac{\lambda}{|\lambda|}\right)\hat{z}_l\right).
$$

Thus the following theorem has been obtained.

Theorem 3.1 *Let $\lambda \in \mathbb{R}^m \setminus \{0\}$ and let $K(\lambda)$ be a bounded on $L_2(\mathbb{C}^n) = L_2(\mathbb{R}^{2n})$ operator of the form* (3.1) *with the Weyl symbol* (3.2) *(see, also* (3.3) *). Then there exist non-singular matrices $\hat{U}_r = \hat{U}_r\left(\frac{\lambda}{|\lambda|}\right)$, $\hat{U}_l = \hat{U}_l\left(\frac{\lambda}{|\lambda|}\right)$ and unitary on $L_2(\mathbb{C}^n)$ operator $\nu (= \nu_2 \circ \nu_1)$ such that the Weyl symbol of the operator $\nu\, K(\lambda)\, \nu^{-1}$ is calculated by the formula*

$$\omega_{K_2(\lambda)} = \hat{k}\left(-\lambda, \sqrt{|\lambda|}\hat{U}_r\left(\frac{\lambda}{|\lambda|}\right)\hat{z}_r;\ \lambda, \sqrt{|\lambda|}\hat{U}_l\left(\frac{\lambda}{|\lambda|}\right)\hat{z}_l\right),$$

where the coordinates of the n–dimensional complex vectors $\hat{z}_r$ and $\hat{z}_l$ are given by the formulas (3.5) *and* (3.6) .

Lemma 3.2 *The components of the matrices $\hat{U}_r(\lambda)$ and $\hat{U}_l(\lambda)$ are uniformly bounded on S^{m-1}.*

The statement of the lemma follows directly from the boundedness of elements of an unitary matrix U', the continuity of the eigen–values of the Hermitian matrix $L(\lambda)$ on S^{m-1}, and from the compactness of the sphere S^{m-1}. □

Lemma 3.3 *Let $B_j(\lambda)$ and $C_j(\lambda)$ $(j = \overline{1,q})$ be bounded on $L_2(\mathbb{C}^n) = L_2(\mathbb{R}^{2n})$ operators with the following Weyl symbols*

$$\omega_{B_j(\lambda)}(\tilde{x}, \tilde{\xi}) = \hat{b}_j(-\lambda,\, -\tilde{\xi} + 2\Lambda(\lambda)\tilde{x}),$$

$$\omega_{C_j(\lambda)}(\tilde{x}, \tilde{\xi}) = \hat{c}_j(\lambda,\, \tilde{\xi} + 2\Lambda(\lambda)\tilde{x}).$$

Then the operators $B_j(\lambda)$ and $C_k(\lambda)$ mutually commute and the Weyl symbol of the operator

$$K(\lambda) = \sum_{j=1}^{q} B_j(\lambda) \cdot C_j(\lambda)$$

is calculated by the formula

$$\omega_{K(\lambda)} = \sum_{j=1}^{q} \omega_{B_j(\lambda)} \cdot \omega_{C_j(\lambda)}.$$

The statements of the lemma are evident for the operators $\nu B_j(\lambda)\nu^{-1}$ and $\nu C_j(\lambda)\nu^{-1}$. Then we use the inverse changes for described symplectic changes of variables.□

4 The boundedness of convolution operators

4.1

As in the theory of pseudodifferential operators it is more convenient to define and to study the right and the left convolution operators on the group $\mathbb{H}_F$ in the terms of their (pseudodifferential) symbols, not in the terms of their kernels. A. Nagel and E. Stein [19] for the case of Heisenberg group and M. Taylor [27] for general step 2 nilpotent Lie groups have calculated the (pseudodifferential) symbol for convolution operators. Their constructions show that in the case of our $\mathbb{H}_F$ group the symbols σ_B and σ_C of the right (2.2) and the left (2.3) convolution operators have the following forms

$$\sigma_B = \hat{b}(-\lambda, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}) = \hat{b}(-\lambda, -\xi + 2\,\mathrm{Im}\,(L(\lambda)\overline{z}), -\eta + 2\,\mathrm{Re}\,(L(\lambda)\overline{z})), \tag{4.1}$$

$$\sigma_C = \hat{c}(\lambda,\, \widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}) = \hat{c}(\lambda,\, \xi + 2\,\mathrm{Im}\,(L(\lambda)\overline{z}),\, \eta + 2\,\mathrm{Re}\,(L(\lambda)\overline{z})), \tag{4.2}$$

where as earlier for $z = x + iy,\ \zeta = \xi + i\eta \in \mathbb{C}^n$ we put $\widetilde{x} = (x, y),\ \widetilde{\xi} = (\xi, \eta) \in \mathbb{R}^{2n}$.

The some formulas have been obtained in Section 2 for Weyl symbols of the right and the left convolution operators (see Corollary 2.8). This coincidence is not accidental. The following lemma describe a class of operators for which all τ–symbols coincide.

Let us recall at first (see, for instance, [24]), that for operator

$$(A\varphi)(x) = (2\pi)^{-l} \int_{\mathbb{R}^{2l}} e^{i(x-y)\cdot\xi} \sigma_\tau((1-\tau)x + \tau y, \xi)\, \varphi(y)\, dy\, d\xi$$

the function $\sigma_\tau(x, \xi)$ is called its τ–symbol, $\tau \in [0{,}1]$. For $\tau = 0$ we have a usual pseudodifferential or a left symbol, for $\tau = \frac{1}{2}$ we have a Weyl symbol.

Lemma 4.1 *Let A be a bounded on $L_2(\mathbb{R}^{m+l})$, $m \geq 0$, operator with a symbol*

$$\sigma(t, x, \lambda, \xi) = \sigma_0(t, x, \lambda, xi) = a(\varepsilon\lambda,\, \varepsilon\xi + \Lambda(\lambda)x)$$

$$(\sigma(x, \xi) \;=\; a(\varepsilon\xi + \Lambda x) \;\; \text{if } m = 0)\,,$$

where $\varepsilon = \pm 1$, $\Lambda = \Lambda(\lambda)$ is a skew-symmetric matrix, $(t, x),\ (\lambda, \xi) \in \mathbb{R}^m \times \mathbb{R}^l$.

Then for all $\tau \in [0{,}1]$ the τ–symbol of operator A does not depend on τ and coincides with $\sigma(t, x, \lambda, \xi)$.

In particular, the Weyl symbol ($\tau = \frac{1}{2}$) of the operator A coincides with its left ($\tau = 0$) symbol.

Let us show at first that

$$(x - y)\cdot(\eta - \Lambda(\lambda)x) = (x - y)(\eta - \Lambda(\lambda)\,((1-\tau)x + \tau y)).$$

Indeed, the skew–symmetry of Λ implies

$$\begin{aligned} (x - y)\cdot(\eta - \Lambda(\lambda)x) &= (x - y)\cdot\eta - x\Lambda(\lambda)y, \\ (x - y)\cdot(\eta - \Lambda(\lambda)((1-\tau)x + \tau y)) &= (x - y)\cdot\eta - x\Lambda(\lambda)\tau y + y\Lambda(\tau)(1-\tau)x = \\ &= (x - y)\cdot\eta - x\Lambda(\lambda)y. \end{aligned}$$

Then

$$\begin{aligned} (A\varphi)(t, x) &= (2\pi)^{-(m+l)} \int_{\mathbb{R}^{2(m+l)}} e^{i(t-u)\cdot\lambda} e^{i(x-y)\cdot\xi} a(\varepsilon\lambda,\, \varepsilon\xi + \Lambda(\lambda)x)\, \varphi(u, y)\, du\, dy\, d\lambda\, d\xi = \\ &= (2\pi)^{-(m+l)} \int_{\mathbb{R}^{2(m+l)}} e^{i(t+u)\cdot\lambda} e^{i\varepsilon(x-y)\cdot(\eta - \Lambda(\lambda)x)} a(\varepsilon\lambda, \eta)\, \varphi(u, y)\, du\, dy\, d\lambda\, d\eta = \\ &= (2\pi)^{-(m+l)} \int_{\mathbb{R}^{2(m+l)}} e^{i(t-u)\cdot\lambda} e^{i\varepsilon(x-y)\cdot(\eta - \Lambda(\lambda)((1-\tau)x + \tau y))} a(\varepsilon\lambda, \eta)\, \varphi(u, y)\, du\, dy\, d\lambda\, d\eta = \end{aligned}$$

$$= (2\pi)^{-(m+l)} \int_{\mathbb{R}^{2(m+l)}} e^{i(t-u)\cdot\lambda} e^{i(x-y)\cdot\xi} a(\varepsilon\lambda, \varepsilon\,\xi + \Lambda(\lambda)((1-\tau)x + \tau y))\, \varphi(u, y)\, du\, dy\, d\lambda\, d\xi.$$

Remark 4.2 *The coincidence of all τ–symbols for one–sided convolution operators on $\mathbb{H}_F$ has been obtained by V. Kisil (see, [16]).*

4.2

Let us consider now some auxiliary statements. Let as usual (t, x) and $(\lambda,\ \xi)$ be the direct and the dual coordinates in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^l$. And let $a(x, \lambda, \xi)$ be a function defined in $\mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^l$ such that for all $\lambda_0 \in \mathbb{R}^m$

$$a_{\lambda_0}(x, \xi) = a(x, \lambda_0, \xi) \in C_b(\mathbb{R}^{2l}).$$

Recall (see, for instance, [25; Ch. IV, § 7]) that the mapping

$$\lambda \in \mathbb{R}^m \quad \longmapsto \quad a_\lambda(x, \xi) \in C_b(\mathbb{R}^{2l}) \tag{4.3}$$

is called measurable (with respect to the Lebesque measure $\mu = d\lambda$ in $\mathbb{R}^m$), if the mapping

$$\lambda \in \mathbb{R}^m \quad \longmapsto \quad a_\lambda(x, \xi) I \in \mathfrak{B}(L_2(\mathbb{R}^{2l}))$$

is strongly measurable, *i.e.* if for all $f \in L_2(\mathbb{R}^{2l})$ the mapping

$$\lambda \in \mathbb{R}^m \quad \longmapsto \quad a_\lambda f \in L_2(\mathbb{R}^{2l}) \tag{4.4}$$

is measurable.

The last statement means (see [25; Ch. IV, §7]) that for all compact set $K \subset \mathbb{R}^m$ and for all $\varepsilon > 0$ there exists a compact subset $K_0 (\subset K)$ such that $\mu(K - K_0) < \varepsilon$ and the mapping (4.4) is continuous on K_0 (with respect to the L_2–norm).

Lemma 4.3 *Let E be a measurable subset in $\mathbb{R}^m$, $\mu(E) = 0$ and*

$$a(x, \lambda, \xi) \in C_b(\mathbb{R}^l \times (\mathbb{R}^m \setminus E) \times \mathbb{R}^l).$$

Then the mapping (4.3) *is measurable.*

Let K be an arbitrary compact in $\mathbb{R}^m$, $\varepsilon > 0$. And let G be an open set in $\mathbb{R}^m$ such that $E \subset G$ and $\mu(G) < \varepsilon$. Then for $K_0 = K \setminus G$ we have: K_0 is compact and $\mu(K \setminus K_0) < \varepsilon$.

To prove the lemma we will show that for an arbitrary function $f(x, \xi) \in L_2(\mathbb{R}^{2l})$ the mapping

$$\lambda \in K_0 \quad \longmapsto \quad a(x, \lambda, \xi)\, f(x, \xi) \in L_2(\mathbb{R}^{2l})$$

is continuous.

Denote by $B^N(r)$ the closed ball in $\mathbb{R}^N$ with a radius r and centered in the origin. And let χ_r be a characteristic function of a ball $B^N(r)$.

It is known that the sequence of the orthogonal projectors $\{P_r\}$, where $r \in \mathbb{N}$, $P_r = \chi_r(x, \xi) I$, strongly converges to the identity operator I on $L_2(\mathbb{R}^{2l})$.

For $\lambda_0, \lambda \in K_0$ we have

$$\begin{aligned} \|a(x, \lambda_0, \xi)\, f(x, \xi) \quad &- \quad a(x, \lambda, \xi) f(x, \xi)\| \le \\ &\le \quad \|a(x, \lambda_0, \xi)\,(I - P_r) f\| + \|a(x, \lambda, \xi)\,(I - P_r) f\| + \\ &+ \quad \|\,[a(x, \lambda_0, \xi) - a(x, \lambda, \xi)\,] P_r f\,\|. \end{aligned} \tag{4.5}$$

Let ε_0 be an arbitrary positive number. Then taking into account the boundedness of the functions $a(x, \lambda_0, \xi)$, $a(x, \lambda, \xi)$ and the strong convergence of P_r to the identity operator we select (and fix) a number r such that

$$\|a(x, \lambda_0, \xi)\,(I - P_r) f\| + \|a(x, \lambda, \xi)\,(I - P_r) f\| < \frac{\varepsilon_0}{2}.$$

Now let us estimate the last summand in (4.5)

$$\| [a(x,\lambda_0,\xi) - a(x,\lambda,\xi)]\, P_r f\| \le \|a(x,\lambda_0,\xi)\chi_r(x,\xi) - a(x,\lambda,\xi)\chi_r(x,\xi)\| \cdot \|f\|.$$

We have

$$a(x,\lambda,\xi)\,\chi_r(x,\xi) \in C(K_0 \times B^N(r)) \cong C(K_0,\, C(B^N(r))) \cong C(K_0) \otimes C(B^N(r)).$$

Thus for $\frac{\varepsilon_0}{2\|f\|}$ there exists $\delta > 0$ such that for all $\lambda \in K_0$ with the property $|\lambda_0 - \lambda| < \delta$ we have

$$\|a(x,\lambda_0,\xi)\,\chi_r(x,\xi) - a(x,\lambda,\xi)\,\chi_r(x,\xi)\| < \frac{\varepsilon_0}{2\|f\|}.\square$$

Let $a(x,\lambda,\xi)$ be a function on $\mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^l$ such that for all $\lambda \in \mathbb{R}^m$ a pseudodifferential operator

$$(A_\lambda \varphi)(x) = (2\pi)^{-\frac{l}{2}} \int_{\mathbb{R}^l} e^{ix\cdot\xi} a(x,\lambda,\xi)\,\hat{\varphi}(\xi)\,d\xi \tag{4.6}$$

with the symbol $a_\lambda(x,\xi) = a(x,\lambda,\xi)$ is bounded on $L_2(\mathbb{R}^l)$.

Lemma 4.4. *Let $a(x,\lambda,\xi)$ be a function on $\mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^l$ such that*

1. *for all* $\lambda \in \mathbb{R}^m$ $a_\lambda(x,\xi) = a(x,\lambda,\xi) \in C_b(\mathbb{R}^{2l})$;
2. *the mapping* (4.3) *is measurable;*
3. *for all $\lambda \in \mathbb{R}^m$ the operator* (4.6) *is bounded on $L_2(\mathbb{R}^l)$ and the function $\lambda \longmapsto \|A_\lambda\|$ is bounded on compacts in $\mathbb{R}^m$.*

Then the operator-function

$$\lambda \in \mathbb{R}^m \quad \longmapsto \quad A_\lambda \in \mathfrak{B}(L_2(\mathbb{R}^l))$$

is (strongly) measurable.

Let us fix a function $\varphi(x) \in L_2(\mathbb{R}^l)$, a compact $K \subset \mathbb{R}^m$ and $\varepsilon < 0$. The sequence $\{P_r\}$ of the orthogonal projectors $P_r = \chi_r(x)I$, where $r \in \mathbb{N}$, $\chi_r(x)$ is a characteristic function of the ball $B^l(r)$, converges strongly to the identity operator. Then the sequence $\{Q_r\}$, where $Q_r = F^{-1}P_rF$, has the same property.

For the points λ_0 and λ form K we have

$$\begin{aligned}\|A_{\lambda_0}\varphi &- A_\lambda\varphi\| \le \|A_{\lambda_0}\varphi - A_{\lambda_0}Q_{r_1}\varphi\| + \\ &+ \|A_\lambda\varphi - A_\lambda Q_{r_1}\varphi\| + \|A_{\lambda_0}Q_{r_1}\varphi - A_\lambda Q_{r_1}\varphi\| \le \\ &\le \{\,\|A_{\lambda_0}\,\varphi - A_{\lambda_0}Q_{r_1}\varphi\| + \|A_\lambda\varphi - A_\lambda Q_{r_1}\varphi\| + \|A_{\lambda_0}Q_{r_1}\varphi - \\ &- P_{r_2}A_{\lambda_0}Q_{r_1}\varphi\| + \|A_\lambda Q_{r_1}\varphi - P_{r_2}A_\lambda Q_{r_1}\varphi\|\,\} + \\ &+ \|P_{r_2}A_{\lambda_0}Q_{r_1}\varphi - P_{r_2}A_\lambda Q_{r_1}\varphi\|.\end{aligned} \tag{4.7}$$

Let ε_0 be a positive number. Then taking into account the strong convergence of the sequences $\{Q_r\}$ and $\{P_r\}$ to the identity operator and the boundedness of norm of A_λ we select r_1 and r_2 such that the first summand in (4.7) (in curly brackets) would be less than $\frac{\varepsilon_0}{2}$.

Now let us estimate the last summand in (4.7)

$$\begin{aligned}
\|P_{r_2}A_{\lambda_0}Q_{r_1}\varphi \quad &- \quad P_{r_2}A_{\lambda}Q_{r_1}\varphi\| = \\
&= \|(2\pi)^{-\frac{l}{2}}\int_{B^l(r_1)} e^{ix\cdot\xi}\chi_{B^l(r_2)}(x)\,[a(x,\lambda_0,\xi)-a(x,\lambda,\xi)]\,\hat{\varphi}(\xi\,)\,d\xi\| = \\
&= (2\pi)^{-\frac{l}{2}}\left(\int_{B^l(r_2)} dx \int_{B^l(r_1)} e^{ix\cdot\xi}\,[a(x,\lambda_0,\xi)-a(x,\lambda,\xi)]\,\hat{\varphi}(\xi)\,d\xi|^2\right)^{\frac{1}{2}} \le \\
&\le (2\pi)^{-\frac{l}{2}}\left(\int_{B^l(r_2)} dx \int_{B^l(r_1)} |\,[a(x,\lambda_0,\xi)-a(x,\lambda,\xi)]\,\hat{\varphi}(\xi)|^2 d\xi \cdot \int_{B^l(r_1)} d\xi\right)^{\frac{1}{2}} = \\
&= M\cdot\|a(x,\lambda_0,\xi)\,f(x,\xi)-a(x,\lambda,\xi)\,f(x,\xi)\|,
\end{aligned}$$

where $M=(2\pi)^{-\frac{1}{2}}\mu(B^l(r_1))^{\frac{1}{2}}$, $f(x,\xi)=\chi_{r_2}(x)\,\hat{\varphi}(\xi)\,\chi_{r_1}(\xi)\in L_2(\mathbb{R}^{2l})$.

The mapping (4.4) is measurable so we can select (for given $f\in L_2(\mathbb{R}^{2l})$, K and ε) a compact subset K_0 of a set K such that $\mu(K\setminus K_0)<\varepsilon$ and the mapping (4.4) is continuous on K_0. Thus for $\frac{\varepsilon_0}{2M}$ we can select $\delta>0$ such that for all $\lambda_0,\lambda\in K_0$ and $|\lambda_0-\lambda|<\delta$ the last summand in (4.7) would be less than $\frac{\varepsilon_0}{2}$.$\square$

Lemma 4.5 *Let E be a measurable subset in $\mathbb{R}^m$,* $\mu(E)=0$ and

$$a(x,\lambda,\xi)\in C_b(\mathbb{R}^l\times(\mathbb{R}^m\setminus E)\times\mathbb{R}^l).$$

If for all $\lambda\in\mathbb{R}^m\setminus E$ the operator A_λ of the form (4.6) *is bounded and the function $\lambda\longmapsto\|A_\lambda\|$ is bounded on compacts in $\mathbb{R}^m$ then the operator function*

$$\lambda\in\mathbb{R}^m\setminus E \quad\longmapsto\quad A_\lambda\in\mathfrak{B}(L_2(\mathbb{R}^l))$$

is strongly continuous in $\mathbb{R}^m\setminus E$.

Let $\lambda_0\in\mathbb{R}^m\setminus E$ and V be some compact neighborhood of λ_0 in $\mathbb{R}\setminus E$ (which is constructed, for example, by the following way: let $B^m(\lambda_0,1)$ be a closed ball centered in the point λ_0 and of the radius 1, and let G_0 be an open set which contains E and such that $\mu(G)<\varepsilon_0$ for some small ε_0; for each point $\tau\in E$ denote by U_τ an open ball centered in τ and with the radius $\frac{1}{2}|\lambda_0-\tau|$, and let $G_1=\cup_{\tau\in E}U_\tau$, then

$$V=B^m(\lambda_0,1)\setminus(G_0\cap G_1)\,).$$

For an arbitrary function $\varphi\in L_2(\mathbb{R}^l)$ and for any $\lambda\in V$ let us estimate

$$\begin{aligned}
\|A_{\lambda_0}\varphi \quad &- \quad A_\lambda\varphi\| \le \{\,\|A_{\lambda_0}\varphi-A_{\lambda_0}Q_{r_1}\varphi\|+\|A_\lambda\varphi-A_\lambda Q_{r_1}\varphi\|+ \\
&+ \quad \|A_{\lambda_0}Q_{r_1}\varphi-P_{r_2}A_{\lambda_0}Q_{r_1}\varphi\|+\|A_\lambda Q_{r_1}\varphi-P_{r_2}A_\lambda Q_{r_1}\varphi\|\,\}+ \\
&+ \quad \|P_{r_2}A_{\lambda_0}Q_{r_1}\varphi-P_{r_2}A_\lambda Q_{r_1}\varphi\|.
\end{aligned}$$

For $\varepsilon<0$ it is possible to select (as it has been done in Lemma 4.4) r_2 and r_1 such that the first summand (in curly bracket) would be lees than $\frac{\varepsilon}{2}$.

Now estimate (see the proof of Lemma 4.4)

$$\begin{aligned}
&\|P_{r_2}A_{\lambda_0}Q_{r_1}\varphi \quad - \quad P_{r_2}A_\lambda Q_{r_1}\varphi\| \le \\
\le \|\,(2\pi)^{-\frac{l}{2}}&\int_{B^l(r_1)} e^{ix\cdot\xi}\chi_{B^l(r_2)}(x)\,[a(x,\lambda_0,\,\xi) \quad - \quad a(x,\lambda,\xi)]\,\hat{\varphi}(\xi)\,d\xi\,\| \le
\end{aligned}$$

$$\leq (2\pi)^{-\frac{l}{2}} \left(\int_{B^l(r_2)} dx \int_{B^l(r_l)} | [a(x,\lambda_0,\ \xi) \quad - \quad a(x,\lambda,\xi)] \ \hat{\varphi}(\xi)|^2 \, d\xi \cdot \int_{B^l(r_1)} d\xi \right)^{\frac{1}{2}} \leq$$

$$\leq M \cdot \|a(x,\lambda_0,\xi)\, f(x,\xi) \quad - \quad a(x,\lambda,\xi)\, f(x,\xi)\| \leq$$

$$\leq M \cdot \|f(x,\xi)\| \cdot \|a(x,\lambda_0,\xi)\chi_{r_2}(x)\chi_{r_1}(\xi) \quad - \quad a(x,\lambda,\xi)\chi_{r_2}(x)\chi_{r_1}(\xi)\|,$$

where $M = (2\pi)^{-\frac{l}{2}} \mu(B^l(r_1))^{1/2}, \ \ f(x,\xi) = \chi_{r_2}(x)\, \hat{\varphi}(\xi)\, \chi_{r_1}(\xi)$.

Further

$$a(x,\lambda,\xi)\, \chi_{r_2}(x)\chi_{r_1}(\xi) \in C(V \times B^l(r_l) \times B^l(r_2)) \cong C(V) \otimes C(B^l(r_2) \times B^l(r_1)).$$

Thus for $\frac{\varepsilon}{2M\cdot\|f(x,\xi)\|}$ there exists $\delta > 0$ such that for all $\lambda \in V$ with the property $|\lambda_0 - \lambda| < \delta$ we have

$$\|a(x,\lambda_0,\xi)\, \chi_{r_2}(x)\, \chi_{r_1}(\xi) - a(x,\lambda,\xi)\, \chi_{r_2}(x)\, \chi_{r_1}(\xi)\|^2 < \frac{\varepsilon}{2M \cdot \|f(x,\xi)\|}.\square$$

Lemma 4.6 *Let E be a measurable subset in $\mathbb{R}^m$, $\mu(E) = 0$, and A_λ be some measurable essentially bounded operator–function in $\mathbb{R}^m \setminus E$. If the function A_λ is strongly continuous in $\mathbb{R}^m \setminus E$, then*

$$\text{ess-sup}_{\ \mathbb{R}^m \setminus E} \|A_\lambda\| = \sup_{\ \mathbb{R}^m \setminus E} \|A_\lambda\|.$$

At first let us prove that there exists a set $E_0 \subset M = \mathbb{R}^m \setminus E$, $\mu(E_0) = 0$ such that

$$\text{ess–sup}_{\ M} \|A_\lambda\| = C = \sup_{\ M \setminus E_0} \|A_\lambda\|.$$

In fact, by the definition of an essential supremum for all $n \in \mathbb{N}$ there exists a measurable set $E_n \subset M$ such that $\mu(E_n) = 0$ and

$$C \leq \sup_{\ M \setminus E_n} \|A_\lambda\| < C + \frac{1}{n}.$$

Than for $E_0 = \cup_n E$ we have $\mu(E_0) = 0$ and for all $n \in \mathbb{N}$

$$C \leq \sup_{\ M \setminus E_0} \|A_\lambda\| < C + \frac{1}{n},$$

thus $\sup_{\ M \setminus E_0} \|A_\lambda\| = C$.

To complete the proof we show that for strongly continuous operator–function A_λ $\|A_{\lambda_0}\| \leq C$ for all $\lambda_0 \in E_0$. The set E_0 has a zero measure, thus for all $\lambda_0 \in E_0$ there exists a sequence $\{\lambda_n\}$ such that for all $n \in \mathbb{N}$ $\lambda_n \in M \setminus E_0$ and $\lim_{\ n\to\infty} \lambda_n = \lambda_0$. Then the sequence of the operators A_{λ_n} strongly converges to the operator A_{λ_0}, and for all $n \in \mathbb{N}$ $\|A_{\lambda_n}\| \leq C$. The uniform boundedness theorem completes the proof.$\square$

Lemma 4.7 *Let $a(x,\lambda,\xi)$ be a function on $\mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^l$ such that*

1. *for all $\lambda \in \mathbb{R}^m$ $a_\lambda(x,\xi) = a(x,\lambda,\xi) \in C_b(\mathbb{R}^{2l})$;*

2. *the mapping* (4.3) *is measurable;*

3. *for all* $\lambda \in \mathbb{R}^m$ *the operator* A_λ *of the form* (4.6) *is bounded on* $L_2(\mathbb{R}^l)$;

4. *the norms of* A_λ *are uniformly bounded.*

Then the operator

$$(Af)(t,x) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{ix\cdot\xi} e^{it\cdot\lambda} a(x,\lambda,\xi)\, \hat{f}(\lambda,\xi)\, d\xi\, d\lambda$$

with the symbol $a(x,\lambda,\xi)$ *is bounded on* $L_2(\mathbb{R}^N)$ *and*

$$\|A\| = \text{ess–sup}_{\ \lambda\in\mathbb{R}^m} \|A_\lambda\|.$$

Moreover the operator A *is decomposable, i.e.*

$$A = (F^{-1}\otimes I)\, \widetilde{A}\, (F\otimes I),$$

where the operator $\widetilde{A} \in \mathfrak{B}(L_2(\mathbb{R}^m)\otimes L_2(\mathbb{R}^l))$ *is defined by the rule*

$$\widetilde{A}\, f(\lambda,x) = (I\otimes A_\lambda) f(\lambda,x).$$

By Lemma 4.4 the operator–function

$$\lambda \in \mathbb{R}^m \quad \longmapsto \quad A_\lambda \in \mathfrak{B}(L_2(\mathbb{R}^l))$$

is measurable. Then (see, for example, [25]) the operator

$$\widetilde{A} f(\lambda,x) = (I\otimes A_\lambda) f(\lambda,x)$$

is well–defined and is bounded on $L_2(\mathbb{R}^{m+l})$, and

$$\|\widetilde{A}\| = \text{ess–sup}_{\ \lambda\in\mathbb{R}^m} \|A_\lambda\|.$$

Therefore the operator

$$(F^{-1}\otimes I)\, \widetilde{A}\, (F\otimes I)$$

(which is unitary equivalent to $\widetilde{A}$) has just the same properties.

Let as calculate

$$(F^{-1}\otimes I)\widetilde{A}(F\otimes I)f =$$

$$\begin{aligned} &= (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{it\cdot\lambda} dx\, (2\pi)^{-\frac{l}{2}} \int_{\mathbb{R}^l} e^{ix\cdot\xi} a(x,\lambda,\xi)\, d\xi\, (2\pi)^{-\frac{l}{2}} \int_{\mathbb{R}^l} e^{-iy\cdot\xi} dy \,\cdot \\ &\cdot\ (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-i\tau\cdot\lambda} f(\tau,y) d\tau = \\ &= (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^{m+l}} e^{ix\cdot\xi} e^{it\cdot\lambda} a(x,\lambda,\xi)\, \hat{f}(\lambda,\xi)\, d\xi\, d\lambda = (Af)(t,x). \square \end{aligned}$$

4.3

Introduce on $\mathbb{H}_F = \mathbb{R}^m \times \mathbb{C}^n = \mathbb{R}^m \times \mathbb{R}^{2n}$ the one–parameter group of the non-isotropic dilations $\delta_\tau, \tau > 0$, (see, for example, [10, 27]) by the rule

$$\delta_\tau(\lambda, \zeta) = (\tau\lambda,\ \tau^{\frac{1}{2}}\zeta), \quad (\lambda,\ \zeta) \in \mathbb{R}^m \times \mathbb{C}^n,$$

or

$$\delta_\tau(\lambda, \widetilde{\xi}) = (\tau\lambda,\ \tau^{\frac{1}{2}}\widetilde{\xi}), \quad (\lambda, \widetilde{\xi}) \in \mathbb{R}^m \times \mathbb{R}^{2n}.$$

The action of this group is transferred onto $\mathbb{H}_F \times \mathbb{H}_F$ in a natural way

$$\delta_\tau(g_1, q_2) = (\delta_\tau(g_1),\ \delta_\tau(g_2)),$$

for all $(g_1, q_2) \in \mathbb{H}_F \times \mathbb{H}_F$.

A function ψ on $\mathbb{H}_F$ (or on $\mathbb{H}_F \times \mathbb{H}_F$) is called a δ_τ–homogeneous function of the degree s if

$$\psi \circ \delta_\tau = \tau^s \cdot \psi.$$

The values of an arbitrary δ_τ–homogeneous function ψ (on $\mathbb{H}_F$) are determined uniquely by its values on the non–isotropic unit sphere Ω in $\mathbb{H}_F$

$$\Omega = \{(\lambda, \widetilde{\xi}) \in \mathbb{H}_F = \mathbb{R}^m \times \mathbb{R}^{2n} : \quad \|\,(\lambda, \widetilde{\xi})\,\| = 1\},$$

where $\|(\lambda, \widetilde{\xi})\|^2 = \sum_{j=1}^m |\lambda_j|^2 + \sum_{k=1}^n (|\xi_k|^4 + |\eta_k|^4)$.

Theorem 4.8 *Let E be a closed subset in $\mathbb{R}^m$ with the measure zero and with the following "cone" property*

$$\forall \lambda \in E \quad \forall \tau > 0 \quad \tau \cdot \lambda \in E,$$

and let $E_\Omega = \Omega \cap (E \times \mathbb{R}^{2n})$. Further let $\widetilde{b}(-\lambda, \widetilde{\xi})$ be a δ_τ –homogeneous function of the degree zero in $\mathbb{H}_F = \mathbb{R}^m \times \mathbb{R}^{2n}$ having $C^\infty(\Omega \setminus E_\Omega)$ restriction onto Ω.

Then a pseudodifferential operator B with the Weyl symbol

$$\omega_B(\widetilde{x}, \lambda, \widetilde{\xi}) = \widetilde{b}(-\lambda,\ -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x})$$

is bounded on $L_2(\mathbb{H}_F)$. Moreover the operator B generates an operator–function

$$B(\cdot) : \quad \mathbb{R}^m \longrightarrow \mathfrak{B}(L_2(\mathbb{R}^{2n})),$$

which is essentially bounded and measurable in $\mathbb{R}^m$ and is strongly continuous in $\mathbb{R}^m \setminus E$. The operator

$$\widetilde{B}\, f(\lambda, \widetilde{\xi}) = (I \otimes B(\lambda))\, f(\lambda, \widetilde{\xi})$$

defined by $B(\cdot)$ is bounded on $L_2(\mathbb{R}^m \times \mathbb{R}^{2n})$ and is unitary equivalent to the operator B:

$$B = (F^{-1} \otimes I)\widetilde{B}(F \otimes I).$$

And in addition

$$\|B\| = \sup\,_{\lambda \in \mathbb{R}^m \setminus E} \|B(\lambda)\|.$$

For $\lambda \in \mathbb{R}^m \setminus E$ let us introduce a pseudodifferential operator $B'(\lambda)$ with the following Weyl symbol

$$\omega_{B'(\lambda)}(\hat{z}_r) = \widetilde{b}(-\lambda, \sqrt{|\lambda|}\hat{U}_r \hat{z}_r) = \widetilde{b}(-\frac{\lambda}{|\lambda|}, \hat{U}_r \hat{z}_r),$$

where the n–dimensional complex vector $\hat{z}_r$ and the $n \times n$ complex matrix $\hat{U}_r$ are given by the formulas (3.4) and (3.6) respectively. The operator $B'(\lambda)$ is bounded on $L_2(\mathbb{R}^{2n})$ by the Calderón–Vaillancourt–Cordes theorem [2, 3, 26]. And the norms of the operators $B'(\lambda)$ for $\lambda \in \mathbb{R}^m \setminus E$ uniformly bounded (Lemma 3.2 has been used here).

Further if we apply the inverse to described in Section 3 symplectic changes of variables we obtain uniformly bounded on $L_2(\mathbb{R}^{2n})$ pseudodifferential operators $B(\lambda)$, $\lambda \in \mathbb{R}^m \setminus E$, with the Weyl symbol

$$\omega_{B(\lambda)}(\widetilde{x}, \widetilde{\xi}) = \widetilde{b}(-\lambda, \, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}).$$

The arguments of Lemma 4.1. shows that this Weyl symbol coincides, in fact, with an arbitrary τ–symbol of $B(\lambda)$ and in particular, with its left ($\tau = 0$) symbol.

Then by Lemma 4.7 a pseudodifferential operator B with the (left) symbol

$$\widetilde{b}(\lambda, \, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x})$$

is bounded on $L_2(\mathbb{R}^m \times \mathbb{R}^{2n})$,

$$\|B\| = \text{ess–sup}_{\lambda \in \mathbb{R}^m} \|B(\lambda)\|$$

and the operator B is decomposable, *i.e.*

$$B = (F^{-1} \otimes I) \, \widetilde{B} \, (F \otimes I),$$

where $\widetilde{B} \, f(\lambda, \widetilde{x}) = (I \otimes B(\lambda)) \, f(\lambda, \widetilde{x})$.

By Lemma 4.1 the (left) symbol of B coincides with its Weyl symbol. By Lemma 4.5 the operator–function $B(\lambda)$ is strongly continuous in $\mathbb{R}^m \setminus E$. And, finally, by Lemma 4.6

$$\|B\| = \sup_{\lambda \in \mathbb{R}^m \setminus E} \|B(\lambda)\|. \square$$

The similar proof can be done for the following theorem for the left convolution operators.

Theorem 4.9 *Let E and E_Ω be as in Theorem 4.8 . And let $\widetilde{c}(\lambda, \widetilde{\xi})$ be a δ_τ–homogeneous function of the degree zero in $\mathbb{H}_F = \mathbb{R}^m \times \mathbb{R}^{2n}$ having $C^\infty(\Omega \setminus E_\Omega)$ restriction onto Ω.*

Then a pseudodifferential operator C with the Weyl symbol

$$\omega_C(\widetilde{x}, \lambda, \widetilde{\xi}) = \widetilde{c}(\lambda, \, \widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x})$$

is bounded on $L_2(\mathbb{H}_F)$. Moreover the operator C generates an operator–function

$$C(\cdot): \;\; \mathbb{R}^m \longrightarrow \mathfrak{B}(L_2(\mathbb{R}^{2n})),$$

which is essentially bounded and measurable in $\mathbb{R}^m$ *and is strongly continuous in* $\mathbb{R}^m \setminus E$. *The operator*

$$\widetilde{C}\, f(\lambda, \widetilde{\xi}) = (I \otimes C(\lambda))\, f(\lambda, \widetilde{\xi})$$

defined by $C(\cdot)$ *is bounded on* $L^2(\mathbb{R}^m \times \mathbb{R}^{2n})$ *and is unitary equivalent to the operator* C*:*

$$C = (F^{-1} \otimes I)\, \widetilde{C}\, (F \otimes I).$$

And in addition

$$\|C\| = \sup{}_{\lambda \in \mathbb{R}^m \setminus E} \|C(\lambda)\|.$$

Let $\mathcal{A}$ $(\subset L_\infty(\Omega))$ be some algebra of functions on Ω. Denote by $H_\delta^s(\mathcal{A})$ the algebra of δ_τ–homogeneous of the degree s functions in $\mathbb{H}_F$ having $\mathcal{A}$ restrictions onto the non–isotropic unit sphere Ω in $\mathbb{H}_F$.

Corollary 4.10 *Let* E *and* E_Ω *be as in Theorem* 4.8. *And let*

$$\hat{k}(g_1, q_2) \in H_\delta^0(C^\infty(\Omega \setminus E_\Omega)) \otimes H_\delta^0(C^\infty(\Omega \setminus E_\Omega)).^*$$

Then a pseudodifferential operator K *with the Weyl symbol* (2.20)

$$\omega_K(\widetilde{x}, \lambda, \widetilde{\xi}) = \widetilde{k}(-\lambda, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x};\ \lambda,\ \widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x})$$

is bounded on $L_2(\mathbb{H}_F)$.

Moreover the operator K *generates an operator–function*

$$K(\cdot): \quad \mathbb{R}^m \longrightarrow \mathfrak{B}(L_2(\mathbb{R}^{2n})),$$

which is essentially bounded and measurable in $\mathbb{R}^m$ *and is strongly continuous in* $\mathbb{R}^m \setminus E$. *The Weyl symbol of an operator* $K(\lambda)$ *is calculated by the formula*

$$\omega_{K(\lambda)} = \widetilde{k}(-\lambda,\ -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x};\ \lambda,\ \widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}).$$

The operator

$$\widetilde{K}\, f(\lambda, \widetilde{\xi}) = (I \otimes K(\lambda))\, f(\lambda, \widetilde{\xi})$$

defined by $K(\cdot)$ *is bounded on* $L_2(\mathbb{R}^m \times \mathbb{R}^{2n})$ *and is unitary equivalent to the operator* K*:*

$$K = (F^{-1} \otimes I)\, \widetilde{K}\, (F \otimes I).$$

And in addition

$$\|K\| = \sup{}_{\lambda \in \mathbb{R}^m \setminus E} \|K(\lambda)\|.$$

The proof follows directly from Theorems 4.8, 4.9, from the arguments of Lemma 3.3 and from the properties of the direct integral construction (see, for example, [25, Ch IV]).□

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*)Here we consider an algebraic (non–closed) tensor product.

5 Structure of the Hardy space and the Szegö projector associated with the Siegel domain

5.1

Let us introduce (see Subsection 1.6) the type II Siegel domain

$$D = \{(w, z) \in \mathbb{C}^{m+n} : \ \mathrm{Im}\ w - F(z, z) \in V\}.$$

The standard CR–manifold

$$B = \{(w, z) \in \mathbb{C}^{m+n} : \ \mathrm{Im}\ w - F(z, z) = 0\}$$

is a distinguished boundary of the domain D. And thus the Hardy space $H_2(D)$ is a subspace of $L_2(B)$.

The mapping

$$\kappa_0 : \ (t, z) \in \mathbb{R}^m \times \mathbb{C}^n \ \longmapsto \ (t + iF(z, z), \ z) \in B$$

identifies $\mathbb{R}^m \times \mathbb{C}^n = \mathbb{H}_F$ with B, and the corresponding mapping of the functional spaces

$$\kappa : \ L_2(B) \ \longrightarrow \ L_2(\mathbb{R}^m \times \mathbb{C}^n) = L_2(\mathbb{H}_F),$$

where $(\kappa f)(t, z) = f(\kappa_0(t, z))$, transfers isometrically the Hardy space $H^2(D)$ onto some subspace $H_2(\mathbb{H}_F) = \kappa(H^2(D))$ of the space $L_2(\mathbb{H}_F)$.

We can give the following equivalent definition of the space $H^2(\mathbb{H}_F)$ (see Subsection 1.6): the space $H^2(\mathbb{H}_F)$ is the closure in $L_2(\mathbb{H}_F)$ of a linear vector space of the functions which satisfy the following tangential Cauchy–Riemann equations on $\mathbb{H}_F$:

$$\overline{Z} f_0 = \left(\frac{\partial}{\partial \overline{z}} - i\, \overline{L(\frac{\partial}{\partial t})\, z} \right) f_0 = 0. \tag{5.1}$$

Recall [9, 21], that the domain D has a group of the biholomorphic automorphisms of the form

$$\omega_{(a,c)} : \ (w, z) \in D \ \longmapsto \ (w + a + 2iF(z, c) + iF(c, c), \ z + c) \in D,$$

for all $(a, c) \in \mathbb{R}^m \times \mathbb{C}^n$. These automorphisms preserve the distinguished boundary B and act on B transitively. The mapping κ_0 transfers this group of automorphisms onto $\mathbb{R}^m \times \mathbb{C}^n$. In this context we have

$$\omega_{(a,c)} : \ (t, z) \in \mathbb{R}^m \times \mathbb{C}^n \ \longmapsto \ (a, c) \cdot (t, z) \in \mathbb{R}^m \times \mathbb{C}^n,$$

where the multiplication means the group multiplication in the group $\mathbb{H}_F$. Biholomorphic automorphisms ω_g, $g = (a, c) \in \mathbb{R}^m \times \mathbb{C}^n$, generate unitary on $L_2(B)$ shift operators

$$(W_g^0 f)(w, z) = f(\omega_g^{-1}(w, z)).$$

These operators preserve the Hardy space $H^2(D)$ and thus commute with Sego projector P_D of the space $L_2(B)$ onto $H^2(D)$:

$$W_g^0 \cdot P_D = P_D \cdot W_g^0,$$

for all $g \in \mathbb{R}^m \times \mathbb{C}^n$. Then the operators $W_g = \kappa W_g^0 \kappa^{-1}$ preserve the space $H^2(\mathbb{H}_F)$ and commute with the orthogonal projector P of the space $L_2(\mathbb{H}_F)$ onto $H^2(\mathbb{H}_F)$.

The operator

$$(W_g f)(h) = f(\omega_g^{-1}(h)) = f(g^{-1}h) = (\pi_l(g)f)(h)$$

coincides with the left shift operator in the group structure of $\mathbb{H}_F$, and thus it is natural that the operator P is a right–side group convolution operator on $\mathbb{H}_F$ (see, for instance, [4, 10]), *i.e.*

$$P = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{H}_F} p(g)\, \pi_r(g)\, dg. \tag{5.2}$$

From (4.1) and Lemma 4.1 if follows that the Weyl symbol of the operator (5.2) is calculated by the formula

$$\omega_P(\tilde{x}, \lambda, \tilde{\xi}) = \hat{p}(-\lambda,\, -\tilde{\xi} + 2\Lambda(\lambda)\tilde{x}), \tag{5.3}$$

where $\hat{p}$ is a Fourier transformation of p.

Putting $z = x + iy$, $\zeta = \xi + i\eta$ we can rewrite the Weyl symbol (5.3) in the complex form

$$\omega_P(z, \lambda, \zeta) = \hat{p}(-\lambda,\, 2i\, \omega_{\overline{Z}}(z, \lambda, \zeta)),$$

where $\omega_{\overline{Z}}(z, \lambda, \zeta) = \frac{1}{2} i\zeta + \overline{L(\lambda)} z$ is a Weyl symbol of the tangential Cauchy–Riemann operator (5.1).

By Lemma 2.2 the operator P generates essentially bounded measurable operator–function

$$P(\cdot) : \;\; \mathbb{R}^m \;\; \longrightarrow \;\; \mathfrak{B}(L_2(\mathbb{C}^n))$$

such that

$$P = (F^{-1} \otimes I)\, (I \otimes P(\lambda))\, (F \otimes I).$$

The results of Subsection 1.6 show that for all $\lambda \in \overline{\mathbb{R}^m \setminus V^*}$ $P(\lambda) = 0$. The Weyl symbol of the operator $P(\lambda)$ for $\lambda \in$ Int V^* has the form

$$\omega_{P(\lambda)}(z, \zeta) = \hat{p}(-\lambda,\, 2i\, \omega_{\overline{Z}(\lambda)}(z, \zeta)),$$

where $\omega_{\overline{Z}(\lambda)}(z, \zeta) = \frac{1}{2}\, i\zeta + \overline{L(\lambda)} z$ is a Weyl symbol of the operator

$$\overline{Z}(\lambda) = \frac{\partial}{\partial \overline{z}} + \overline{L(\lambda)} z.$$

Moreover the operator $P(\lambda)$ is an orthogonal projector of $L_2(\mathbb{C}^n)$ onto the space $H^2(\lambda)$ which is a closure in $L_2(\mathbb{C}^n)$ of a linear vector space of the functions satisfying the equation $\overline{Z}(\lambda)\varphi = 0$.

For all $\lambda \in \text{Int } V^*$ the Hermitian matrix $L(\frac{\lambda}{|\lambda|})$ can be reduced to a diagonal form

$$UL(\frac{\lambda}{|\lambda|})U^{-1} = \text{diag } (\varepsilon_1^2, \varepsilon_2^2, \ldots, \varepsilon_n^2) = e^2(\varepsilon)$$

with the help of some unitary matrix $U = U(\frac{\lambda}{|\lambda|})$.

Let us introduce now the following symplectic in $T^*(\mathbb{C}) = T^*(\mathbb{R}^{2n})$ changes of variables (see Section 3)

$$z = U'z_1, \quad \zeta = U'\zeta_1,$$

and

$$x_1 = \frac{1}{2\sqrt{|\lambda|}}\, e^{-1}(\varepsilon)\,(x+y), \quad y_1 = \frac{1}{2\sqrt{|\lambda|}} e^{-1}(\varepsilon)\,(\xi - \eta),$$

$$\xi_1 = \sqrt{|\lambda|}\, e(\varepsilon)\,(\xi + \eta), \quad \eta_1 = \sqrt{|\lambda|}\, e(\varepsilon)\,(-x + y).$$

Denote by $\nu_1 = \nu_1(\lambda)$ and unitary on $L_2(\mathbb{C}^n)$ operator which realizes the composition of these symplectic changes of variables. Then (see Section 3) we have

$$\nu_1(\lambda) \circ \overline{Z}(\lambda) \circ \nu_1^{-1}(\lambda) = \sqrt{|\lambda|}\, e(\varepsilon) \left(\frac{\partial}{\partial x} + x \right).$$

And thus the operator $\nu_1(\lambda)$ maps isometrically the space $H^2(\lambda)$ onto the space $\nu_1(\lambda)\,(H^2(\lambda)) = H_1^2(\lambda)$ which is the closure in $L_2(\mathbb{C}^n)$ of a linear vector space of the functions satisfying the equation

$$\left(\frac{\partial}{\partial x} + x \right) \varphi(x, y) = 0. \tag{5.4}$$

And the operator $P_1(\lambda) = \nu_1(\lambda) P(\lambda) \nu_1^{-1}(\lambda)$ is the orthogonal projector from $L_2(\mathbb{C}^n)$ onto $H_1^2(\lambda)$, and its Weyl symbol is calculated by the formula

$$\omega_{P_1(\lambda)} = \hat{p}(-\lambda,\, 2i\sqrt{|\lambda|}\, U' e(\varepsilon)\,(x + i\xi)). \tag{5.5}$$

The equation (5.4) can be solved easily. Its general solution has the form

$$\pi^{-\frac{n}{4}} e^{-\frac{x^2}{2}} \cdot g(y),$$

where $g(y)$ is an arbitrary function from $L_2(\mathbb{R}^n)$. The function $l(x) = \pi^{-\frac{n}{4}} e^{-\frac{x^2}{2}}$ belongs to $L_2(\mathbb{R}^n)$ and $\|l(x)\| = 1$.

Denote by L_0 the one–dimensional subspace of $L_2(\mathbb{R}^n)$ generated by $l(x)$. Then

$$H_1^2(\lambda) = L_0 \otimes L_2(\mathbb{R}^n)$$

and

$$P_1(\lambda) = P_0 \otimes I,$$

where

$$(P_0 f)(x) = \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x^2+u^2)} f(u)\, du \tag{5.6}$$

is the one–dimensional orthogonal projector of $L_2(\mathbb{R}^n)$ onto L_0. Introduce now the (unitary) operator $\nu_2 : L_2(\mathbb{R}^m \times \mathbb{C}^n) \longrightarrow L_2(\mathbb{R}^m \times \mathbb{C}^n)$ by the following rule:

$$\nu_2\, f(\lambda, z) = (I \otimes \chi_{V^*}(\lambda)\nu_1(\lambda) + (1 - \chi_{V^*}(\lambda))I) f(\lambda, z).$$

The following theorem gives the result we have obtained.

Theorem 5.1 *The unitary operator $\nu = \nu_2\,(F \otimes I)$ gives an isometrical isomorphism of the space*

$$L_2(\mathbb{R}^m \times \mathbb{C}^n) = L_2(\mathbb{R}^m) \otimes L_2(\mathbb{R}^n) \otimes L_2(\mathbb{R}^n)$$

under which

1. *the space $H^2(\mathbb{H}_F)$ is mapped onto*
$$L_2(V^*) \otimes L_0 \otimes L_2(\mathbb{R}^n);$$
2. $\nu P \nu^{-1} = \chi_{V^*}(\lambda) I \otimes P_0 \otimes I,$

where $\chi_{V^*}(\lambda)$ is a characteristic function of a dual cone V^*, P_0 is the one–dimensional orthogonal projector (5.6) *onto the one–dimensional subspace L_0 in $L_2(\mathbb{R}^n)$ generated by the function $l(x) = \pi^{-\frac{n}{4}} e^{-\frac{x^2}{2}}$.*

The following corollary gives an explicit isomorphic description of the Hardy space $H^2(D)$ and the Szegö projector P_D for the type II Siegel domain.

Corollary 5.1 *The operator $\nu_D = \nu \circ \kappa$ maps isometrically $L_2(B)$ onto $L_2(\mathbb{R}^m) \otimes L_2(\mathbb{R}^n) \otimes L_2(\mathbb{R}^n)$ and under this mapping we have*

1. $\nu_D(H^2(D)) = L_2(V^*) \otimes L_0 \otimes L_2(\mathbb{R}^n);$
2. $\nu_D P_D \nu_D^{-1} = \chi_{V^*}(\lambda) I \otimes P_0 \otimes I.$

5.2

We will calculate now the Weyl symbol of the projector P (5.2). Let us start from the Weyl symbol of the operator P_0 (5.6).

We have

$$\begin{aligned} \pi^{\frac{n}{2}} &= \int_{\mathbb{R}^n} \Pi_{j=1}^n e^{-((x_j - u_j)/2))^2} d\xi = \int_{\mathbb{R}^n} e^{-\xi^2 + i(x-u)\cdot\xi + ((x-u)/2)^2} d\xi = \\ &= \int_{\mathbb{R}^n} e^{-\xi^2 + i(x-u)\cdot\xi + (x^2+u^2)/2 - ((x+u)/2)^2} d\xi. \end{aligned}$$

Substitute it into (5.6)

$$\begin{aligned} (P_0 f)(x) &= \pi^{-n} \int_{\mathbb{R}^n} \pi^{\frac{n}{2}} e^{-(x^2+u^2)/2} f(u)\, du = \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-u)\cdot\xi} \cdot 2^n e^{-((x+u)/2)^2 - \xi^2} f(u)\, du\, d\xi. \end{aligned}$$

Thus we have

$$\omega_{P_0}(x, \xi) = 2^n \cdot e^{-(x^2+\xi^2)}. \tag{5.7}$$

Comparing (5.7) with (5.5) we obtain

$$\hat{p}(-\lambda,\, 2i\sqrt{|\lambda|}U'e(\varepsilon)(x+i\xi)) = 2^n \cdot e^{-(x^2+\xi^2)},$$

or

$$\hat{p}(-\lambda,\, 2iU'e(\varepsilon)(x+i\xi) = 2^n \cdot e^{-\frac{x^2+\xi^2}{|\lambda|}}.$$

Taking into account that $\hat{p}=0$ for $\lambda \in \overline{\mathbb{R}^m \setminus V^*}$ we have

$$\hat{p}(\lambda,\, 2iU'e(\varepsilon)(x+i\xi)) = \chi_{V^*}(\lambda) \cdot 2^n \cdot e^{-\frac{x^2+\xi^2}{|\lambda|}}. \tag{5.8}$$

In particular it implies that the function $\hat{p}$ is a δ_τ–homogeneous one of the degree zero.

Let us put $\alpha = 2\,iU'e(\varepsilon)(x+i\xi)$, then $x+i\xi = -\frac{i}{2}e^{-1}(\varepsilon)\overline{U}\alpha$ and

$$\begin{aligned} x^2+\xi^2 &= (x+i\xi)'\cdot(x-i\xi) = \frac{1}{4}\alpha' U^{-1}e^{-1}(\varepsilon)\cdot e^{-1}(\varepsilon)U\overline{\alpha} = \\ &= \frac{1}{4}\alpha' U^{-1}e^{-2}(\varepsilon)U\overline{\alpha} = \frac{1}{4}\alpha' L^{-1}(\frac{\lambda}{|\lambda|})\overline{\alpha}. \end{aligned}$$

Thus

$$\hat{p}(-\lambda,\, \alpha) = \chi_{V^*}(\lambda)\, 2^n\, e^{-\frac{\alpha' L^{-1}(\frac{\lambda}{|\lambda|})\overline{\alpha}}{4|\lambda|}},$$

and

$$\begin{aligned} \omega_{P(\lambda)}(z,\lambda) &= \hat{p}(-\lambda,\, 2i(\frac{1}{2}i\zeta + \overline{L(\lambda)z})) = \hat{p}(-\lambda, -(\zeta - 2i|\lambda|\overline{L(\frac{\lambda}{|\lambda|})z})) = \\ &= \chi_{V^*}(\lambda)\, 2^n\, e^{-\frac{(\zeta-2i|\lambda|\overline{L(\frac{\lambda}{|\lambda|})})' L^{-1}(\frac{\lambda}{|\lambda|})\overline{(\zeta-2i|\lambda|\overline{L(\frac{\lambda}{|\lambda|})}z)}}{4|\lambda|}}. \end{aligned}$$

Taking into account $L^* = \overline{L}' = L$ we have

$$\begin{aligned} &(\zeta' - 2i|\lambda|z'L)L^{-1}(\overline{\zeta} + 2i|\lambda|L\overline{z}) = \\ &= \zeta' L^{-1}\overline{\zeta} + 4|\lambda|^2 z'L\overline{z} + 2i|\lambda|(\zeta'\overline{z} - z'\zeta) = \\ &= \zeta' L^{-1}(\frac{\lambda}{|\lambda|})\zeta' + 4|\lambda|\,\lambda\cdot F(z,z) + 4|\lambda|(x\eta - y\xi). \end{aligned}$$

Recall that $z = x+iy$, $\zeta = \xi + i\eta$. And finally

$$\omega_{P(\lambda)}(x,y,\xi,\eta) = \chi_{V^*}(\lambda)\cdot 2^n \cdot e^{-\lambda\cdot F(z,z)-(x\eta-y\xi)}\cdot e^{-\frac{\zeta' L^{-1}(\frac{\lambda}{|\lambda|})\overline{\zeta}}{4|\lambda|}}.$$

Putting $E = \partial V^* (\subset \mathbb{R}^m)$ in Theorem 4.8 we obtain

Theorem 5.2 *The Weyl symbol of the orthogonal projector* $P:\, L_2(\mathbb{H}_F) \longrightarrow H^2(\mathbb{H}_F)$ *is calculated by the formula*

$$\begin{aligned} \omega_P(x,u;\lambda,\xi,\eta) &= \chi_{V^*}(\lambda)\cdot 2^n \cdot e^{-\overline{z}'L(\lambda)\overline{z}-(x\eta-y\xi)}\cdot e^{\frac{1}{4}\zeta' L^{-1}(\lambda)\overline{\zeta}} = \\ &= \chi_{V^*}(\lambda)\cdot 2^n \cdot e^{-\lambda\cdot F(z,z)-(x\eta-y\xi)}\cdot e^{-\frac{1}{4}\zeta'(\lambda\cdot F)^{-1}\overline{\zeta}}. \end{aligned}$$

Corollary 5.3 *For the Heisenberg group* $\mathbb{H}_n = \mathbb{R}\times\mathbb{C}^n$ *the Weyl of the orthogonal projector* $P:\, L_2(\mathbb{H}_n) \to \mathbb{H}^2(\mathbb{H}_n)$ *is calculated by the formula*

$$\omega_P(x,y;\,\lambda,\xi,\eta) = \chi_{\mathbb{R}_+}(\lambda)\cdot 2^n \cdot e^{-\lambda|z|^2-(x\eta-y\xi)}\cdot e^{-\frac{|\zeta|^2}{4\lambda}}.$$

5.3

We will give now a formula for partial (with respect to t) Fourier transformation $\hat{p}_I(-\lambda, z)$ of the kernel of the projector P (5.2).

Put $w = x + i\xi \in \mathbb{C}^n$ and $\widetilde{w} = (x, \xi) \in \mathbb{R}^{2n}$, then (5.8)

$$\hat{p}(-\lambda, 2iU'e(\varepsilon)w) = \chi_{V^*}(\lambda) \cdot 2^n \cdot e^{-\frac{|w|^2}{|\lambda|}},$$

or "in the real form"

$$\hat{p}(-\lambda, A\widetilde{w}) = X_{V^*}(\lambda) \cdot 2^n \cdot e^{-\frac{|\widetilde{w}|^2}{|\lambda|}}, \tag{5.9}$$

where (for $U = U_1 + iU_2$))

$$A = -2 \begin{pmatrix} U_2' & U_1' \\ -U_1' & U_2' \end{pmatrix} \begin{pmatrix} e(\varepsilon) & 0 \\ 0 & e(\varepsilon) \end{pmatrix}.$$

We have

$$\begin{aligned} \left[(I \otimes F^{-1}_{\widetilde{w}\to\widetilde{\omega}})\hat{p}(-\lambda, A\widetilde{w})\right](\widetilde{\omega}) &= |\det A^{-2}| \left[(I \otimes F^{-1}_{\widetilde{w}\to\widetilde{\omega}})\hat{p}(-\lambda, \widetilde{w})\right](A^{-1'}\widetilde{\omega}) = \\ &= |\det A^{-1}| \cdot \hat{p}_I(-\lambda, A^{-1'}\widetilde{\omega}), \end{aligned} \tag{5.10}$$

where $\hat{p}_I = (F \otimes I)p = (I \otimes F^{-1})\hat{p}$.

The matrix U is unitary, thus

$$|\det A| = \det \begin{pmatrix} 2e(\varepsilon) & 0 \\ 0 & 2\varepsilon(\varepsilon) \end{pmatrix} = \det 4e^2(\varepsilon) = \det 4L(\frac{\lambda}{|\lambda|}).$$

On the other hand, putting $B = \frac{1}{\sqrt{|\lambda|}} e_{2n}$ we have

$$\left[(I \otimes F^{-1}_{\widetilde{w}\to\widetilde{\omega}})\chi_{V^*}(\lambda) \cdot 2^n \cdot e^{-\frac{|\widetilde{w}|^2}{|\lambda|}}\right](\widetilde{\omega}) =$$

$$\begin{aligned} &= \left[(I \otimes F^{-1}_{\widetilde{w}\to\widetilde{\omega}})\chi_{V^*}(\lambda) \cdot 2^n \cdot e^{-|B\widetilde{w}|^2}\right](\widetilde{\omega}) = \\ &= |\det B^{-1}| \cdot \left[(I \otimes F^{-1}_{\widetilde{w}\to\widetilde{\omega}}) \cdot 2^n \cdot \chi_{V^*} e^{-|\widetilde{w}|^2}\right](\sqrt{|\lambda|}\widetilde{\omega}) = \\ &= |\lambda|^n e^{-\frac{1}{4}|\lambda|\cdot|\widetilde{\omega}|^2} \cdot \chi_{V^*}(\lambda). \end{aligned} \tag{5.11}$$

Formulas (5.9), (5.10) and (5.11) imply

$$\hat{p}_I(-\lambda, A^{-1'}\widetilde{\omega}) = |\det A| \cdot e^{-\frac{1}{4}|\lambda|\cdot|\widetilde{\omega}|^2} \cdot \chi_{V*}(\lambda). \tag{5.12}$$

Put now $A^{-1'}\widetilde{\omega} = (x, y) = \widetilde{x}$, then $\widetilde{\omega} = A'\widetilde{x}$, where

$$A' = -2 \begin{pmatrix} e(\varepsilon) & 0 \\ 0 & e(\varepsilon) \end{pmatrix} \begin{pmatrix} U_2 & -U_1 \\ U_1 & U_2 \end{pmatrix}.$$

And thus $\omega = -2ie(\varepsilon)\overline{U}z$, where $z = x + iy$.

Then

$$|\widetilde{\omega}|^2 = \omega' \cdot \widetilde{\omega} = 4z'\overline{U}'e(\varepsilon)^2U\overline{z} = 4z'L(\frac{\lambda}{|\lambda|})\overline{z} = 4 \cdot \frac{\lambda}{|\lambda|} \cdot F(z,z).$$

Substituting it into (5.12) we have

$$\hat{p}_I(\lambda, z) = \chi_{V*}(\lambda)e^{-\lambda \cdot F(z,z)} \cdot \det\ (4\lambda \cdot F) = \chi_{V*}(\lambda) \cdot e^{-z'L(\lambda)\overline{z}} \cdot \det\ 4L(\lambda).$$

Theorem 5.4 *The partial (with respect to t) Fourier transformation $\hat{p}_I(-\lambda, z)$ of the kernel of the projector P (5.2) is calculated by the formula*

$$\hat{p}_I(-x, z) = \chi_{V*}(\lambda)e^{-\lambda \cdot F(z,z)}\det\ (4\lambda \cdot F) = \chi_{V*}(\lambda)e^{-z'L(\lambda)\overline{z}}\det\ 4L(\lambda).$$

Remark *This result has been obtained in* [4] *in a different way.*

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