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Convolution operators on standard CR-manifolds II. Algebras of convolution operators on the Heisenberg group

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Various algebras generated by one–sided and two–sided convolution operators with homogeneous kernels and on the Heisenberg group are considered. Isomorphic description of these algebras are given. Their spectra and all irreducible representations are described.

As an application an algebra generated by Toeplitz operators on the Heisenberg group is studied.

0 Introduction

In the paper we study the convolution operator algebras on the Heisenberg group $\mathbb{H}_n$ – the space $\mathbb{R}^N = \mathbb{R} \times \mathbb{C}^n, N = 1 + 2n$, with the following group operation

$$(t_1, z_1) \cdot (t_2, z_2) = (t_1 + t_2 + 2\text{Im}\ z_1 \cdot \overline{z}_2, z_1 + z_2).$$

The noncommutativity of this operation induces (in contrast to the standard summation operation in $\mathbb{R}^N$) the existence of two shift operators: the right and the left shift operators which are defined on $L_2(\mathbb{H}_n)$ by the rule

$$\begin{aligned} (\pi_r(g)\varphi)(h) &= \varphi(hg), \\ (\pi_l(g)\varphi)(h) &= \varphi(g^{-1}h). \end{aligned}$$

Integrating these shift operators with some kernel functions we arrive to the right and the left group convolution operators

$$B = (2\pi)^{-\frac{N}{2}} \int\limits_{\mathbb{H}_n} b(g)\pi_r(g)dg, \tag{0.1}$$

$$C = (2\pi)^{-\frac{N}{2}} \int\limits_{\mathbb{H}_n} c(g)\pi_l(g)dg. \tag{0.2}$$

The convolution operators on the Heisenberg group have been studying intensively by many authors (see, for example, the books [Fo, Ge, Ta2] which contain expansive bibliography).

Usually the investigation of the convolution operators on $\mathbb{H}_n$ has been based on the group Fourier transformation and that is why only either left– invariant (0.1) or right–invariant (0.2) operators have been considered.

This paper continuous the investigation of [VT] on which another approach for the study of convolution operators has been used. This approach is based on the direct integral construction and symplectic changes of variables and allows to study the operators (0.1) and (0.2) simultaneously. Moreover it turns out to be useful to consider the so–called two–sided convolution operators

$$K = (2\pi)^{-N} \int\limits_{\mathbb{H}_n \times \mathbb{H}_n} k(g_1, g_2)\pi_r(g_2)\pi_l(g_2)dg_1dg_2. \tag{0.3}$$

This work is used essentially the results from [VT]. Section 1 contains necessary material from [VT].

The main result of Section 2 is the general local principle for C^*–algebras based on the Dauns–Hofmann construction [DH,H]. It gives a flexible tool for the description of C^*–algebras and allows to take into consideration the peculiarities of each concrete algebra. Many of previously considered local principles can be interpreted as special cases of the one described in Section 2.

All constructions of first two sections are given for the more general nilpotent group $\mathbb{H}_F$, and only from Section 3 the results are restricted for the Heisenberg group $\mathbb{H}_n$. Nevertheless the main results of the article are concentrated in Section 3 and 4. Various algebras generated by one– sided (0.1), (0.2) and two–sided (0.3) convolution operators are considered. The explicit isomorphic description of these algebras are given, the spectra and all irreducible representations of these algebras are described.

The Szegö orthogonal projector onto Hardy space associated with the Heisenberg group (considered as the distinguished boundary of some Siegel domain) belongs to algebras of Section 3. We give in Section 4 the description of an algebra generated by Toeplitz operators associated with this Siegel domain.

1 Preliminaries.

1.1 Let $F(z_1, z_2)$ be a Hermitian form defined on $\mathbb{C}^n$ and valued in $\mathbb{C}^m$. Denote by $\mathbb{H}_F$ the space $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{C}^n$ endowed with the following group operation

$$(t_1, z_1) \cdot (t_2, z_2) = (t_1 + t_2 + 2\mathrm{Im}\ F(z_1, z_2), z_1 + z_2), \tag{1.1}$$

where $(t_k, z_k) \in \mathbb{H}_F = \mathbb{R}^m \times \mathbb{C}^n, \quad k = 1, 2.$

It is known (see, for instance [OV, Ta2]) that $\mathbb{H}_F$ is a step two nilpotent Lie group and its (two–sided) Haar measure coincides with the Lebesgue measure on $\mathbb{R}^N$. For $m = 1$ and $F(z_1, z_2) = z_1 \cdot \overline{z}_2 \quad \mathbb{H}_F$ is just the Heisenberg group.

Introduce the group convolution operators on $\mathbb{H}_F$. Let π_r and π_l be the right and the left regular representations of the group $\mathbb{H}_F$ on $L_2(\mathbb{H}_F)$. The operators $\pi_r(g)$ and $\pi_l(g)$, for $g \in \mathbb{H}_F$, have the following form

$$(\pi_r(g)f)(h) = f(hg), \tag{1.2}$$
$$(\pi_l(g)f)(h) = f(g^{-1}h). \tag{1.3}$$

As usual the representations π_r and π_l are extended to the representations of the group algebra of $\mathbb{H}_F$. In particular the right and the left group convolution operators which correspond to functions $b, c \in L_1(\mathbb{H}_F)$ have the form

$$B = \pi_r(b) = (2\pi)^{-\frac{N}{2}} \int\limits_{\mathbb{H}_F} b(g)\pi_r(g)dg, \tag{1.4}$$

$$C = \pi_l(c) = (2\pi)^{-\frac{N}{2}} \int\limits_{\mathbb{H}_F} c(g)\pi_l(g)dg. \tag{1.5}$$

Introduce also the “two–sided” group convolution operator, i.e. for a function $k \in L_1(\mathbb{H}_F \times \mathbb{H}_F)$ define

$$K = (2\pi)^{-N} \int\limits_{\mathbb{H}_F \times \mathbb{H}_F} k(g_1, g_2)\pi_r(g_1)\pi_l(g_2)dg_1 dg_2. \tag{1.6}$$

Note, that the operators (1.2) and (1.3) are mutually commuting, thus they may be written in (1.6) in an arbitrary order.

1.2 Let $(F_1, \dots, F_m)$ be the coordinate notation of the vector form F. Each $F_\nu, \nu = \overline{1, m}$, is a scalar Hermitian form and thus

$$F_\nu(z_1, z_2) = z_1' F_\nu \overline{z}_2.$$

As usual a form and a corresponding Hermitian matrix are denoted by the same letter. For $\lambda \in \mathbb{R}^m$ denote by $L(\lambda)$ the following scalar Hermitian form

$$\begin{aligned} L(\lambda)(z_1, z_2) &= \lambda \cdot F(z_1, z_2) = \sum_{\nu=1}^{m} \lambda_\nu F_\nu(z_1, z_2) = \\ = z_1' L(\lambda)\overline{z}_2 &= z_1'(\sum_{\nu=1}^{m} \lambda_\nu F_\nu)\overline{z}_2. \end{aligned}$$

Introduce also the following $2n \times 2n$ matrix

$$\Lambda(\lambda) = \begin{pmatrix} \mathrm{Im}\ L(\lambda) & -\mathrm{Re}\ L(\lambda) \\ \mathrm{Re}\ L(\lambda) & \mathrm{Im}\ L(\lambda) \end{pmatrix}.$$

Note that the matrices $L(\lambda)$ and $\Lambda(\lambda)$ are homogeneous of the first order with respect to λ.

We will use the following notations for the direct and the dual coordinates of the points of

$$\mathbb{H}_F = \mathbb{R}^m \times \mathbb{C}^n = \mathbb{R}^m \times \mathbb{R}^{2n} = \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n :$$

$$g = (t, z) = (t, \widetilde{x}) = (t, x, y)$$

and

$$\gamma = (\lambda, \zeta) = (\lambda, \widetilde{\xi}) = (\lambda, \xi, \eta).$$

Here, for example, $z = x + iy \in \mathbb{C}^n$, $\widetilde{x} = (x, y) \in \mathbb{R}^{2n}$, $t \in \mathbb{R}^m$.

Theorem 1.1 ([VT]) *The right (1.4), the left (1.5) and the two–sided (1.6) convolution operators are pseudodifferential operators on $L_2(\mathbb{H}_F) = L_2(\mathbb{R}^N)$ and have the following Weyl symbols, respectively,*

$$\begin{aligned}
\omega_B(t,\widetilde{x},\lambda,\widetilde{\xi}) &= \hat{b}(-\lambda,-\widetilde{\xi}+2\Lambda(\lambda)\widetilde{x}) = && (1.7)\\
&= \hat{b}(-\lambda,-\xi+2\mathrm{Im}\ L(\lambda)\overline{z},-\eta+2\mathrm{Re}\ L(\lambda)\overline{z}),\\
\omega_C(t,\widetilde{x},\lambda,\widetilde{\xi}) &= \hat{c}(\lambda,\widetilde{\xi}+2\Lambda(\lambda)\widetilde{x}) = && (1.8)\\
&= \hat{c}(\lambda,\xi+2\mathrm{Im}\ L(\lambda)\overline{z},\eta+2\mathrm{Re}\ L(\lambda)\overline{z}),\\
\omega_K(t,\widetilde{x},\lambda,\widetilde{\xi}) &= \hat{k}(-\lambda,-\widetilde{\xi}+2\Lambda(\lambda)\widetilde{x},\lambda,\widetilde{\xi}+2\Lambda(\lambda)\widetilde{x}) =\\
&= \hat{k}(-\lambda,-\xi-2\mathrm{Im}\ L(\lambda)\overline{z},-\eta+2\mathrm{Re}\ L(\lambda)\overline{z}, && (1.9)\\
&\quad \lambda,\xi+2\mathrm{Im}\ L(\lambda)\overline{z},\eta+2\mathrm{Re}\ L(\lambda)\overline{z}),
\end{aligned}$$

where $\hat{b},\hat{c}$ and $\hat{k}$ are the (Euclidean) Fourier transforms of the kernels b,c and k of the corresponding convolution operators.

Recall ([Hö, Sh]), that acting on the space $L_2(\mathbb{R}^N)$ pseudodifferential operator A with a Weyl symbol $\omega_A(g,\gamma)$ has the form

$$(Af)(g) = (2\pi)^{-N}\int\limits_{\mathbb{R}^N} e^{i(g-h)\cdot\gamma}\omega_A(\frac{1}{2}(g+h),\gamma)f(h)dhd\gamma.$$

As in the theory of pseudodifferential operators it is more convenient to define and to study the convolution operators on the group $\mathbb{H}_F$ in the terms of their (pseudodifferential) symbols, not in the terms of their kernels.

Lemma 1.2 ([VT]) *Let B and C be bounded on $L_2(\mathbb{R}^N)$ pseudodifferential operators with the Weyl symbols (1.7) and (1.8) respectively. Then the Weyl symbols of these operators coincide with their left (pseudodifferential) symbols, moreover for all $\tau \in [0,1]$ their τ–symbols do not depend on τ and coincide with (1.7) and (1.8) respectively.*

For the definition of the τ–symbol see, for example, [Sh].

1.3 Introduce on $\mathbb{H}_F = \mathbb{R}^m \times \mathbb{C}^n = \mathbb{R}^m \times \mathbb{R}^{2n}$ the one–parameter group of the nonisotropic dilations $\delta_\tau, \tau > 0$, (see, for example, [Go Ta1]) by the rule

$$\delta_\tau(\tau,\zeta) = (\tau\lambda,\tau^{\frac{1}{2}}\zeta), \quad (\lambda,\zeta)\in\mathbb{R}^m\times\mathbb{C}^n,$$

or

$$\delta_\tau(\lambda,\widetilde{\xi}) = (\tau\lambda,\tau^{\frac{1}{2}}\widetilde{\xi}), \quad (\lambda\widetilde{\xi})\in\mathbb{R}^m\times\mathbb{R}^{2n}.$$

A function ψ on $\mathbb{H}_F$ is called a δ_τ–homogeneous function of the degree $s\in\mathbb{R}$ if

$$\psi\circ\delta_\tau = \tau^s\cdot\psi.$$

The values of an arbitrary δ_τ–homogeneous function ψ is determined uniquely by its values on the nonisotropic unit sphere Ω in $\mathbb{H}_F$

$$\Omega = \{(\lambda,\widetilde{\xi})\in\mathbb{H}_F = \mathbb{R}^m\times\mathbb{R}^{2n}: \quad \|(\lambda,\widetilde{\xi})\| = 1\}, \tag{1.10}$$

where $\|(\lambda,\widetilde{\xi})\|^2 = \sum\limits_{\nu=1}^{m}|\lambda_\nu|^2 + \sum\limits_{k=1}^{n}(|\xi_k|^4+|\eta_k|^4)$.

For some algebra $\mathcal{A}(\subset L_\infty(\Omega))$ of functions on Ω denote by $H^s_\delta(\mathcal{A})$ the algebra of all δ_τ–homogeneous of the degree s functions in $\mathbb{H}_F$ having $\mathcal{A}$ restrictions onto Ω.

Theorem 1.3 ([VT]) *Let E be a closed subset in $\mathbb{R}^m$ with the measure zero and with the "cone" property:*

$$\forall\lambda\in E \qquad \forall\tau>0 \qquad \tau\lambda\in E,$$

and let $E_\Omega=\Omega\cap(E\times\mathbb{R}^{2n})$. Further let $\widetilde{b}(-\lambda,\widetilde{\xi})$ and $\widetilde{c}(\lambda,\widetilde{\xi})$ be functions from $H^\circ_\delta(C^\infty(\Omega\backslash E_\Omega))$, $(\widetilde{k}(-\lambda,\widetilde{\xi},\lambda,\widetilde{\xi})\in H^\circ_\delta(C^\infty(\Omega\backslash E_\Omega))\otimes H^\circ_\delta(C^\infty(\Omega\backslash E_\Omega))^*$. Then pseudodifferential operators B,C and K with the Weyl symbols

$$\begin{aligned}\omega_B(\widetilde{x},\lambda,\widetilde{\xi}) &= \widetilde{b}(-\lambda,-\widetilde{\xi}+2\Lambda(\lambda)\widetilde{x}),\\ \omega_C(\widetilde{x},\lambda,\widetilde{\xi}) &= \widetilde{c}(\lambda,\widetilde{\xi}+2\Lambda(\lambda)\widetilde{x})\end{aligned}$$

and

$$\omega_K(\widetilde{x},\lambda,\widetilde{\xi})=\widetilde{k}(-\lambda,-\widetilde{\xi}+2\Lambda(\lambda)\widetilde{x};\lambda,\widetilde{\xi}+2\Lambda(\lambda)\widetilde{x}),$$

respectively, are bounded on $L_2(\mathbb{H}_F)$.

Moreover each this operator, say $A\in\{B,C,K\}$, generates an operator–function

$$A(\cdot):\quad \mathbb{R}^m\to\mathfrak{B}(L_2(\mathbb{R}^{2n}))$$

which is essentially bounded and measurable in $\mathbb{R}^m$ and is strongly continuous in $\mathbb{R}^m\backslash E$. The operator

$$\widetilde{A}f(\lambda,\widetilde{\xi})=(I\otimes A(\lambda))f(\lambda,\widetilde{\xi})$$

defined by $A(\cdot)$ is bounded on $L_2(\mathbb{R}^m\times\mathbb{R}^{2n})$ and is unitary equivalent to the operator A:

$$A=(F^{-1}\otimes I)\widetilde{A}(F\otimes I),$$

where F and F^{-1} are direct and inverse Fourier transformation on $L_2(\mathbb{R}^m)$.

And in addition

$$\|A\|=\sup_{\lambda\in\mathbb{R}^m\backslash E}\|A(\lambda)\|.$$

Remark 1.4 *The Weyl symbol of the operator $A(\lambda)$ for $\lambda\in\mathbb{R}^m\backslash E$ is calculated by the formula*

$$\omega_{A(\lambda)}(\widetilde{x},\widetilde{\xi})=\omega_A(\widetilde{x},\lambda,\widetilde{\xi}).$$

2 General local principle.

2.1 Consider the space $\mathbb{R}^N=\mathbb{R}^m\times\mathbb{R}^l$ and denote by (t,x) and (λ,ξ) the direct and dual coordinates in this space, respectively. We need now the results from [VT, Lemmas 4.5–4.7] which can be summarized in the following proposition.

Proposition 2.1 *Let E be a measurable subset in $\mathbb{R}^m$ with $\mu(E)=0$. Consider a function $a(x,\lambda,\xi)$ defined in $\mathbb{R}^l\times\mathbb{R}^m\times\mathbb{R}^l$ with the properties*

* Here we consider an algebraic (non-closed) tensor product.

1. $a(x,\lambda,\xi) \in C_b(\mathbb{R}^l \times (\mathbb{R}^m \backslash E) \times \mathbb{R}^l)$;
2. *for all* $\lambda \in \mathbb{R}^m \backslash E$ *the operator* A_λ

$$(A_\lambda \varphi)(x) = (2\pi)^{-\frac{l}{2}} \int\limits_{\mathbb{R}^e} e^{ix\cdot\xi} a(x,\lambda,\xi)\hat{\varphi}(\xi)d\xi$$

   is bounded on $L_2(\mathbb{R}^l)$;
3. $\sup\limits_{\lambda \in \mathbb{R}^m \backslash E} \|A_\lambda\| < +\infty$.

Then

1. *The operator–function* $A(\cdot): \quad \mathbb{R}^m \to \mathfrak{B}(L_2(\mathbb{R}^l))$

$$\lambda \in \mathbb{R}^m \longmapsto \begin{cases} A_\lambda, & \text{for} \quad \lambda \in \mathbb{R}^m \backslash E \\ 0, & \text{for} \quad \lambda \in E \end{cases}$$

   is measurable in $\mathbb{R}^m$ *and is strongly continuous in* $\mathbb{R}^m \backslash E$;
2. *The operator* A

$$(Af)(t,x) = (2\pi)^{-\frac{N}{2}} \int\limits_{\mathbb{R}^N} e^{ix\cdot\xi} \cdot e^{it\cdot\lambda} \cdot a(x,\lambda,\xi)\hat{f}(\lambda,\xi)d\xi d\lambda$$

   with the symbol $a(x,\lambda,\xi)$ *is bounded on* $L_2(\mathbb{R}^N)$ *and*

$$\|A\| = \sup_{\lambda \in \mathbb{R}^m} \|A_{(\lambda)}\|.$$

3. *Operator* A *is decomposable, i.e.*

$$A = (F^{-1} \otimes I)\widetilde{A}(F \otimes I)$$

   where

$$\widetilde{A}f(\lambda,x) = (I \otimes A(\lambda))f(\lambda,x).$$

2.2 Let E be a measurable subset in $\mathbb{R}^m$. Consider a set $\mathcal{A}$ of functions defined on $\mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^l$ with the properties 1–3 of Proposition 2.1.

Then the mapping

$$\text{Op} \ : \quad a \in \mathcal{A} \longmapsto A \in \mathfrak{B}(L_2(\mathbb{R}^N))$$

is well defined.

We assume also that

4. For all $a \in \mathcal{A}$ there exists $a^\# \in \mathcal{A}$ such that

$$\text{Op}\ (a^\#) = (\text{Op}\ (a))^*.$$

Denote by $\mathcal{R} = \mathcal{R}(\mathcal{A})$ the C^*–algebra generated by all the operators Op (a), for $a \in \mathcal{A}$.

The decomposability of the operators Op (a) implies that

$$\mathcal{R} \subset (F^{-1} \otimes I)(L_\infty(\mathbb{R}^m) \otimes \mathfrak{B}(L_2(\mathbb{R}^l)))(F \otimes I).$$

Let $\mathcal{R}_0$ be a dense subalgebra in $\mathcal{R}$ consisting of all finite sum of products of operators of the form Op (a), i.e. of all the operators

$$\sum_{k=1}^{p} \prod_{j=1}^{n_p} A_{jk},$$

where $A_{j,k} = \mathrm{Op}\ (a_{j,k})$, all $a_{j,k} \in \mathcal{A}$.

Lemma 2.2 *For each fixed $\lambda \in \mathbb{R}^m$ the mapping*

$$\pi_\lambda : \quad \mathcal{R}_0 \longrightarrow \mathfrak{B}(L_2(\mathbb{R}^l))$$

defined by the rule

$$\pi_\lambda : \quad \sum_{k=1}^{p} \prod_{j=1}^{n_p} A_{jk} \longmapsto \sum_{k=1}^{p} \prod_{j=1}^{n_k} A_{jk}(\lambda)$$

is a $*$–homomorphism. And for all $A \in \mathcal{R}_0$

$$\|\pi_\lambda(A)\| \leq \|A\|.$$

The statement of lemma is trivial for $\lambda \in E$. Therefore consider the case $\lambda \in \mathbb{R}^m \backslash E$. To prove the lemma we take two operators A_1 and A_2 from $\mathcal{R}_0$ with the property: operator–functions

$$\lambda \in \mathbb{R}^m \longmapsto A_k(\lambda) \in \mathfrak{B}(L_2(\mathbb{R}^l)) \tag{2.1}$$

are bounded on $\mathbb{R}^m$ and strongly continuous in $\mathbb{R}^m \backslash E$.

Note that this property is definitely fulfilled for the generators Op $(a), a \in \mathcal{A}$, of the algebra $\mathcal{R}_0$.

We should prove that

$$\begin{aligned} (A_1 + A_2)(\lambda) &= A_1(\lambda) + A_2(\lambda) \\ (A_1 \cdot A_2)(\lambda) &= A_1(\lambda) \cdot A_2(\lambda) \end{aligned}$$

and that the operator–functions

$$\begin{aligned} \lambda &\longmapsto (A_1 + A_2)(\lambda) \\ \lambda &\longmapsto (A_1 \cdot A_2)(\lambda) \end{aligned}$$

are bounded and strongly continuous in $\mathbb{R}^m \backslash E$.

These properties are simple corollaries of the strongly continuity of the operator–functions (2.1) and of the properties of the direct integral construction.

Then by Lemma 4.6 from [VT]

$$\|\pi_\lambda(A)\| \le \sup_\lambda \|\pi_\lambda(A)\| = \|A\|.$$

Corollary 2.3 *The homomorphism π_λ is extended to the representation*

$$\pi_\lambda : \quad \mathcal{R} \to \mathfrak{B}(L_2(\mathbb{R}^l))$$

of the C^*–algebra $\mathcal{R}$ on the space $L_2(\mathbb{R}^l)$.

Denote by $J(\lambda)$ the kernel of the representation π_λ.

Lemma 2.4 *We have*

$$\bigcap_{\lambda \in \mathbb{R}^m \setminus E} J(\lambda) = \{0\}.$$

Really $A \in J(\lambda)$ if and only if $A_\lambda = \pi_\lambda(A) = 0$. Thus $A \in \bigcap_{\lambda \in \mathbb{R}^m \setminus E} J(\lambda)$ implies $A_\lambda = 0$ for all $\lambda \in \mathbb{R}^m \setminus E$, which implies $A = 0$.

2.3 The Dauns–Hofmann construction (see, for references, [DH], [H]) may be used to a general approach to local principles in the theory of C^*–algebras. Our aim here is to describe the general local principle based on this construction.

Let $\mathcal{R}$ be a C^*–algebra, and let $J_T = \{J(t) : t \in T\}$ be a system of closed two–sided ideals of $\mathcal{R}$ indexed by points of some set T.

For each $t \in T$ introduce the quotient algebra

$$\mathcal{R}(t) = \mathcal{R}/J(t).$$

We denote by $a(t)$ the image of $a \in \mathcal{R}$ in $\mathcal{R}(t)$. Let

$$E = \bigsqcup_{t \in T} \mathcal{R}(t)$$

be the disjointed union of the algebras $\mathcal{R}(t)$. Denote by

$$p : \quad E \longrightarrow T$$

the following projection mapping

$$p : \quad a(t) \longmapsto t.$$

Of course, $p^{-1}(t) = \mathcal{R}(t)$.

Introduce the topologies on E and T which turn the triple $\xi = (p, E, T)$ into a C^*–bundle. Each element $a \in \mathcal{R}$ generates a section $\tilde{a} : T \to E$ of ξ by the rule

$$\tilde{a} : \quad t \longmapsto a(t).$$

Denote by $\widetilde{\mathcal{R}}$ the set of all these sections.

For each $\varepsilon > 0$ and arbitrary $\tilde{a} \in \widetilde{\mathcal{R}}$ introduce the set

$$U(\tilde{a}, \varepsilon) = \{x \in E : \|x - \tilde{a}(p(x))\| < \varepsilon\},$$

here $\|\cdot\|$ means the norm in the C^*–algebra $\mathcal{R}$.

Provide the set E with the topology the prebase of which coincides with all the sets $U(\tilde{a},\varepsilon)$.

Remark 2.5 *The topology in each fiber $\xi(t) = \mathcal{R}(t)$ which is generated by the quotient–norm*

$$\|a(t)\| = \inf_{z\in J(t)} \|a+z\|$$

coincides with the topology in $\xi(t)$ which is generated by the restriction of the topology in E onto $\xi(t)$.

Provide the base T of the bundle $\xi = (p, E, T)$ with the quotient topology: the strongest topology for which the projection $p : E \to T$ is continuous. This topology is usually called the $*$–bundle topology and has the following properties (see for details [DH, H]):

1. the mapping $p : E \to T$ is open,
2. this topology coincides with the weakest topology for which all the mappings $\tilde{a} : T \to E$ are continuous,
3. the prebase of this topology can be define by all the sets

$$V(\tilde{a},\varepsilon) = \{t \in T : \|\tilde{a}(t)\| < \varepsilon\}.$$

The described C^*–bundle $\xi = (p, E, T)$ is called the canonical C^*–bundle defined by the C^*–algebra $\mathcal{R}$ and by the system of ideals J_T.

Let now $\xi = (p, E, T)$ be some C^*–bundle. As usual by $\Gamma^b(\xi)$ we denote the C^*–algebra of all its bounded continuous sections σ with respect to the component–wise operations and the norm

$$\|\sigma\| = \sup_{t\in T} \|\sigma(t)\|.$$

Theorem 2.6 [H,V] *Let $\mathcal{R}$ be a C^*–algebra, $J_T = \{J(t) : t \in T\}$ be a system of its closed two–sided ideals, $\xi = (p, E, T)$ be the canonical C^*–bundle defined by $\mathcal{R}$ and J_T, $\Gamma^b(\xi)$ be C^*–algebra of bounded continuous sections of ξ.*

Then the mapping

$$\tilde{\pi} : a \in \mathcal{R} \longmapsto \tilde{a} \in \Gamma^b(\xi)$$

is a morphism of C^*–algebras $\mathcal{R}$ and $\Gamma^b(\xi)$ under which

1. $\ker \tilde{\pi} = \bigcap\limits_{t\in T} J(t)$,
2. $\mathrm{Im}\ \tilde{\pi} = \widetilde{\mathcal{R}}$.

In particular the mapping $\tilde{\pi} : \mathcal{R} \to \widetilde{\mathcal{R}}$ is isometric $*$–isomorphism if and only if

$$\bigcap_{t\in T} J(t) = \{0\}.$$

The construction of the C^*–bundle defined by a C^*–algebra $\mathcal{R}$ and a system of its ideals J_T together with Theorem 2.6 gives, in fact, the general conception of local principles in the theory of C^*–algebras:

In the case when $\bigcap_{t\in T} J(t) = \{0\}$ we have the system of so–called local algebras $\mathcal{R}(t) = \mathcal{R}/J(t)$ parameterized by points $t \in T$. Elements a_1 and a_2 from the algebra $\mathcal{R}$ is called local equivalent at the point $t \in T : a_1 \overset{t}{\sim} a_2$, if $a_1 - a_2 \in J(t)$. The natural projection

$$\pi_t : \mathcal{R} \to \mathcal{R}(t)$$

identifies the local equivalent at the point t elements of the algebra $\mathcal{R}$.

The element $a(t) = \pi_t(a)$ is called the local representative of $a \in \mathcal{R}$ in the local algebra $\mathcal{R}(t)$.

In this connection

1. $\|a\| = \sup_t \|a(t)\|$,
2. the algebra $\mathcal{R}$ is reestablished by its local descriptions by some canonical procedure as the algebra of continuous sections of the canonically defined C^*–bundle over T.

3 The case of the Heisenberg group.

3.1 Recall (see, Subsection 1.1), that the Heisenberg group $\mathbb{H}_n$ coincides with the space $\mathbb{R}^N = \mathbb{R} \times \mathbb{C}^n$ endowed with the following group operation

$$(t_1, z_1) \cdot (t_2, z_2) = (t_1 + t_2 + 2\mathrm{Im}\ z_1 \cdot \overline{z}_2,\ z_1 + z_2),$$

where $(t_k, z_k) \in \mathbb{H}_n = \mathbb{R} \times \mathbb{C}^n, \quad k = 1, 2.$

In this case formulas (1.7)–(1.9) take the form:

$$\begin{aligned}
\omega_B(t, \tilde{x}, \lambda, \tilde{\xi}) &= \hat{b}(-\lambda, -\xi - 2\lambda y, -\eta + 2\lambda x) \\
\omega_C(t, \tilde{x}, \lambda, \tilde{\xi}) &= \hat{c}(\lambda, \xi - 2\lambda y, \eta + 2\lambda x) \\
\omega_K(t, \tilde{x}, \lambda, \tilde{\xi}) &= \hat{k}(-\lambda, -\xi - 2\lambda y, -\eta + 2\lambda x; \lambda, \xi - 2\lambda y, \eta + 2\lambda x),
\end{aligned}$$

where $\tilde{x} = (x, y) \in \mathbb{R}^{2n}, \quad \tilde{\xi} = (\xi, \eta) \in \mathbb{R}^{2n}, \quad \lambda \in \mathbb{R}.$

Consider now $\mathbb{H}_n = \mathbb{R} \times \mathbb{R}^{2n}$, its nonisotropic unit sphere Ω (see (1.10)), and the algebra $C^\infty(\Omega)$.

Denote by χ_+ the characteristic function of the "upper half sphere", i.e.

$$\chi_+(\lambda, \xi, \eta) = \begin{cases} 1, & \lambda \geq 0 \\ 0, & \lambda < 0 \end{cases},$$

where $(\lambda, \xi, \eta) \in \Omega$, and by $PC^\infty(\Omega)$ the algebra generated by χ_+ and functions from $C^\infty(\Omega)$.

Let $E = \{0\} \subset \mathbb{R}$ and $E_\Omega = \Omega \cap (E \times \mathbb{R}^n)$.

It is obvious that

$$PC^\infty(\Omega) \subset C_b^\infty(\Omega \backslash E_\Omega)$$

and

$$H_\delta^\circ(PC^\infty(\Omega)) \subset H_\delta^\circ(C_b^\infty(\Omega \backslash E_\Omega)).$$

3.2 By Theorem 1.3 all the operators having the following Weyl symbols

$$\begin{aligned} \omega_B &= \widetilde{b}(-\lambda, -\xi - 2\lambda y, -\eta + 2\lambda x), && (3.1)\\ \omega_C &= \widetilde{c}(\lambda, \xi - 2\lambda y, \eta + 2\lambda x), && (3.2) \end{aligned}$$

where $\widetilde{b}, \widetilde{c} \in H_\delta^\circ(PC^\infty(\Omega))$, are bounded on $L_2(\mathbb{H}_n)$, and by Lemma 2.1 their Weyl symbols coincide with their left (pseudodifferential) symbols. Moreover the class of these symbols satisfies all the conditions 1–4 of Section 2.

Denote by $\mathcal{R} = \mathcal{R}(H_\delta^\circ(PC(\Omega)))$ the C^*–algebra generated by all these operators:

$$\begin{aligned} B &= \text{Op}\ (\widetilde{b}(-\lambda, -\xi - 2\lambda y, -\eta + 2\lambda x)), && (3.3)\\ C &= \text{Op}\ (\widetilde{c}(\lambda, \xi - 2\lambda y, \eta + 2\lambda x)), && (3.4) \end{aligned}$$

where $\widetilde{b}, \widetilde{c} \in H_\delta^\circ(PC^\infty(\Omega))$.

From Theorem 1.3 and [VT, Lemma 3.3] it follows that the algebra $\mathcal{R}$ is generated also by all operators K with the Weyl symbols

$$\omega_K(\widetilde{x}, \lambda, \widetilde{\xi}) = \widetilde{k}(-\lambda, -\xi - 2\lambda y, -\eta + 2\lambda x; \lambda, \xi - 2\lambda y, \eta + 2\lambda x), \qquad (3.5)$$

where $\widetilde{k} \in H_\delta^\circ(PC^\infty(\Omega)) \otimes H_\delta^\circ(PC^\infty(\Omega))$.

3.3 We apply now the local principle of Subsection 2.3 to the algebra $\mathcal{R}$. We set $T = \mathbb{R}\backslash\{0\}$ and let for each $\lambda \in \mathbb{R}\backslash\{0\}$ an ideal $J(\lambda)$ be the ideal defined right after Corollary 2.3. Lemma 2.4 says that

$$\bigcap_{\lambda \in \mathbb{R}\backslash\{0\}} J(\lambda) = \{0\}$$

and thus we can use the machinery of Subsection 2.3. By Theorem 1.3 and Remark 1.4 each local algebra $\mathcal{R}(\lambda)$, $\lambda \in \mathbb{R}\backslash\{0\}$, are generated by the operators with the Weyl symbols

$$\begin{aligned} \omega_{B(\lambda)}(\widetilde{x}, \widetilde{\xi}) &= \omega_B(\widetilde{x}, \lambda, \widetilde{\xi})\\ \omega_{C(\lambda)}(\widetilde{x}, \widetilde{\xi}) &= \omega_C(\widetilde{x}, \lambda, \widetilde{\xi}), \end{aligned}$$

where ω_B and ω_C are given by (3.1), (3.2), or by the operators with the Weyl symbols

$$\omega_{K(\lambda)} = \omega_K(\widetilde{x}, \lambda, \widetilde{\xi}),$$

where ω_K is given by (3.5).

As it will be proved in Subsection 3.4 all local algebras $\mathcal{R}(\lambda)$ for $\lambda > 0$ are isomorphic and the same for $\lambda < 0$. In fact we will have only two different local algebras and thus the base of our C^*–bundle will be just two–points set with a discrete topology. The bundle space E will be disjointed union of these two different local algebras.

3.4 The algebra $\mathcal{R}(\lambda)$, $\lambda \in \mathbb{R}\backslash E = \mathbb{R}\backslash\{0\}$ coincides with C^*–algebra generated by all acting on the space $L_2(\mathbb{R}^{2n})$ operators $K(\lambda)$ with the following Weyl symbols

$$\omega_{K(\lambda)}(\widetilde{x}, \widetilde{\xi}) = \widetilde{k}(-\lambda, -\xi - 2\lambda y, -\eta + 2\lambda x; \lambda, \xi - 2\lambda y, \eta + 2\lambda x),$$

where $\widetilde{k} \in H_\delta^\circ(PC^\infty(\Omega)) \otimes H_\delta^\circ(PC^\infty(\Omega))$.

Fix $\lambda > 0$. Introduce in $T^*(\mathbb{R}^{2n})$ the following symplectic change of variables

$$x_1 = \frac{1}{2\sqrt{|\lambda|}}\xi + y\sqrt{|\lambda|}, \qquad y_1 = \frac{1}{2\sqrt{|\lambda|}}\eta + x\sqrt{|\lambda|},$$
$$\xi_1 = \frac{1}{2\sqrt{|\lambda|}}\eta - x\sqrt{|\lambda|}, \qquad \eta_1 = \frac{1}{2\sqrt{|\lambda|}}\xi - y\sqrt{|\lambda|}.$$

Let $U = U(\lambda)$ be the unitary on $L_2(\mathbb{R}^{2n})$ operator realizing this change of variables. Then (omitting subindices "1" and taking into account the homogeneity of $\widetilde{k}$) we have

$$\begin{aligned} & \omega_{UK(\lambda)U^*}(x, y; \xi, \eta) = \\ = \; & \widetilde{k}(-\lambda, -2\sqrt{|\lambda|}x, -2\sqrt{|\lambda|}\xi; \lambda, 2\sqrt{|\lambda|}\eta, 2\sqrt{|\lambda|}y) = \\ = \; & \widetilde{k}(-1, -2x, -2\xi; 1, 2\eta, 2y). \end{aligned} \tag{3.6}$$

Follow H. Cordes (see, for example, [Co]) denote by $CS(\mathbb{R}^k)$ the algebra of functions in $\mathbb{R}^k$ which is generated by the following functions

$$s_0(x) = \frac{1}{\sqrt{1+x^2}}, \quad s_j = \frac{x_j}{\sqrt{1+x^2}}, \quad j = \overline{1, k}, \quad x \in \mathbb{R}^k.$$

The algebra $CS(\mathbb{R}^k)$ consists of all functions $a(x)$ which are continuous in $\mathbb{R}^k$ and have radial limit values at the infinity

$$a_\infty(\theta) = \lim_{\rho \to +\infty} a(\rho \cdot \theta), \quad \rho \in \mathbb{R}_+, \quad \theta \in S^{k-1}, \tag{3.7}$$

and moreover the function $a_\infty(\theta), \quad \theta \in S^{k-1}$, is continuous on S^{k-1}.

Lemma 3.1 *If* $\widetilde{b}, \widetilde{c} \in H_\delta^\circ(PC^\infty(\Omega)), \quad \widetilde{k} \in H_\delta^\infty(PC^\infty(\Omega)) \otimes H_\delta^\circ(PC^\infty(\Omega))$, *then*

$$\widetilde{b}(-1, -2x, -2\xi) \in CS(\mathbb{R}^{2n}), \quad \widetilde{c}(1, 2\eta, 2y) \in CS(\mathbb{R}^{2n}),$$
$$\widetilde{k}(-1, -2x, -2\xi; 1, 2\eta, 2y) \in CS(\mathbb{R}^{2n}) \otimes CS(\mathbb{R}^{2n}).$$

The proof is the simple corollary of the representation:

$$\begin{aligned} \widetilde{k}(-1, -2x, -2\xi; 1, 2\eta, 2y) \; &= \\ &= \; \widetilde{k}\Big(\frac{-1}{1+x^2+\xi^2}, \frac{-2x}{\sqrt{1+x^2+\xi^2}}, \frac{-2\xi}{\sqrt{1+x^2+\xi^2}}; \\ & \qquad \frac{1}{\sqrt{1+y^2+\eta^2}}, \frac{2\eta}{\sqrt{1+y^2+\eta^2}}, \frac{2y}{\sqrt{1+y^2+\eta^2}}\Big). \end{aligned}$$

Denote by $\Psi(CS(\mathbb{R}^{2n}))$ the C^*–algebra generated by all acting on the space $L_2(\mathbb{R}^n)$ operators

$$(Af)(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(\frac{x+y}{2}, \xi) f(y)\, dy\, d\xi, \tag{3.8}$$

with (sufficiently smooth) Weyl symbols $a(x,\xi) \in CS(\mathbb{R}^{2n})$.

The combination of the results of [Gu,Co] gives the following:

Theorem 3.2 *The algebra of (Fredholm) symbols*

$$\mathrm{Sym}\ \Psi(CS(\mathbb{R}^{2n})) = \Psi(CS(\mathbb{R}^{2n}))/\mathcal{K},$$

where $\mathcal{K}$ is the ideal of all compact on $L_2(\mathbb{R}^n)$ operators, of the algebra $\Psi(CS(\mathbb{R}^{2n}))$ is isomorphic and isometric to the algebra $C(S^{2n-1})$. If we identify them then the symbol–homomorphism

$$\mathrm{sym}\ :\quad \Psi(CS(\mathbb{R}^{2n})) \longrightarrow C(S^{2n-1})$$

is generated by the following mapping of the generators (3.8) of the algebra $\Psi(CS(\mathbb{R}^{2n}))$:

$$\mathrm{sym}\ A = a_\infty(\theta), \quad \theta \in S^{2n-1}, \tag{3.9}$$

where the function $a_\infty(\theta)$ is defined by the Weyl symbol of the operator A by (3.7).

Theorem 3.3 *For all $\lambda > 0$ there exists C^*–homomorphism π_+ (which does not depend on λ)*

$$\pi_+ :\quad \mathcal{R} \longrightarrow \Psi(CS(\mathbb{R}^{2n})) \otimes \Psi(CS(\mathbb{R}^{2n})),$$

which is generated by the following mapping of the generators of the algebra $\mathcal{R}$:

$$\begin{array}{rcll} \pi_+ : & B & \longmapsto & B_+ \otimes I, \\ \pi_+ : & C & \longmapsto & I \otimes C_+, \end{array}$$

where acting on the space $L_2(\mathbb{R}^n)$ operators B_+ and C_+ have the following Weyl symbols

$$\begin{aligned} \omega_{B_+}(x,\xi) &= \widetilde{b}(-1,-2x,-2\xi), \\ \omega_{C_+}(y,\eta) &= \widetilde{c}(1,2\eta,2y). \end{aligned} \tag{3.10}$$

The proof is the simple combination of formulas (3.6) and Lemma 3.1.

Let now $\lambda < 0$. Introduce in $T^*(\mathbb{R}^{2n})$ the following symplectic change of variables

$$\begin{aligned} x_1 &= \frac{1}{2\sqrt{|\lambda|}}\xi - y\sqrt{|\lambda|}, \qquad & y_1 &= \frac{1}{2\sqrt{|\lambda|}}\eta - x\sqrt{|\lambda|}, \\ \xi_1 &= -\frac{1}{2\sqrt{|\lambda|}}\eta - x\sqrt{|\lambda|}, \qquad & \eta_1 &= -\frac{1}{2\sqrt{|\lambda|}}\xi - y\sqrt{|\lambda|}. \end{aligned}$$

Let $U = U(\lambda)$ be the unitary on $L_2(\mathbb{R}^{2n})$ operator realizing this change of variables. Then (omitting subindices "1" and taking into account the homogeneity of $\widetilde{k}$) we have

$$\omega_{UK(\lambda)U^*}(x,y;\xi,\eta) = \tag{3.11}$$

$$\begin{aligned} &= \widetilde{k}(-\lambda, -2\sqrt{|\lambda|}x, +2\sqrt{|\lambda|}\xi; \lambda, -2\sqrt{|\lambda|}\eta, 2\sqrt{|\lambda|}y) = \\ &= \widetilde{k}(1,-2x,+2\xi;-1,-2\eta,2y). \end{aligned}$$

Combining these formulas with Lemma 3.1 we obtain

Theorem 3.4 *For all $\lambda < 0$ there exists C^*–homomorphism π_- (which does not depend on λ)*

$$\pi_- : \quad \mathcal{R} \longrightarrow \Psi(CS(\mathbb{R}^{2n})) \otimes \Psi(CS(\mathbb{R}^{2n})),$$

which is generated by the following mapping of the generators of the algebra $\mathcal{R}$:

$$\begin{array}{llll} \pi_- : & B & \longmapsto & B_- \otimes I, \\ \pi_- : & C & \longmapsto & I \otimes C_-, \end{array}$$

where acting on the space $L_2(\mathbb{R}^n)$ operators B_- and C_- have the following Weyl symbols

$$\begin{aligned} \omega_{B_-}(x,\xi) &= \widetilde{b}(1,-2x,2\xi) \\ \omega_{C_-}(y,\eta) &= \widetilde{c}(-1,-2\eta,2y). \end{aligned} \tag{3.12}$$

Now Theorems 3.3, 3.4 together with our local principle give

Theorem 3.5 *The algebra $\mathcal{R}$ generated by operators of the form (3.3) and (3.4) is isometrically $*$–isomorphic to the algebra $(\Psi(CS(\mathbb{R}^{2n})) \otimes \Psi(CS(\mathbb{R}^{2n})))^2$. The isomorphism*

$$\pi : \quad \mathcal{R} \longrightarrow (\Psi(CS(\mathbb{R}^{2n})) \otimes \Psi(CS(\mathbb{R}^{2n})))^2$$

is generated by the following mapping of generators (3.3), (3.4)

$$\begin{aligned} \pi : B &\longrightarrow (B_+ \otimes I, B_- \otimes I), \\ \pi : C &\longrightarrow (I \otimes C_+, I \otimes C_-), \end{aligned} \tag{3.13}$$

where the Weyl symbols of the operators B_+, C_+, B_- and C_- are given by formulas (3.10) and (3.12).

Consider now the subalgebra $\mathcal{R}_0 = \mathcal{R}(H_\delta^\circ(C(\Omega)))$ of the algebra $\mathcal{R}$ which is generated by all the operators (3.3), (3.4) where $\widetilde{b}, \widetilde{c} \in H_\delta^\circ(C^\infty(\Omega))$.

Corollary 3.6 *The algebra $\mathcal{R}_0$ is isometrically imbedded into the algebra*

$$(\Psi(CS(\mathbb{R}^{2n})) \otimes \Psi(CS(\mathbb{R}^{2n})))^2.$$

The imbedding

$$\pi : \quad \mathcal{R}_0 \longrightarrow (\Psi(CS(\mathbb{R}^{2n})) \otimes \Psi(CS(\mathbb{R}^{2n})))^2$$

is generated by the mapping (3.13). The image of the mapping π consists of all the pairs (A_+, A_-) for which

$$(\mathrm{sym}\ \otimes \mathrm{sym}\ )A_+(x,-\xi;y,-\eta) = (\mathrm{sym}\ \otimes \mathrm{sym}\ )A_-(x,\xi;y,\eta),$$

where $(x,\xi;y,\eta) \in S^{2n-1} \times S^{2n-1}$, and the sym *homomorphism is given by the formula (3.9).*

Denote by $\mathcal{R}_r = \mathcal{R}_r(H_\delta^\circ(PC(\Omega))$ the C^*–algebra generated by all the (right convolution) operators

$$B = Op(\widetilde{b}(-\lambda,-\xi-2\lambda y,-\eta+2\lambda x)), \tag{3.14}$$

where $\widetilde{b} \in H_\delta^\circ(PC^\infty(\Omega))$.

Corollary 3.7 *The algebra $\mathcal{R}_r$ is isometrically isomorphic to the algebra $(\Psi(CS(\mathbb{R}^{2n})))^2$. The isomorphism*

$$\pi_r : \quad \mathcal{R}_r \longrightarrow (\Psi(CS(\mathbb{R}^{2n})))^2$$

is generated by the following mapping of the generators (3.14),

$$\pi_r : \quad B \longmapsto (B_+, B_-), \tag{3.15}$$

where the Weyl symbols of the operators B_+, B_- are calculated by the formulas (3.10), (3.12).

Let now $\mathcal{R}_{r,0} = \mathcal{R}_r(H_\delta^\circ(C(\Omega)))$ be the subalgebra of $\mathcal{R}_r$, which is generated by all the operators (3.14) with $\widetilde{b} \in H_\delta^\circ(C^\infty(\Omega))$.

Corollary 3.8 *The algebra $\mathcal{R}_{r,0}$ is isometrically imbedded into the algebra $(\Psi(CS(\mathbb{R}^{2n})))^2$. The imbedding*

$$\pi_r : \quad \mathcal{R}_{r,0} \longrightarrow (\Psi(CS(\mathbb{R}^{2n})))^2$$

is generated by the mapping (3.15). The image of the mapping π_r consists of all the pairs (A_+, A_-) for which

$$(\mathrm{sym}\ A_+)(x, -\xi) = (\mathrm{sym}\ A_-)(x, \xi),$$

where $(x, \xi) \in S^{2n-1}$ and the sym *homomorphism is given by the formula (3.9).*

It is clear that analogous corollaries can be done for algebras $\mathcal{R}_l(H_\delta^\circ(PC(\Omega)))$ and $\mathcal{R}_l(H_\delta^\circ(C(\Omega)))$ generated by the (left convolution) operators of the form (3.4).

3.5 We give here the description of irreducible representations and of the spectrum of the above considered algebras. We will use the following general facts.

Let $\mathcal{R}$ be a C^*–algebra of operators acting on some Hilbert space H, moreover let $\mathcal{R}$ contains the ideal $\mathcal{K}$ of all compact on H operators and let the symbol algebra Sym $\mathcal{R} = \mathcal{R}/\mathcal{K}$ be isomorphic to the algebra $C(M)$ for some compact set M.

Denote by

$$\mathrm{sym}\ : \quad \mathcal{R} \longrightarrow \mathrm{Sym}\ \mathcal{R} = C(M)$$

the natural projection. Then $\mathcal{R}$ is an (irreducible) GCR–algebra and the consideration of its composition series $0 \subset \mathcal{K} \subset \mathcal{R}$ shows that all its irreducible representations (up to unitary equivalence) are the following:

a) the identical (infinite dimensional) representation

$$\pi_i : \quad \mathcal{R} \to \mathcal{R},$$

b) the one–dimensional representations $\pi_t : \quad \mathcal{R} \to \mathbb{C}$ generated by points $t \in M$ and defined by the rule

$$\pi_t : \quad A \longmapsto (\mathrm{sym}\ A)(t).$$

The spectrum of the algebra $\mathcal{R}$ has the form

$$sp\ \mathcal{R} = \{i\} \cup M.$$

The point i is an open everywhere dense in $sp\ \mathcal{R}$ set with respect to the Jacobson topology. The set M is closed and the restriction of the Jacobson topology onto M coincides with the topology of compact M.

Let now $\mathcal{R}_1$ and $\mathcal{R}_2$ be to C^*–operator algebras with the properties: $\mathcal{K}_j \subset \mathcal{R}_j$ and Sym $\mathcal{R}_j = \mathcal{R}_j/\mathcal{K}_j \cong C(M_j),\ j = 1, 2$. Then the algebra $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$ is an (irreducible) GCR–algebra. The consideration of its composition series

$$0 \subset \mathcal{K}_1 \otimes \mathcal{K}_2 \subset \mathcal{K}_1 \otimes \mathcal{R}_2 \subset \mathcal{R}_1 \otimes \mathcal{R}_2$$

shows that $sp\ \mathcal{R} = sp\ \mathcal{R}_1 \times sp\ \mathcal{R}_2$ with the direct product topology. All irreducible representations of the algebra $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$ are the following:

a) the identical representation

$$\pi_i = \pi_{i_1} \otimes \pi_{i_2} : \quad \mathcal{R} \to \mathcal{R},$$

b) the infinite dimensional representations

$$\pi_{t_1} \otimes \pi_{i_2} : \quad \mathcal{R} \to \mathcal{R}_2, \qquad t_1 \in M_1,$$

and

$$\pi_{i_1} \otimes \pi_{t_2} : \quad \mathcal{R} \to \mathcal{R}_1, \qquad t_2 \in M_2,$$

defined by the rule

$$\begin{array}{lcll} \pi_{t_1} \otimes \pi_{i_2} : & A & \longmapsto & (\mathrm{sym}\ \otimes \pi_{i_2})A(t_1), \\ \pi_{i_1} \otimes \pi_{t_2} : & A & \longmapsto & (\pi_{i_1} \otimes \mathrm{sym}\ )A(t_2), \end{array}$$

c) the one–dimensional representations

$$\pi_{t_1} \otimes \pi_{t_2} : \quad \mathcal{R} \to \mathbb{C}, \qquad (t_1, t_2) \in M_1 \times M_2,$$

defined by the rule

$$\pi_{t_2} \otimes \pi_{t_2} : A \longmapsto (\mathrm{sym}\ \otimes \mathrm{sym}\ )A(t_1, t_2).$$

Let finally $\mathcal{R}$ be a C^*–algebra and $\mathcal{R}_0$ be its C^*–subalgebra such that the restriction $\pi|_{\mathcal{R}_0}$ is irreducible for each $\pi \in sp\ \mathcal{R}$. Then [Di, 3.9.7] the relation $\pi_1|_{\mathcal{R}_0} \cong \pi_2|\mathcal{R}$ onto $sp\ \mathcal{R}$ is an equivalence relation (say Eq) and the space $sp\ \mathcal{R}_0$ is canonically identified with $sp\ \mathcal{R}/_{Eq}$. These facts allow to give, as corollaries of Theorem 3.5 and corollaries 3.6–3.8, the description of the spectra and irreducible representations of all algebras of this Section.

We consider here (as models examples) the algebras $\mathcal{R}$ and $\mathcal{R}_{r,0}$ only.

Corollary 3.9 *All (up to unitary equivalence) irreducible representations of the algebra $\mathcal{R} = \mathcal{R}(H_\delta^\circ(PC(\Omega))$ are the following:*

a) *two infinite dimensional representations in the space* $L_2(\mathbb{R}^{2n}) = L_2(\mathbb{R}^n) \otimes L_2(\mathbb{R}^n)$

$$\pi_\pm : \qquad \mathcal{R} \longrightarrow \Psi(CS(\mathbb{R}^{2n})) \otimes \Psi(CS(\mathbb{R}^{2n})),$$

which are defined on the generators of the algebra $\mathcal{R}$ *by the rule (see (3.13)):*

$$\pi_\pm : \quad B \cdot C \longmapsto B_\pm \otimes C_\pm;$$

b) *two series of infinite dimensional representations in the space* $L_2(\mathbb{R}^n)$

$$\pi'_\pm(x,\xi) : \quad \mathcal{R} \longrightarrow \Psi(CS(\mathbb{R}^{2n})), \quad (x,\xi) \in S^{2n-1}_{1,\pm} \quad (= S^{2n-1}),$$

which are defined by the rule

$$\pi'_\pm(x,\xi) : \quad A \longmapsto ((sym \otimes id) \circ \pi_\pm) A(x,\xi);$$

c) *two series of infinite dimensional representations in the space* $L_2(\mathbb{R}^n)$

$$\pi''_\pm(y,\eta) : \quad \mathcal{R} \longrightarrow \Psi(CS(\mathbb{R}^{2n})), \quad (y,\eta) \in S^{2n-1}_{2,\pm} \quad (= S^{2n-1}),$$

which are defined by the rule

$$\pi''_\pm(y,\eta) : \quad A \longmapsto ((id \otimes \mathrm{sym}\ ) \circ \pi_2) A(y,\eta);$$

d) *two series of one–dimensional representations parameterized by points* $(x,\xi;y,\eta) \in S^{2n-1}_{1,\pm} \times S^{2n-1}_{2,\pm}$ *and defined by the rule*

$$\pi_\pm(x,\xi;y,\eta) : \quad A \longmapsto ((\mathrm{sym}\ \otimes \mathrm{sym}\ ) \circ \pi_\pm) A(x,\xi;y,\eta).$$

The spectrum sp $\mathcal{R}$ *is the disjoint union*

$$sp\ \mathcal{R} = \mathfrak{M}_+ \bigsqcup \mathfrak{M}_-$$

of two equal sets

$$\mathfrak{M} = \mathfrak{M}_+ = \mathfrak{M}_- = \{i\} \cup S^{2n-1}_1 \cup S^{2n-1}_2 \cup S^{2n-1}_1 \times S^{2n-1}_2.$$

In the Jacobson topology the point $\{i\}$ *is open, the closure of* $\{i\}$ *coincides with* $\mathfrak{M}$. *Let now* u_1 *be a subset of* S^{2n-1}_1 *then its closure is the union of* $\overline{u}_1 (\subset S^{2n-1}_1)$ *and* $\overline{u}_1 \times S^{2n-1}_2 (\subset S^{2n-1}_1 \times S^{2n-1}_2)$. *The same for* $u_2 \subset S^{2n-1}_2$. *The set* $S^{2n-1}_1 \times S^{2n-1}_2$ *is closed in the Jacobson topology and the restriction of it onto* $S^{2n-1}_1 \times S^{2n-1}_2$ *coincides with the natural topology of the product of two spheres.*

Corollary 3.10 *The spectrum of the algebra* $\mathcal{R}_{r,0}$ *of the right convolutions has the form* $sp\mathcal{R}_{r,0} = \{i_+, i_-\} \cup S^{2n-1}$. *In the Jacobson topology the points* $i_\pm$ *are open and the closure of (for example)* $\{i_+\}$ *coincides with* $\{i_+) \cup S^{2n-1}$. *The restriction of the Jacobson topology onto* S^{2n-1} *coincides with the natural sphere's topology. The infinite dimensional representations*

$$\pi_\pm : \quad \mathcal{R}_{r,0} \longrightarrow \Psi(CS(\mathbb{R}^{2n}))$$

correspond to the points $i_\pm$. And these representations are defined on the generators of $\mathcal{R}_{r,0}$ by the rule

$$\pi_\pm : \quad B \longmapsto B_\pm.$$

The one–dimensional representations correspond to the points $(x,\xi) \in S^{2n-1}$. And these representations are defined by the rule

$$\pi(x,\xi): \quad B \longmapsto (\mathrm{sym}\ B_+)(x,\xi) \quad (= \mathrm{sym}\ B_-(x,-\xi)).$$

4 Toeplitz operators with convolution presymbols on the Heisenberg group.

4.1 Consider the following type II Siegel domains [OV, Ta1]

$$D = \{(w,z) \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Im}\ w - z \cdot \overline{z} > 0\}.$$

Its distinguished boundary

$$B = \{(w,z) \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Im}\ w - z \cdot \overline{z} = 0\}$$

has a natural Lie group structure, which after identification of B with $\mathbb{R} \times \mathbb{C}^n$ with the help of mapping

$$\kappa_0 : (t,z) \in \mathbb{R} \times \mathbb{C}^n \longmapsto (t + iz \cdot \overline{z},\ z) \in B$$

coincides with the Heisenberg group $\mathbb{H}_n$.

The mapping

$$\kappa : \quad L_2(B) \longrightarrow L_2(\mathbb{R} \times \mathbb{C}^n) = L_2(\mathbb{H}_n),$$

where $(\kappa f)(t,z) = f(\kappa_0(t,z))$ transfers isometrically the Hardy space $H^2(D)$ ($\subset L_2(\mathcal{B})$) onto the subspace $H^2(\mathbb{H}_n) = \kappa(H^2(D))$ of the space $L_2(\mathbb{H}_n)$.

The Szegö projector P of the space $L_2(\mathbb{H}_F)$ onto the $H^2(\mathbb{H}_F)$ is a right convolution operator of the type (3.14) with the following Weyl symbol (for details, see [VT])

$$\begin{aligned} \omega_P(\lambda, \widetilde{x}, \widetilde{\xi}) &= \chi_+(\lambda) \cdot 2^n \cdot e^{-\frac{|\zeta - 2i\lambda z|^2}{4|\lambda|}} = \\ &= \chi_+(\lambda) \cdot 2^n \cdot e^{-\frac{(\xi + 2\lambda y)^2 + (\eta - 2\lambda x)^2}{4|\lambda|}}, \end{aligned} \tag{4.1}$$

where $\widetilde{x} = (x,y), \widetilde{\xi} = (\xi,\eta) \in \mathbb{R}^{2n}, \quad z = x + iy, \zeta = \xi + i\eta \in \mathbb{C}^n$.

Theorem 4.1 [VT] *There exists unitary on $L_2(\mathbb{H}_n)$ operator ν which gives an isometrical isomorphism of the space*

$$L_2(\mathbb{H}_n) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}^n) \otimes L_2(\mathbb{R}^n)$$

and under this isomorphism

1. *the space* $H^2(\mathbb{H}_n)$ *is mapped onto*

$$L_2(\mathbb{R}_+) \otimes L_0 \otimes L_2(\mathbb{R}^n); \tag{4.2}$$

2.

$$\nu P \nu^{-1} = \chi_+(\lambda) I \otimes P_0 \otimes I, \tag{4.3}$$

where $\chi_+(\lambda)$ is a characteristic function of the $\mathbb{R}_+ = \{\lambda \in \mathbb{R} : \lambda \geq 0\}$,

$$(P_0 f)(x) = \pi^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}(x^2+u^2)} f(u) du$$

is the one–dimensional orthogonal projector of $L_2(\mathbb{R}^n)$ *onto the one– dimensional subspace* L_0 *generated by the function* $l(x) = \pi^{-\frac{n}{2}} e^{-\frac{x^2}{2}}$.

4.2 Denote by $\mathcal{T}(\mathcal{R})$ the algebra generated by all Toeplitz operators

$$T_K : \quad f \in H^2(\mathbb{H}_n) \longmapsto PKf \in H^2(\mathbb{H}_n) \tag{4.4}$$

with presymbols K from the algebra $\mathcal{R} = (\mathcal{R}(H^\circ_\delta(PC(\Omega)))$ considered in Section 3.

Theorem 4.2 *The algebra* $\mathcal{T}(\mathcal{R})$ *is isometrically isomorphic to the algebra* $\Psi(CS(\mathbb{R}^{2n}))$. *The isomorphism*

$$\tau : \quad \mathcal{T}(\mathcal{R}) \to \Psi(CS(\mathbb{R}^{2n}))$$

is generated by the following mapping of generators (4.4) of the algebra $\mathcal{T}(\mathcal{R})$

$$\tau : \quad T_K \longmapsto \hat{K},$$

where the Weyl symbol of the operator $\hat{K} \in \Psi(CS(\mathbb{R}^{2n}))$ *is calculated by the formula*

$$\omega_{\hat{K}}(y,\eta) = \pi^{-n} \int\limits_{\mathbb{R}^{2n}} e^{-(x^2+\xi^2)} \widetilde{k}(-1, -2x, -2\xi; 1, 2\eta, 2y) dx d\xi,$$

where (see (3.5))

$$\omega_K(\widetilde{x}, \lambda, \widetilde{\xi}) = \widetilde{k}(-\lambda, -\xi - 2\lambda y, -\eta + 2\lambda x; \lambda, \xi - 2\lambda y, \eta + 2\lambda x)$$

is the Weyl symbol of the operator $K \in \mathcal{R}(H^\circ_\delta(PC(\Omega)))$.

Really, by (4.1) the projector P belongs to the algebra $\mathcal{R}$ and thus $\mathcal{T}(\mathcal{R})$ can be considered as a restriction onto (invariant subspace) $H^2(\mathbb{H}_n)$ of the subalgebra $\mathcal{R}_T$ of the algebra $\mathcal{R}$, which is generated by all acting on $L_2(\mathbb{H}_n)$ operators PKP, where $K \in \mathcal{R}$.

Theorems 3.5 and 4.1 together with (4.1) show that for the isomorphism π of Theorem 3.5 we have

$$\pi : P \longmapsto (P_+, P_-) = (P_0 \otimes I, 0).$$

And thus the algebra $\mathcal{R}_T$ is isomorphic to the subalgebra of

$$(\Psi(CS(\mathbb{R}^{2n})) \otimes \Psi(CS(\mathbb{R}^{2n})))^2$$

generated by all elements

$$((PKP)_+,0) = ((P_0 \otimes I)K_+(P_0 \otimes I),0).$$

Note (see (4.2), (4.3)) that each operator

$$(P_0 \otimes I)K_+(P_0 \otimes I)$$

acts, in fact, on the space $L_0 \otimes L_2(\mathbb{R}^n)$ and the algebra generated by the operators

$$(P_0 \otimes I)K_+(P_0 \otimes I)|_{L_0 \otimes L_2(\mathbb{R}^n)}$$

coincides with $\mathbb{C}I \otimes \Psi(CS(\mathbb{R}^{2n}))$.

Thus we have a natural isomorphism

$$\tau_0 : \quad \mathcal{T}(\mathcal{R}) \longrightarrow \mathbb{C}I \otimes \Psi(CS(\mathbb{R}^{2n}))$$

which is generated by the following mapping of generators (4.4) of the algebra $\mathcal{T}(\mathcal{R})$

$$\tau_0 : \quad T_K \longmapsto (P_0 \otimes I)K_+(P_0 \otimes I)|_{L_0 \otimes L_2(\mathbb{R}^n)}.$$

There is a natural isomorphism iso between the algebra $\mathbb{C}I \otimes \Psi(CS(\mathbb{R}^{2n}))$ of operators acting on $L_0 \otimes L_2(\mathbb{R}^n)$ and the algebra $\Psi(CS(\mathbb{R}^{2n}))$ of operators acting on $L_2(\mathbb{R}^n)$ which is defined by the rule

$$iso : \quad cI \otimes A \longmapsto cA,$$

where $c \in \mathbb{C}, A \in \Psi(CS(\mathbb{R}^{2n}))$.

And thus we have an isomorphism

$$\tau : \quad \mathcal{T}(\mathcal{R}) \xrightarrow{\tau_0} \mathbb{C}I \otimes \Psi(CS(\mathbb{R}^{2n})) \xrightarrow{iso} \Psi(CS(\mathbb{R}^{2n})),$$

which is generated by the following mapping of generators (4.4) of the algebra $\mathcal{T}(\mathcal{R})$

$$\tau : T_K \longmapsto \hat{K},$$

where $(\hat{K}\varphi)(y) = \int\limits_{\mathbb{R}^n} (K_+ l_0 \cdot \varphi)(x,y) \cdot \overline{l_0(x)} dx$,

and $l_0(x) \cdot \varphi(y) \in L_0 \otimes L_2(\mathbb{R}^n)$.

Calculate now

$$\begin{aligned}
(\hat{K}\varphi)(y) &= \int\limits_{\mathbb{R}^n} (K_+ l_0 \cdot \varphi)(x,y) \cdot \overline{l_0(x)} dx = \\
&= \pi^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} (K_+ e^{-\frac{x^2}{2}} \cdot \varphi(y)) \cdot e^{-\frac{x^2}{2}} dx = \\
&= (2\pi)^{-2n} \pi^{-\frac{n}{2}} \int\limits_{\mathbb{R}^{5n}} e^{i(x-u)\cdot\xi + i(y-v)\cdot\eta} \omega_{K_+}(\frac{x+u}{2}, \frac{y+v}{2}, \xi, \eta). \qquad (4.5) \\
&\quad \cdot e^{-\frac{1}{2}u^2} \cdot e^{-\frac{1}{2}x^2} \cdot \varphi(y) du dv d\xi d\eta dx = \\
&= (2\pi)^{-n} \cdot 2^n \cdot \pi^{-\frac{3}{2}n} \int\limits_{\mathbb{R}^{5n}} e^{i(x-u)\cdot\xi + i(y-v)\cdot\eta} \omega_{K_+}(\frac{x+u}{2}, \frac{y+v}{2}, \xi, \eta) \cdot \\
&\quad \cdot e^{-\frac{1}{2}(x^2+u^2)} \varphi(v) dx du dv d\xi d\eta.
\end{aligned}$$

Let us make the following change of variables

$$p = \frac{x+u}{2}, \quad q = \frac{u-x}{2},$$

or $x = p - q,\ u = p + q$.

Than (4.5) takes a form

$$(\hat{K}\varphi)(y) = (2\pi)^{-n} \quad \cdot \quad \pi^{-\frac{3}{2}n} \int\limits_{\mathbb{R}^{5n}} e^{-i2q\cdot\xi} \cdot e^{i(y-v)\cdot\eta}\omega_{K_+}(p, \frac{y+v}{2}, \xi, \eta) \cdot$$
$$\cdot \quad e^{-(p^2+q^2)}\varphi(v)dpdqd\xi dvd\eta.$$

From the evident equality

$$\pi^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-(q^2+2iq\xi)}dv = e^{-\xi^2} \cdot \pi^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-(v+i\xi)^2}dv = e^{-\xi^2}$$

we have that

$$(\hat{K}\varphi)(y) = (2\pi)^{-n}\pi^{-n} \int\limits_{\mathbb{R}^{4n}} e^{-(p^2+\xi^2)}e^{i(y-v)\cdot\eta}\omega_{K_+}(p, \frac{y+v}{2}, \xi, \eta)\cdot$$
$$\cdot\varphi(v)dpd\xi dvd\eta.$$

Or, finally,

$$(\hat{K}\varphi)(y) = (2\pi)^{-n} \quad \cdot \quad \int\limits_{\mathbb{R}^{2n}} e^{i(y-v)\cdot\eta}[\pi^{-n} \int\limits_{\mathbb{R}^{2n}} e^{-(x^2+\xi^2)}\omega_{K_+}(x, \frac{y+v}{2}, \xi, \eta)dxd\xi]$$
$$\cdot \quad \varphi(v)dvd\eta\cdot$$

Thus the Weyl symbol of the operator $\hat{K}$ is calculated by the formula

$$\begin{aligned} \omega_{\hat{K}}(y,\eta) &= \pi^{-n} \int\limits_{\mathbb{R}^n} e^{-(x^2+\xi^2)}\omega_{K_+}(x, y; \xi, \eta)dxd\xi = \\ &= \pi^{-n} \int\limits_{\mathbb{R}^n} e^{-(x^2+\xi^2)}\widetilde{k}(-1, -2x, -2\xi; 1, 2\eta, 2y)dxd\xi. \end{aligned}$$

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