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Quaternionic ψ-hyperholomorphic functions, singular integral operators and boundary value problems. I. ψ-hyperholomorphic function theory

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Quaternion-valued functions of 3 and 4 real variables are studied which form exact analogues of the class of all holomorphic (or anti-holomorphic) functions and which are called hyperholomorphic. The set of such analogues is parametrizable by the group $O_3(\mathbb{R})$ or $O_3^+(\mathbb{R})$ depending on the number of variables. Connections between various classes of hyperholomorphic functions are established, and some basic facts of the function theory are given. Properties of the singular integral operator with the quaternionic Cauchy kernel are treated and commented in detail.

Among various ways to extend one-dimensional complex analysis to the multidimensional, in our opinion Clifford analysis (i.e. a study of functions with values in a Clifford algebra) is the most important and adequate on. The exact structural analogy between Clifford analysis and one-dimensional complex analysis together with successful applications to a number of problems in physics have caused an intensified development of Clifford analysis. A series of fundamental and (in certain cases) final results were obtained (see, e.g. [BDS], [DSS], [GS]).

Some classical questions in complex one-dimensional theory – in particular the theory of boundary value problems and of singular integral operator – have not been developed sufficiently for Clifford analysis.

This paper is devoted to a systematic study of such questions.

The skew–field of quaternions $\mathbb{H}$ gives an example of a non-commutative Clifford algebra with minimal dimension. It serves as a very convenient model of general Clifford constructions. For this reason $\mathbb{H}$-valued functions are considered below, although many assertions allow quite evident generalizations for Clifford algebra-valued functions.

We study $\mathbb{H}$-valued functions of three and of four real variables. We define the function classes (parametrizable by the group $O_3(\mathbb{R})$ or by $O_3^+(\mathbb{R})$ depending on the number of variables) which are exact analogues of the class of holomorphic (or anti-holomorphic) functions. We clarify the connections among these classes as well as the corresponding analogues of the Cauchy integral–type integral and of the singular integration operator with Cauchy kernel.

It should be mentioned that we do not attempt to give a complete review of papers on this theme. Some bibliographical remarks will be made at appropriate places.

For a reader who is already acquainted with quaternionic or Clifford analysis, most of the results formulated in Sections 2 – 3 found familiar. However, for the sake of convenience and of completeness,we have listed them; in most cases their proofs are omitted.

1 Some basic notations and definitions

1.1 The following notations will be used.

$\mathbb{R}^4_y$ – the four-dimensional Euclidean space with a fixed orthonormalized (canonical) basis

$$\psi_{st} := \{i_0,\, i_1,\, i_2,\, i_3\}\,.$$

$\mathbb{H}$ – the non-commutative skew–field of real quaternions which is built from $\mathbb{R}^4_y$ in such a way that i_0 is its unity and $i_1,\ i_2,\ i_3$ are the imaginary units. If $y = \sum_{k=0}^{3} y_k\, i_k$ is a quaternion then $\vec{y} = (y_0,\, y_1,\, y_2,\, y_3)$ is the corresponding vector from $\mathbb{R}^4_y$.

$\mathbb{H}(\mathbb{C}) := \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ – the corresponding algebra of complex quaternions.

$\mathbb{H}^m := \mathbb{H} \times \ldots \times \mathbb{H}$ (m times, $m \in \{3;4\}$).

$\mathbb{R}^m_x$ – the m-dimensional Euclidean space with a fixed orthonormal basis which does not necessarily coincide with ψ_{st} (when $m=4$) or with its vector part (when $m=3$).

Ω – a connected domain in $\mathbb{R}^m_x$; its topological boundary $\partial\Omega =: \Gamma$ is supposed to be an $(m-1)$-dimensional compact smooth (or Liapounov) surface.

$\mathbb{B}^m(x_0;\mathbb{R})$–closed ball in $\mathbb{R}^m_x$ centered at a point x_0 and with radius $R > 0$.

$\mathbb{S}^{m-1} := \partial\mathbb{B}(0;1)$ – the unit sphere in $\mathbb{R}^m_x$.

$\operatorname{int} X$ – the interior of a set $X \subset \mathbb{R}^m_x$.

$\langle\cdot,\cdot\rangle_m = \langle\cdot,\cdot\rangle_{\mathbb{R}^m}$ – the scalar product in $\mathbb{R}^m$.

$\mathbf{0}_X$ – the function $x \in X \longmapsto 0 \in \mathbb{R}$.

$\mathbb{1}_X$ – the function $x \in X \longmapsto 1 \in \mathbb{R}$.

δ_{jk} – the Kronecker symbol (Kronecker delta).

$\mathbb{R}^{p\times q}$(respectively $\mathbb{C}^{p\times q}$)–the set of all $p \times q$-matrices with real (respectively complex) components.

E_p – the unit $p \times p$-matrix.

1.2 Let $f = \sum_{k=0}^{3} f_k\, i_k : \Omega \longrightarrow \mathbb{H}$, and $\vec{f} = (f_0,\, f_1,\, f_2,\, f_3)$ is the corresponding $\mathbb{R}^4$-valued function. For $p \in \mathbb{N} \cup \{0\}$ we define as usual

$$\begin{aligned} C^p(\Omega;\mathbb{R}^4) &:= \left\{\vec{f}\,|\ f_k \in C^p(\Omega;\mathbb{R})\right\}, \\ C^p(\Omega;\mathbb{H}) &:= \left\{f\,|\ \vec{f} \in C^p(\Omega;\mathbb{R}^4)\right\}. \end{aligned}$$

For an $\mathbb{H}$-valued function a we denote M^a (respectively ${}^aM$) the right (respectively left) multiplication operator by a. We shall use the operator of the quaternionic conjugation $Z : f \longmapsto \overline{f}$ and the partial differentiation operator $\partial_k = \partial_{x_k} = \frac{\partial}{\partial x_k}$.

1.3 Let $\psi := \{\psi^{4-m}, \ldots, \psi^3\} \in \mathbb{H}^m$, $\overline{\psi} := \{\overline{\psi}^{4-m}, \ldots, \overline{\psi}^3\}$. We define on $C^1(\Omega, \mathbb{H})$ the operators ${}^{\psi}D$ and D^{ψ} by the formulas

$$\begin{aligned} {}^{\psi}D[f] &:= \left(\sum_{k=4-m}^{3} {}^{\psi^k}M \cdot \partial_k \right)[f] = \sum_{k=4-m}^{3} \psi^k \cdot \partial_k f, \\ D^{\psi}[f] &:= \left(\sum_{k=4-m}^{3} M^{\psi^k} \cdot \partial_k \right)[f] = \sum_{k=4-m}^{3} \partial_k f \cdot \psi^k. \end{aligned}$$

Let us put the question: what are the conditions on ψ for the equalities

$${}^{\psi}D \cdot {}^{\overline{\psi}}D = {}^{\overline{\psi}}D \cdot {}^{\psi}D = D^{\psi} \cdot D^{\overline{\psi}} = D^{\overline{\psi}} \cdot D^{\psi} = \Delta_{\mathbb{H}} \tag{1.1}$$

to be true, where $\Delta_{\mathbb{H}}[f] := \sum\limits_{k=4-m}^{3} \Delta[f_k] \cdot i_k$, Δ being the m-dimensional Laplace operator? It turns out that (1.1) is valid if and only if

$$\psi^j \overline{\psi}^k + \psi^k \cdot \overline{\psi}^j = 2\delta_{jk} \tag{1.2}$$

for any $j,\ k$. The sufficiency of (1.2) has been proved in [VS1] and [N1], moreover in [N1] necessity has been proved also. The equalities (1.2) are equivalent to

$$\langle \vec{\psi^j}, \vec{\psi^k} \rangle_{\mathbb{R}^4} = \delta_{jk}, \tag{1.3}$$

which means the orthonormality of the set $\vec{\psi}$. Any set ψ of quaternions with the property (1.2) we shall call "structural set".

It is evident that ψ and $\overline{\psi}$ are structural sets simultaneously .

If ψ is a structural set then we shall call ${}^{\psi}D$ and D^{ψ} respectively the left and the right-Cauchy-Riemann operator (C.- R.O.). We have

$${}^{\psi}D = Z D^{\overline{\psi}} Z, \ D^{\psi} = Z\, {}^{\overline{\psi}}D Z, \tag{1.4}$$

so it is sufficient to study the "left" (or the "right") case only.

We have also

$$\begin{aligned} {}^{\psi}D[f \cdot g] &= {}^{\psi}D[f] \cdot g + \sum_{k=4-m}^{3} \psi^k \cdot f \cdot \partial_k g, \\ D^{\psi}[f \cdot g] &= \sum_{k=4-m}^{3} \partial_k f \cdot g \cdot \psi^k + f \cdot D^{\psi}[g]. \end{aligned} \tag{1.5}$$

It follows from (1.5) in particular that the left-(respectively right) C.-R.O. is linear in the right (respectively left) $\mathbb{H}$-module $C^1(\Omega, \mathbb{H})$.

2 Hyperholomorphic functions

2.1 Let us fix a structural set ψ and a domain Ω. We introduce the sets

$$\begin{array}{rcl} {}^{\psi}\mathfrak{M}(\Omega) & := & \ker {}^{\psi}D = \left\{ f \in C^1(\Omega;\mathbb{H}) |\ {}^{\psi}D[f] = \mathbf{0}_{\Omega} \right\}, \\ \mathfrak{M}^{\psi}(\Omega) & := & \ker D^{\psi} = \left\{ f \in C^1(\Omega;\mathbb{H}) |\ D^{\psi}[f] = \mathbf{0}_{\Omega} \right\}. \end{array} \tag{2.1}$$

We shall call elements of these sets left-(respectively, right-) ψ- hyperholomorphic in a domain Ω $\mathbb{H}$-valued functions of m- dimensional variable, shortering this name if misunderstanding is excluded. The sets (2.1) will be called briefly the classes of the hyperholomorphy sometimes. Any class of left-(respectively, right-)-hyperholomorphic functions is a right (respectively, left) $\mathbb{H}$-submodule of the $\mathbb{H}$-module $C^1(\Omega;\mathbb{H})$. The equalities (1.1) mean that the coordinate functions of hyperholomorphic functions of class $C^2(\Omega;\mathbb{H})$ are necessarily harmonic functions in Ω.

2.2 Let $f_0 : \Omega \longrightarrow \mathbb{R}$ be a harmonic function of m variables and put $g(x) := {}^{\overline{\psi}}D[f_0](x) = \sum\limits_{k=4-m}^{3} \overline{\psi}^k \cdot \partial_k f_0$. Then ${}^{\psi}D[g] = {}^{\psi}D \cdot {}^{\overline{\psi}}D[f_0] = \Delta[f_0] = \mathbf{0}_{\Omega}$; similarly $D^{\psi}[g] = \mathbf{0}_{\Omega}$. We have shown that the classes of hyperholomorphy are wide enough and that

$${}^{\psi}\mathfrak{M}^{\psi}(\Omega) := {}^{\psi}\mathfrak{M}(\Omega) \cap \mathfrak{M}^{\psi}(\Omega) \neq \emptyset.$$

It is clear that various structural sets can generate the same class of hyperholomorphy as in shown by trivial example ${}^{\psi}\mathfrak{M}(\Omega) = {}^{-\psi}\mathfrak{M}(\Omega)$. This theme will be discussed in detail in Section 4.

2.3 Let f be a complex-valued function of a complex variable $z = x + iy$. Then as it is well known

$$df = \frac{\partial f}{\partial z} \cdot dz + \frac{\partial f}{\partial \overline{z}} \cdot d\overline{z}, \tag{2.2}$$

where df is the differential of f. This equality is one of the starting points to introduce the notion of a holomorphic function. Now we will show that analogously one can introduce the notion of hyperholomorphic function for $m = 4$.

First we introduce the $\mathbb{H}$-valued differential forms

$$\sigma^{(2)}_{\psi,x} := \operatorname{sgn}\ \psi \cdot \left(\psi^1 dx^2 \wedge dx^3 - \psi^2 dx^1 \wedge dx^3 + \psi^3 dx^1 \wedge dx^2 \right), \tag{2.3}$$

$$\begin{array}{rcl} \sigma^{(3)}_{y,x} & := & -\operatorname{sgn}\ \psi \cdot (dx^1 \wedge dx^2 \wedge dx^3 - \psi^1 dx^0 \wedge dx^2 \wedge dx^3 + \\ & & +\psi^2 dx^0 \wedge dx^1 \wedge dx^3 - \psi^3 dx^0 \wedge dx^1 \wedge dx^2), \end{array} \tag{2.4}$$

where ψ^1, ψ^2, ψ^3 are purely imaginary quaternions, sgn $\psi := 1$ or sgn $\psi := -1$ depending on the fact: the bases $\{\vec{\psi^1}, \vec{\psi^2}, \vec{\psi^3}\}$ and $\{\vec{i_1}, \vec{i_2}, \vec{i_3}\}$ in $\mathbb{R}^3$ have the same orientation or not. Then one can check the equality

$$d\left(\sigma^{(2)}_{\psi,x} f\right) = \sigma^{(3)}_{\overline{\psi},x} \cdot {}^{\psi}D[f] + \sigma^{(3)}_{\psi,x} \cdot {}^{\overline{\psi}}D[f], \tag{2.5}$$

where $f \in C^1(\Omega, \mathbb{H})$ and d is the exterior differentiation operator of a differential form. The equality (2.5) is an exact analog of (2.2).

Further let us denote by ${}'f(x_0)$ – call the left-ψ-derivative of f at a point x_0 – the quaternion satisfying the equality

$$d(\sigma^{(2)}_{\psi,x} \cdot f) = \sigma^{(3)}_{\psi,x} \cdot {}'f(x_0). \tag{2.6}$$

Thus the function f is left-ψ-hyperholomorphic if it has a left-ψ-derivative in Ω and in this case

$$'f(x) = {}^{\overline{\psi}}D[f](x). \tag{2.7}$$

2.4 Apparently for the first time ψ-hyperholomorphic functions (with $\psi = (i_0, i_1, i_2, i_3)$ for $m = 4$ and $\psi = (i_1, i_2, i_3)$ for $m = 3$) were investigated by Gr. C. Moisil [M1], [M2] (see [M3, pp. 23–25, 80-85] also), who used a quaternionic language only in the four-dimensional case ($m = 4$) ([M2], while for $m = 3$ he used a matrix one [M3]). R. Fueter in [F1], [F2] and in other papers and V.S. Vinogradov [Vi1] complemented essentially the results from [M2]. Note the papers [Su], [D], [Sp1], [Sp2] also. An extension of these ideas for Clifford algebra-valued functions can be found in [BDS], [DSS], [R1], [GM], [R2], [Sp3], [L] for example. V. Iftimie [I] investigated the case $\psi = \{\varepsilon_0 i_{p_0}, \ldots, \varepsilon_m i_{p_m}\}$, where $\varepsilon_0, \ldots, \varepsilon_m$ are real numbers.

For $\mathbb{H}$-valued functions a general definition of the class ${}^{\psi}\mathfrak{M}$ was introduced independently in [VS1] and [N1] (see also [N2] and [VS2], [VS3]), for a corresponding extension to Clifford analysis see [Sh1], [Sh2].

3 Quaternionic Cauchy integral-type integrals (C.i.t.i) and some of their properties

3.1 Hyperholomorphic functions have the exact analogues of the main theorems of one-dimensional complex analysis. Now we give some of them which we need for further purposes.

If $\sigma_{\psi,x}^{(m-1)}$ is defined by (2.3) or (2.4) and if we denote as usual by $d\widehat{x}^{(k)}$ the differential form $dx^{4-m} \wedge \ldots \wedge dx^3$ with the multiple dx^k omited then

$$\sigma_{\psi,x}^{(m-1)} = -\mathrm{sgn}\ \psi \sum_{k=4-m}^{3} (-1)^k \psi^k d\widehat{x}^k. \tag{3.1}$$

If ds_{m-1} is the differential form of the $(m-1)$-dimensional volume in $\mathbb{R}^m$ then we have

$$|\sigma_{\psi,x}^{(m-1)}| = ds_{m-1}.$$

This fact allows us to call $\sigma_{\psi,x}^{(m-1)}$ the $\mathbb{H}$-valued differential form of the $(m-1)$-dimensional volume in $\mathbb{R}^m$ corresponding to ψ.

Let $\{f, g\} \subset C^1(\overline{\Omega}; \mathbb{H})$. Then direct computation leads to

$$d(g\sigma_{\psi,x}^{(m-1)} f) =$$

$$= (-1)^{m-1}\mathrm{sgn}\ \psi \left(D^{\psi}[g] \cdot f + g \cdot {}^{\psi}D[f]\right) dx, \tag{3.2}$$

where dx is the differential form of the m-dimensional volume in $\mathbb{R}^m$.

The equality (3.2) is the base for a series of essential constructions further on. Thus the general Stokes formula gives an integral form of (3.2):

$$\int_{\partial\Omega} g(x)\sigma_{\psi,x}^{(m-1)} f(x) =$$

$$= (-1)^{m-1}\mathrm{sgn}\ \psi \int\limits_{\Omega} \left(D^{\psi}[g](x)\cdot f(x) + g(x)^{\psi}D[f](x) \right) dx. \tag{3.3}$$

One may call it *the quaternionic Stokes' formula.* *The general* (that is for two functions) *Cauchy integral theorem* follows in turn from (3.3): *if f is a left-ψ-hyperholomorphic in $\overline{\Omega}$ and g is right-ψ-hyperholomorphic in $\overline{\Omega}$ then*

$$\int\limits_{\partial\Omega} g(x)\sigma_{\psi,x}^{(m-1)} f(x) = 0. \tag{3.4}$$

3.2 Now we introduce the Cauchy kernel $\mathcal{K}_{\psi}$ generated by a structural set ψ. Let

$$\begin{aligned} \mathcal{K}_{\psi}(x) &:= -\frac{1}{(m-2)|\mathbb{S}^{m-1}|}\cdot {}^{\overline{\psi}}D[\theta_m](x) = \\ &= \frac{1}{|\mathbb{S}^{m-1}|\cdot|x|^m}\sum_{k=4-m}^{3} \overline{\psi}^k\cdot x^k, \end{aligned}$$

where

$$\begin{aligned} \theta_m &: \quad x\in\mathbb{R}^m\backslash\{0\} \longmapsto \frac{1}{|x|^{m-2}}; \\ |\mathbb{S}^{m-1}| &= \begin{cases} 4\pi, & \text{if } m=3, \\ 2\pi^2, & \text{if } m=4, \end{cases} \end{aligned}$$

is the $(m-1)$-dimensional volume of the unit sphere $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$. Then obviously:

1) $\mathcal{K}_{\psi} \in C^{\infty}\left(\mathbb{R}^m\backslash\{0\},\mathbb{H}\right)$

2) $\mathcal{K}_{\psi} \in {}^{\psi}\mathfrak{M}^{\psi}(\mathbb{R}^m\backslash\{0\})$

3) Let $z\in\mathbb{R}^m$, and let $\mathcal{K}_{\psi,z}(x) := \mathcal{K}_{\psi}(z-x)$. Then

$$\mathcal{K}_{\psi,z} \in {}^{\psi}\mathfrak{M}^{\psi}(\mathbb{R}^m\backslash\{z\}).$$

The second property is a mere corollary of the harmonicity of θ_m.

3.3 Let x_0 be fixed and let $\varepsilon > 0$ be small and let $\Omega_{\varepsilon} := \Omega\backslash\mathbb{B}^m(x_0,\varepsilon)$. Using formula (3.3) for Ω_{ε} and the Cauchy kernel properties we obtain by letting $\varepsilon\to 0$ the *analog of the* well-known *Borel-Pompeiu formula*: if $\{f,g\}\subset C^1(\overline{\Omega},\mathbb{H})$, then

$$\begin{aligned} &(-1)^{m-1}\mathrm{sgn}\ \psi \int\limits_{\partial\Omega} \left(\mathcal{K}_{\psi}(\tau-x)\sigma_{\psi,\tau}^{(m-1)} f(\tau) + g(\tau)\sigma_{\psi,\tau}^{(m-1)}\mathcal{K}_{\psi}(\tau-x) \right) - \\ &- \int_{\Omega} \left(\mathcal{K}_{\psi}(y-x)\cdot {}^{\psi}D[f](y) + D^{\psi}[g](y)\cdot\mathcal{K}_{\psi}(y-x) \right) dy = \\ &= \begin{cases} f(x)+g(x), & \text{if} \quad x\in\Omega, \\ 0, & \text{if} \quad x\in\mathbb{R}^m\backslash\overline{\Omega}. \end{cases} \end{aligned} \tag{3.5}$$

If $g \in \mathfrak{M}^{\psi}(\overline{\Omega})$ and $f \in {}^{\psi}\mathfrak{M}(\overline{\Omega})$ then (3.5) gives us the Cauchy integral formula for a function which is a sum of a left and a right-ψ-hyperholomorphic functions. Assuming additionally that $f = \mathbf{0}_{\overline{\Omega}}$ or $g = \mathbf{0}_{\overline{\Omega}}$ we obtain *the Cauchy integral formula* for a right and for a left-ψ-hyperholomorphic functions respectively:

$$(-1)^{m-1}\mathrm{sgn}\ \psi \int_{\partial\Omega} g(\tau)\sigma_{\psi,\tau}^{(m-1)}\mathcal{K}_{\psi}(\tau - x) = \begin{cases} g(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^m \setminus \overline{\Omega}, \end{cases} \tag{3.6}$$

and

$$(-1)^{m-1}\mathrm{sgn}\ \psi \int_{\partial\Omega} \mathcal{K}_{\psi}(\tau - x)\sigma_{\psi,\tau}^{(m-1)} f(\tau) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^m \setminus \overline{\Omega}. \end{cases} \tag{3.7}$$

One can consider the upper lines of these formulas as the representations of a right and of a left-ψ-hyperholomorphic in Ω functions by their respective right and left Cauchy integrals.

If $g = f = \mathbf{1}_{\overline{\Omega}}$ in (3.6) and (3.7) then

$$\begin{aligned} &(-1)^{m-1}\mathrm{sgn}\,\psi \int_{\partial\Omega} \mathcal{K}_{\psi}(\varepsilon - x)\sigma_{\psi,\tau}^{(m-1)} = \\ &= (-1)^{m-1}\mathrm{sgn}\,\psi \int_{\partial\Omega} \sigma_{\psi,\tau}^{(m-1)}\mathcal{K}_{\psi}(\tau - x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \in \mathbb{R}^m \setminus \overline{\Omega}. \end{cases} \end{aligned} \tag{3.8}$$

3.4 Now let $f \in C(\partial\Omega; \mathbb{H})$. Then for $x \notin \partial\Omega$ we define

$$\begin{aligned} x &\longmapsto {}^{\psi}K[f](x) := (-1)^{m-1}\mathrm{sgn}\,\psi \int_{\partial\Omega} \mathcal{K}_{\psi}(\tau - x)\sigma_{\psi,\tau}^{(m-1)} f(\tau), \\ x &\longmapsto K^{\psi}[f](x) := (-1)^{m-1}\mathrm{sgn}\,\psi \int_{\partial\Omega} f(\tau)\sigma_{\psi,\tau}^{(m-1)}\mathcal{K}_{\psi}(\tau - x). \end{aligned} \tag{3.9}$$

Those functions will be called respectively the left and the right Cauchy integral-type integral (C.i.t.i.) of the function f and the maps

$${}^{\psi}K : f \longmapsto {}^{\psi}K[f],\ K^{\psi} : f \longmapsto K^{\psi}[f]$$

are called C.i.t.i. operators.

When f is a restriction to $\partial\Omega$ of a function from ${}^{\psi}\mathfrak{M}(\overline{\Omega})$ then of course the left C.i.t.i of f turns into its left Cauchy integral. Analogously for the right case.

It follows immediately from the Leibnitz rule that both C.i.t.i. of f belong to $C^{\infty}(\mathbb{R}^m \setminus \partial\Omega; \mathbb{H})$ and that

$$\begin{aligned} {}^{\psi}D_x[{}^{\psi}K[f]](x) &= (-1)^{m-1}\mathrm{sgn}\,\psi \int_{\partial\Omega} {}^{\psi}D_x[\mathcal{K}_{\psi,x}](\tau)\sigma_{\psi,\tau}^{(m-1)} f(\tau) = \\ &= (-1)^{m-1}\mathrm{sgn}\,\psi \int_{\partial\Omega} {}^{\psi}D_{\tau}[\mathcal{K}_{\psi}](\tau - x)\sigma_{\psi,\tau}^{(m-1)} f(\tau) = 0, \end{aligned}$$

analogously $D^{\psi}[K^{\psi}[f]](x) = 0$. Consequently the left (the right) C.i.t.i. of a continuous function f is a left-(right-)-ψ-hyperholomorphic function in $\mathbb{R}^m \setminus \partial\Omega$.

3.5 Now we proceed to study boundary properties of the C.i.t.i. of a fixed function f. As usual it is convenient to start from the case of a Hölderian density f. Let for $0 < \mu \leq 1$,

$$\begin{aligned} H_\mu(\Gamma; \mathbb{H}) &:= \{f| \ |f(t_1) - f(t_2)| \leq L_f \cdot |t_1 - t_2|^\mu; \ \forall \{t_1 t_2\} \subset \Gamma, \\ &\quad L_f = \text{const}\}, \end{aligned}$$

and let $f \in H_\mu(\Gamma; \mathbb{H})$. We show that the integral ${}^\psi K[f](t)$ is defined in the sense of the principal value for $t \in \Gamma$. To simplify the statement we assume $m = 3$ and $\operatorname{sgn} \psi = 1$. For $\varepsilon > 0$ we put

$$\begin{aligned} \gamma_\varepsilon &:= \Gamma \cap \mathbb{B}^3(t;\varepsilon); \quad \Omega_\varepsilon^+ := \Omega^+ \backslash \mathbb{B}^3(t,\varepsilon); \\ \Lambda_\varepsilon &:= \overline{\partial\Omega_\varepsilon^+ \backslash \mathbb{B}^3(t,\varepsilon)}; \ A_t(\varepsilon) := \int\limits_{\Lambda_\varepsilon} \mathcal{K}_\psi(\tau - t)\sigma^{(2)}_{\psi,\tau} f(\tau). \end{aligned}$$

We have: $A_t(\varepsilon) = \int\limits_{\Lambda_\varepsilon} \mathcal{K}_\psi(\tau - t)\sigma^{(2)}_{\psi,\tau}(f(\tau) - f(t)) +$

$$+ \left(\int\limits_{\Lambda_\varepsilon} \mathcal{K}_\psi(\tau - t)\sigma^{(2)}_{\psi,\tau} \right) f(t) =: M + N.$$

Let us transform N:

$$N = \left(\int\limits_{\Lambda_\varepsilon \cup (\partial\mathbb{B}^3(t,\varepsilon)\backslash\Omega^+)} \mathcal{K}_\psi(\tau - t)\sigma^{(2)}_{\psi,\tau} \right) f(t) - \left(\int\limits_{\partial\mathbb{B}^3(t,\varepsilon)\backslash\Omega^+} \mathcal{K}_\psi(\tau - t)\sigma^{(2)}_{\psi,\tau} \right) \cdot f(t) =: N_1 + N_2.$$

According to (3.8) for any $\varepsilon > 0$ $N_1 = f(t)$. Changing the variables in N_2 to spherical coordinates we obtain

$$\begin{aligned} N_2 &= \int\limits_{\Delta_\varepsilon} \frac{1}{4\pi\varepsilon^2} \overline{w} \cdot \varepsilon^2 \cdot \sin\theta_1 \cdot w \cdot d\theta_1 \wedge d\theta_2 \cdot f(t) = \\ &= \frac{1}{4\pi} \int\limits_{\Delta_\varepsilon} \sin\theta_1 d\theta_1 \wedge d\theta_2 f(t), \end{aligned}$$

where

$$\begin{aligned} \Delta_\varepsilon &:= \{(\theta_1, \theta_2)| \ \theta_1 \in [0, 2\pi], \ \theta_2 \in [0; \varphi(\theta_1, \varepsilon)]\}, \\ w &:= \psi^1 \cdot \sin\theta_1 \cos\theta_2 + \psi^2 \sin\theta_1 \cdot \sin\theta_2 + \psi^3 \cdot \cos\theta_1. \end{aligned}$$

It follows from the smoothness of the surface Γ that with a decreasing of $\varepsilon > 0$ part $\partial\mathbb{B}^3(t,\varepsilon)\backslash\Omega$ of the sphere $\partial\mathbb{B}^3(t,\varepsilon)$ will differ from hemisphere less and less. This means

$$\varphi(\theta_1, \varepsilon) \to \frac{\pi}{2}, \ \Delta_\varepsilon \to \Delta := [0; 2\pi] \times [0; \frac{\pi}{2}]$$

when $\varepsilon \to 0$. Hence

$$\lim_{\varepsilon \to 0} N_2 = \frac{1}{4\pi} \int\limits_{\Delta} \sin\theta_1 d\theta_1 \wedge d\theta_2 f(t) = \frac{1}{2} f(t).$$

Thus $N = \frac{1}{2}f(t)$. As for M, from the estimate

$$\begin{aligned} |\mathcal{K}_\psi(\tau - t) \cdot \sigma^{(2)}_{\psi,\tau} \cdot (f(\tau) - f(t))| &\leq \frac{\text{const}}{|\tau - t|^2} \cdot |\sigma^{(2)}_{\psi,\tau}| \cdot |\tau - t|^\mu \leq \\ &\leq \frac{\text{const}}{|\tau_t|^{2-\mu}} ds_2 \end{aligned}$$

we conclude that the integral M exists as an improper integral. Hence $\lim\limits_{\varepsilon\to 0} A_t(\varepsilon)$ exists.

Thus we have finally: if $t \in \Gamma$ then

$$\begin{aligned} {}^\psi K_\Gamma[f](t) &:= \tfrac{1}{2}f(t) + (-1)^{m-1}\text{sgn}\,\psi \int\limits_\Gamma \mathcal{K}_\psi(\tau - t) \cdot \sigma^{(m-1)}_{\psi,\tau} \cdot (f(\tau) - f(t)), \\ K^\psi_\Gamma[f](t) &:= \tfrac{1}{2}f(t) + (-1)^{m-1}\text{sgn}\,\psi \int\limits_\Gamma (f(\tau) - f(t))\sigma^{(m-1)}_{\psi,\tau}\mathcal{K}_\psi(\tau - t). \end{aligned} \tag{3.10}$$

It is clear that (3.10) is true also for a non-connected surface Γ. Moreover Γ may be a surface with a boundary γ, but $t \notin \gamma$ then.

Note also that if $f = \mathbb{1}_\Gamma$ then it follows from (3.10) that

$$\begin{aligned} &(-1)^{m-1}\text{sgn}\,\psi \int\limits_\Gamma \mathcal{K}_\psi(\tau - t)\sigma^{(m-1)}_{\psi,\tau} = \\ &= (-1)^m \text{sgn}\,\psi \int\limits_\Gamma \sigma^{(m-1)}_{\psi,\tau}\mathcal{K}_\psi(\tau - t) = \frac{1}{2}. \end{aligned} \tag{3.11}$$

In addition to the operators ${}^\psi K_\Gamma$ and K^ψ_Γ defined in (3.11) it is convenient to consider the operators

$$ {}^\psi S_\Gamma := 2 \cdot {}^\psi K_\Gamma; \; S^\psi_\Gamma := 2 \cdot K^\psi_\Gamma. \tag{3.12}$$

We call them, like in the complex case, the operators of (respectively left or right) singular integration with the quaternionic Cauchy kernel along a surface Γ.

3.6 It was established earlier that C.i.t.i. of a Hölderian function is defined in the whole $\mathbb{R}^m_x$. Let us consider now the behavior of the functions ${}^\psi K[f]$ and $K^\psi[f]$ when $x \to t$, $x \in \Omega =: \Omega^+$ or $x \in \Omega^- := \mathbb{R}^m \backslash \overline{\Omega}$, $t \in \Gamma$.

We have

$$\begin{aligned} {}^\psi K[f](x) = (-1)^{m-1}\text{sgn}\,\psi \int\limits_\Gamma \mathcal{K}_\psi(\tau - x) \cdot \sigma^{(m-1)}_{\psi,\tau} \cdot (f(\tau) - f(t)) + \\ + (-1)^{m-1}\text{sgn}\,\psi \int\limits_\Gamma \mathcal{K}_\psi(\tau - x)\sigma^{(m-1)}_{\psi,\tau} \cdot f(t). \end{aligned}$$

The second term here equals $f(t)$ if $x \in \Omega^+$ and equals zero if $x \in \Omega^-$ so that

$$\begin{aligned} &{}^\psi K[f](x) = \\ &= \begin{cases} f(t) + (-1)^{m-1}\text{sgn}\,\psi \int\limits_\Gamma \mathcal{K}_\psi(\tau - x) \cdot \sigma^{(m-1)}_{\psi,\tau} \cdot (f(\tau) - f(t)), \; x \in \Omega^+, \\ (-1)^{m-1}\text{sgn}\,\psi \cdot \int_\Gamma \mathcal{K}_\psi(\tau - x) \cdot \sigma^{(m-1)}_{\psi,\tau} \cdot (f(\tau) - f(t)), \; x \in \Omega^-. \end{cases} \end{aligned} \tag{3.13}$$

These are analogous formulas for $K^{\psi}[f](x)$ as well. The integrals ${}^{\psi}B(x)$ and $B^{\psi}(x)$ in both formulas exist also in the improper sense when $x = t$. Let us denote the corresponding values ${}^{\psi}B(t)$ and $B^{\psi}(t)$. One can show (using arguments similar to the ones in 3.5) that $\lim_{\Omega^{\pm}\ni x\to t} {}^{\psi}B(x) =: {}^{\psi}B^{\pm}(t)$ and $\lim_{\Omega^{\pm}\ni x\to t} B^{\psi}(x) =: B^{\psi\pm}(t)$. Moreover

$$\begin{array}{rcl} {}^{\psi}B^{\pm}(t) & = & -\frac{1}{2}f(t) + {}^{\psi}K[f](t), \\ B^{\psi\pm}(t) & = & -\frac{1}{2}f(t) + K^{\psi}[f](t). \end{array} \tag{3.14}$$

This means in particular that $\{{}^{\psi}B, B^{\psi}\} \subset C(\mathbb{R}^m, \mathbb{H})$. Finally combining (3.13) and (3.14) we obtain *the Sokhotski-Plemelj formulas*

$$\begin{array}{rcl} \lim\limits_{\Omega^{\pm}\ni x\to t} {}^{\psi}K[f](x) & =: & {}^{\psi}K[f]^{\pm}(t) = \pm\frac{1}{2}f(t) + {}^{\psi}K[f](t), \\ \lim\limits_{\Omega^{\pm}\ni x\to t} K^{\psi}[f](x) & =: & K^{\psi}[f]^{\pm}(t) = \pm\frac{1}{2}f(t) + K^{\psi}[f](t), \end{array} \tag{3.15}$$

where ${}^{\psi}K[f](t)$ and $K^{\psi}[f](t)$ were defined by (3.14).

There are very useful equivalent formulas:

$$\begin{array}{rcl} {}^{\psi}K[f]^{\pm}(t) & = & \pm\frac{1}{2}f(t) + \frac{1}{2}\cdot {}^{\psi}S[f](t), \\ K^{\psi}[f]^{\pm}(t) & = & \pm\frac{1}{2}f(t) + \frac{1}{2}S^{\psi}[f](t). \end{array} \tag{3.16}$$

3.7 In 3.4 wee supposed that $f \in C(\partial\Omega, \mathbb{H})$, hence the C.i.t.i. of f could be understood as the Riemann integral. Now let $f \in L_p(\partial\Omega, \mathbb{H})$. Then the C.i.t.i. of f has to be understood as the Lebesgue integral preserving the results from 3.4–3.6 with the corresponding changes. For instance the limit values of ${}^{\psi}K[f]$ and $K^{\psi}[f]$ exist almost everywhere (with respect to Lebesgue surface measure) on $\partial\Omega$, and the Sokhotski-Plemelj formulae are true almost everywhere only. Note that the questions related to these discussed above have been examined in [B], [Vi2] for instance.

3.8 The Sokhotski-Plemelj formulae allow (just as in the one-dimensional complex analysis) to solve some simple but essential boundary value problems. Let $f \in H_\mu(\partial\Omega, \mathbb{H})$ be given; than we would like to find functions

$$\begin{array}{rcl} \Phi_1^{\pm} & \in & {}^{\psi}\mathfrak{M}(\Omega^{\pm}) \cap H_\mu(\overline{\Omega}^{\pm}); \\ \Phi_2^{\pm} & \in & \mathfrak{M}^{\psi}(\Omega^{\pm}) \cap H_\mu(\overline{\Omega}^{\pm}), \end{array}$$

such that boundary values statisfy

$$\Phi_1^+ + \Phi_2^+ - \Phi_1^- - \Phi_2^- = f. \tag{3.17}$$

The content of the previous pages yields that one of the solutions to this problem is given by the C.i.t.i. of f:

$$\Phi_1^{\pm} = \frac{1}{2}{}^{\psi}K[f],\ \Phi_2^{\pm} = \frac{1}{2}K^{\psi}[f]. \tag{3.18}$$

Actually, the formulae (3.18) give the solution of such two "jump problems" for left-and for right-ψ-hyperholomorphic functions separately:

$$\Phi_1^+ - \Phi_1^- = f, \tag{3.19}$$

$$\Phi_2^+ - \Phi_2^- = f. \tag{3.20}$$

One can show that the Liouville theorem is true for hyperholomorphic functions. It thus follows that if we are interested in solutions of (3.19) and (3.20) vanishing at the infinity then the solution (3.18) is the only one.

Let $\{a,b,c,d\} \subset \mathbb{H}$ and consider boundary problems with the next boundary conditions:

$$\begin{aligned} \Phi_1^+ \cdot a + b \cdot \Phi_2^+ + \Phi_1^- \cdot c + d \cdot \Phi_2^- &= f, \\ \Phi_1^+ \cdot a + \Phi_1^- \cdot c &= f, \\ b \cdot \Phi_2^+ + d \cdot \Phi_2^- &= f. \end{aligned}$$

Obviously one can directly reduce these problems to (3.17), (3.19), (3.20).

3.9 Let Γ be a smooth surface (without boundary) in $\mathbb{R}^m$ bounding a finite domain Ω^+ and an infinite domain Ω^-. Let $f \in H_\mu(\Gamma;\mathbb{H})$. Next assertions follow directly from 3.6.

Assertion A. *In order that a function f be the boundary value (that is a restriction on Γ) of a function $\widetilde{f}$ from*

$${}^{\psi}\mathfrak{M}(\Omega^+) \cap H_\mu(\overline{\Omega}^+;\mathbb{H})$$

the following necessary and sufficient conditions should be satisfied:

$$\begin{array}{ccc} -\frac{1}{2}f(t) & + & {}^{\psi}K[f](t) = 0 \\ \forall t & \in & \Gamma \end{array} \tag{3.21}$$

Assertion B. *In order that a function f be the boundary value of a function $\widetilde{f}$ from*

$${}^{\psi}\mathfrak{M}(\Omega^-) \cap H_\mu(\overline{\Omega}^-;\mathbb{H}),\ \widetilde{f}(\infty) = 0,$$

the following necessary and sufficient conditions should be satisfied:

$$\begin{array}{ccc} \frac{1}{2}f(t) & + & {}^{\psi}K[f](t) = 0 \\ \forall t & \in & \Gamma \end{array} \tag{3.22}$$

To formulate corresponding assertions for $f \in L_p(\Gamma;\mathbb{H})$ the following definition is useful: if X is any space consisting of $\mathbb{H}$-valued functions defined on Γ then ${}^{\psi}\mathfrak{A}(\Omega^\pm, X)$ denote the classes of functions $\widetilde{f}^\pm$ such that:

1. $\widetilde{f}^\pm \in {}^{\psi}\mathfrak{M}(\Omega^\pm)$
2. when $\Omega^\pm \ni x \to t \in \Gamma$ there exist
$$\lim \widetilde{f}^\pm(x) =: \widetilde{f}^\pm(t)$$
(along, non-tangential paths at any rate) everywhere or almost everywhere on Γ generating function from X.
3. $\widetilde{f}^\pm$ is represented by a C.i.t.i. with a density from X.

If $X = L_p(\Gamma;\mathbb{H})$ then we shall write ${}^{\psi}\mathfrak{A}_p(\Omega^\pm)$.

Assertion C. *In order that the values of function $f \in L_p(\Gamma;\mathbb{H})$ coincide almost everywhere on Γ with limit values of $\widetilde{f}^\pm \in {}^{\psi}\mathfrak{A}_p(\Omega^+)$ the following necessary and sufficient condition should be realized almost everywhere on Γ:*

$$\frac{1}{2}f(t) + {}^{\psi}K[f](t) = 0.$$

It is clear how one can reformulate Assertion C for the case Ω^- and the right–ψ–hyperholomorphic case.

3.10 Let us make here several remarks on $L_p(\Gamma,\mathbb{H}),\ p > 1$. First of all there are two equivalent definitions of L_p as a set:

1. $L_p(\Gamma;\mathbb{H}) := \left\{ f = \sum_{k=0}^{3} f_k\, i_k | \quad \int_{\Gamma} |f(t)|^p ds_{m-1}(t) < +\infty \right\}.$

2. $L_p(\Gamma;\mathbb{H}) := \left\{ f = \sum_{k=0}^{3} f^k\, i_k | \quad f^k \in L_p(\Gamma;\mathbb{R}),\ k \in \{0,1,2,3\} \right\}.$

It is clear that one can introduce a structure of both left and right $\mathbb{H}$-modulus in $L_p(\Gamma;\mathbb{H})$. We shall make more porecise what we mean in concrete situations. In both cases $L_p(\Gamma;\mathbb{H})$ is a Banach module.

Let L_p provided with a right structure. Then classes ${}^{\psi}\mathfrak{A}_p(\Omega^{\pm})$ introduced in 3.9 generate sub-modules of the $\mathbb{H}$-module $L_p(\Gamma;\mathbb{H})$. One can show that for $\Omega^{+} = \mathbb{B}^m(0;1)$ the definition of the class ${}^{\psi}\mathfrak{A}_p(\Omega^{+})$ is a natural hyperholomorphic analog of the complex Hardy space $H_p(\mathbb{B}^m)$ definition. That is why we shall denote it as ${}^{\psi}\mathcal{H}_p(\mathbb{B}^m;\mathbb{H})$. Analogously for $\mathcal{H}_p^{\psi}(\mathbb{B}^m)$. As usual we identify an element of ${}^{\psi}\mathcal{H}_p(\mathbb{B}^m;\mathbb{H})$ with a function from $L_p(\Gamma;\mathbb{H})$ generated by its limit values and so ${}^{\psi}\mathcal{H}_p(\mathbb{B}^m;\mathbb{H})$ becomes a sub-module of $L_p(\Gamma;\mathbb{H})$ just in this sense.

3.11 In the sequel we shall frequently use the notion of a right (left) Hilbert $\mathbb{H}$–module. For convenience we only recall some properties of such modules and we refer the reader to [Ist] for forther information, even in more general cases.

The inner product on a Hilbert $\mathbb{H}$– module H will be denoted by $\langle \cdot, \cdot \rangle$; for a (non-empty) subset H_0 of H, its orthogonal complement is denoted by $H_0^{\perp}$.

The Riesz theorem tells us that a continuous linear functional χ in a right (respectively left) Hilbert $\mathbb{H}$-module H acts by the rule $\chi(a) = \langle g_{\chi}, a \rangle$ (respectively $\chi(a) = \langle a, g_{\chi} \rangle$) for $\forall a \in H$ where $g_{\chi} \in H$ is uniquely defined and $|\chi| = |g_{\chi}|$. Correspondingly an antilinear functional χ in a right (or left) Hilbert $\mathbb{H}$-module H acts by the rule $\chi(a) = \langle a, g_{\chi} \rangle$ (or $\chi(a) = \langle g_{\chi}, a \rangle$) for $\forall a \in H$ where g_{χ} is the same as above.

The set of all continuous linear functions in a right (respectively left) Hilbert $\mathbb{H}$-module H is a left (respectively right) Hilbert $\mathbb{H}$-module H^* with the scalar product

$$\langle \xi, \chi \rangle_{H^*} := \langle g_{\xi}, g_{\chi} \rangle_{H}.$$

It also follows from the Riesz theorem that for any right (respectively left) Hilbert $\mathbb{H}$-module H there exists an antilinear invertible operator $B : H \longrightarrow H^*$ with the property

$$\langle x, y \rangle_{H} = \langle Bx, By \rangle_{H^*},$$

which means that B is an isometric bijection and so B is an isomorphism in this sense.

Let H be a right or a left Hilbert $\mathbb{H}$-module, and let S be a linear operator in H. Then the adjoint operator S^* acts in a (left or right correspondingly) $\mathbb{H}$-module H^* by the rule

$$S^*[\xi](x) \ := \ \xi(S[x]) \quad \forall \xi \in H^*,\ \forall x \in H.$$

By the Riesz theorem the last equality takes the form: for the right case

$$\langle B^{-1} S^*[\xi], x \rangle \ = \ \langle B^{-1}[\xi], S[x] \rangle,$$

and for the left case

$$\langle x, B^{-1} S^*[\xi] \rangle \ = \ \langle S[x], B^{-1}[\xi] \rangle.$$

Let H be a right $\mathbb{H}$-module. An operator S is called self-adjoint if

$$\langle B^{-1}[\xi], S[x] \rangle \ = \ \langle S B^{-1}[\xi], x \rangle,$$

that is $SB^{-1} = B^{-1}S^*$ and this means
$$BSB^{-1} = S^*.$$
The same holds for the left case. The notion of a self-adjoint operator can be defined in another but equivalent way. Let H be a right $\mathbb{H}$-module, denote
$$B^{-1}[\xi] =: g_\xi \in H, \;\; B^{-1}S^*[\xi] =: g_{S^*[\xi]} \in H,$$
then
$$\langle g_\xi, S[x]\rangle = \langle g_{S^*[\xi]}, x\rangle.$$
Consequently a (left) linear operator S^* generates a map
$$S' \,:\, g_\xi \in H \longrightarrow g_{S^*[\xi]} \in H$$
which is linear (that is right-linear) in H. The operators S' and S^* are connected by the evident equality
$$S^* = BS'B^{-1}.$$
It is clear that S is self-adjoint in H in the sense of the above definition if and only if
$$S' = S.$$
3.12 Let us consider $L_2(\Gamma;\mathbb{H})$ as a right $\mathbb{H}$–module. Then
$$\langle f, g\rangle := \int_\Gamma \overline{f(t)} \cdot g(t)\, ds_{m-1}$$
defines a scalar product on it turning $L_2(\Gamma;\mathbb{H})$ into a right Hilbert $\mathbb{H}$-module. In the left $L_2(\Gamma;\mathbb{H})$ a scalar product is given by
$$\langle f, g\rangle := \int_\Gamma f(t) \cdot \overline{g(t)}\, ds_{m-1}.$$
The hyperholomorphic Hardy spaces ${}^\psi\mathcal{H}_2(\Omega,\mathbb{H})$ and $\mathcal{H}_2^\psi(\Omega,\mathbb{H})$ introduced in 3.10 form Hilbert submodules in the corresponding $L_2(\Gamma;\mathbb{H})$.

Let us note that if $A : L_2(\Gamma,\mathbb{H}) \to L_2(\Gamma;\mathbb{H})$ is a self-adjoint operator in the sense of 3.11 then we have
$$\int_\Gamma \overline{A[f](t)} \cdot g(t)\, ds_{m-1} = \int_\Gamma \overline{f(t)} \cdot A[g](t) ds_{m-1}$$
or
$$\int_\Gamma A[f](t) \cdot \overline{g(t)}\, ds_{m-1} = \int_\Gamma f(t) \cdot \overline{A[g](t)}\, ds_{m-1}.$$

4 Dependence of some main objects on a structural set

4.1 Now we return to the discussion concerning the connections between the classes of hyperholomorphy and the structural sets generating them.

Let ψ be a structural set and let $h \in \mathbb{H}$, $|h| = 1$. As the multiplication operator (both from the left and from the right) by h is an orthogonal transformation of $\mathbb{R}^4$ the sets $y := h \cdot \psi$ and $\widetilde{y} := \psi \cdot h$

are structural also. In this case we shall call y and ψ (respectively $\widetilde{y}$ and ψ) left-(respectively right-) equivalent.

4.2 We have

$$ {}^{h\psi}D = \sum_{k=4-m}^{3} h \cdot \psi^k \cdot \partial_k, \;\; D^{\psi h} = M^h \cdot D^{\psi} \tag{4.1} $$

Thus

$$ {}^{\psi}\mathfrak{M}(\Omega) = {}^{h\psi}\mathfrak{M}(\Omega), \;\; \mathfrak{M}^{\psi}(\Omega) = \mathfrak{M}^{\psi h}(\Omega). \tag{4.2} $$

In exactly the same way, from

$$ D^{h\psi} = D^{\psi} \cdot M^h, \;\; {}^{\psi h}D = {}^{\psi}D \cdot {}^{h}M \tag{4.3} $$

it follows that

$$ \mathfrak{M}^{\psi}(\Omega) = \mathfrak{M}^{h\psi}(\Omega)h; \; {}^{\psi}\mathfrak{M}(\Omega) = h \cdot {}^{\psi h}\mathfrak{M}(\Omega). \tag{4.4} $$

Thus a class of hyperholomorphy is not changed under:

1. multiplication of left-(respectively right-) ψ-hyperholomorphic functions from the right (respectively from the left) by constants from $\mathbb{H}$,
2. replacing a structural set by an equivalent one (see (4.2)),
3. simultaneous multiplication "from the wrong side" by $h \in \mathbb{S}^3 \subset \mathbb{H}$ of all functions of a class and of a structural set (see (4.4)).

We note also that from (1.4) we have

$$ {}^{\psi}\mathfrak{M}(\Omega) = Z(\mathfrak{M}^{\overline{\psi}}(\Omega)), \;\; \mathfrak{M}^{\psi}(\Omega) = Z({}^{\overline{\psi}}\mathfrak{M}(\Omega)). $$

Now let $a \in \mathbb{H}\backslash\{0\}$. Since all classes of hyperholomorphy are $\mathbb{R}$-linear spaces, for $h := \frac{1}{|a|} \cdot a$ we have: if $f \in {}^{\psi}\mathfrak{M}(\Omega)$ then $af \notin {}^{\psi}\mathfrak{M}(\Omega)$ in general but $af = |a| \cdot hf \in {}^{\psi \overline{h}}\mathfrak{M}(\Omega)$. So by multiplying a left-ψ-hyperholomorphic function by quaternion from the left we get a left-hyperholomorphic function again but from another class generally saying. The same holds in the right case.

4.3 We shall describe now how an affine change of variables (with an orthogonal matrix) affects on a C.-R.O. with fixed ψ. This will allow us to point out the connection between arbitrary ψ-hyperholomorphy and the standard hyperholomorphy.

Let $\Omega \subset \mathbb{R}^m_x$, and let $x = \gamma(y) := a \cdot y + b$ where

$$ a = (a_{kq}) \in O_m(\mathbb{R}), \;\; b \in \mathbb{R}^m, \;\; y \in \gamma^{-1}(\Omega) =: \Xi. $$

Let us denote by W_γ the operator of the change of variable: $W_\gamma[f](y) = g = (f \circ \gamma)(y)$. As one can easily see we have

$$ (W_\gamma \cdot {}^{\psi}D_x)[f](y) = \sum_{q=4-m}^{3} \left(\sum_{k=4-m}^{3} \psi^k \cdot a_{qk} \right) \cdot \partial_{y^q} g(y). $$

It turns out that the set $\varphi := \{\varphi^{4-m}, \ldots, \varphi^3\}$, with

$$ \varphi^q := \sum_{k=4-m}^{3} \psi^k \cdot a_{qk}, \tag{4.5} $$

is structural. Consequently the C.-R.O. ${}^{\psi}D_x$ turns into C.-R.O. ${}^{\varphi}D_y$:

$$W_\gamma \cdot {}^{\psi}D_x = {}^{\varphi}D_y \cdot W_\gamma \tag{4.6}$$

and

$$f \in {}^{\psi}\mathfrak{M}(\Omega) \Longleftrightarrow f \circ \gamma \in {}^{\varphi}\mathfrak{M}(\Xi). \tag{4.7}$$

4.4 As was established in 4.1 equivalent structural sets define the same class of hyperholomorphy. Later on we shall speak sometimes about "structural 3-tuples (4-tuples)" instead of "structural sets" when we must distinguish between $m = 3$ and $m = 4$. Firstly suppose $m = 3$. We show that among all equivalent structural 3-tuples there is one which, in the space $\mathbb{R}^4_y$ corresponding to $\mathbb{H}$, lies in the hyperplane

$$\widetilde{\mathbb{R}}^3_y := \left\{(0, y^1, y^2, y^3) |\ (y^1, y^2, y^3) \in \mathbb{R}^3\right\}.$$

Really, for any fixed structural 3-tuple ψ there are exactly two quaternions $\pm\psi^0$ such that sets of vectors $\{\pm\vec{\psi^0}, \vec{\psi^1}, \vec{\psi^2}, \vec{\psi^3}\}$ form orthonormalized bases in $\mathbb{R}^4_y$. Let ψ^0 be chosen in such a way that the basis $\{\vec{\psi^0}, \vec{\psi^1}\vec{\psi^2}, \vec{\psi^3}\}$ is co-oriented with the canonical basis in $\mathbb{R}^4_y$ and denote $h := \overline{\psi}^0$. Then $y^0 := h\psi^0 = i_0$, whence a 3-tuple $y := h \cdot \psi$ which is left equivalent to a 3-tuple ψ lies in the hyperplane $\widetilde{\mathbb{R}}^3_y$.

Analogously if $\lambda := \psi h$ then a 3-tuple λ which is right equivalent to ψ lies in $\widetilde{\mathbb{R}}^3_y$.

It then follows that for $m = 3$ a class of hyperholomorphy is defined in fact by a certain orthonormal basis in $\mathbb{R}^3$ (but not by three orthonormal vectors in $\mathbb{R}^4$).

It is clear that at the same time we can conclude that for $m = 4$ a class of hyperholomorphy is defined by a structural 4-tuple consisting of the unit and the orthonormalized basis in $\widetilde{\mathbb{R}}^3$, i.e., it is defined just by this basis.

However it is important to note that for $m = 4$ (in contrast to $m = 3$) a basis in $\mathbb{R}^3$ is not obliged to have positive orientation. Thus for $m = 3$ the set of classes of hyperholomorphy is parametrized by the elements of the group $O_3^+(\mathbb{R})$ while for $m = 4$ by the elements of the group $O_3(\mathbb{R})$.

In the sequel the structural set ψ is chosen in accordance with the procedure described. In the opposite case a special remark is made explicitly.

Now we point out the way for establishing connections between different classes of hyperholomorphy with the help of a change of variables. Let $\psi = \{\psi^1,\ \psi^2,\ \psi^3\}$ be a structural 3-tuple and let

$$a := \left(\psi^k_j\right)^3_{k,j=1}. \tag{4.8}$$

Then $a \in 0_3^+(\mathbb{R})$. Consider the change of variable $x = a \cdot y$ with matrix (4.8). Formula (4.5) gives

$$\begin{aligned} \varphi^q &= \sum_{k=1}^{3} \psi^k \psi^k_q = i_q, \\ q &\in \{1, 2, 3\}. \end{aligned} \tag{4.9}$$

For $m = 4$ let $\psi = \{i_0,\ \psi^1,\ \psi^2,\ \psi^3\}$, where $\{\psi^1,\ \psi^2,\ \psi^3\} \subset \widetilde{\mathbb{R}}^3$, and where,in general the matrix a in (4.8) belongs to $O_3(\mathbb{R})$ (not to $O_3^+(\mathbb{R})$). Then the change of variables $x = \widetilde{a} \cdot y$, where

$$\begin{aligned} \widetilde{a} &= \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \\ \text{gives}\quad \widetilde{\varphi} &= \{i_0,\ i_1,\ i_2,\ i_3\} = \psi_{st}, \end{aligned} \tag{4.10}$$

and the standard set ψ_{st} corresponds to the standard embedding of $\mathbb{R}^m$ into $\mathbb{H}$: for $m = 3$ $\mathbb{R}^3$ is identified with $\widetilde{\mathbb{R}}^3$ corresponding coordinate axes coinciding; for $m = 4$ just the same thing. Besides inequitable arrangement of coordinates in $\mathbb{H}$ occasioned by the existence of the quaternionic multiplication is taking into account.

Summarazing we thus have by a linear change of variables γ that

$$\begin{array}{rcl} f \in {}^{\psi}\mathfrak{M}(\Omega) & \Longleftrightarrow & f \circ \gamma \in {}^{st}\mathfrak{M}(\gamma^{-1}(\Omega)), \\ f \in \mathfrak{M}^{\psi}(\Omega) & \Longleftrightarrow & f \circ \gamma \in \mathfrak{M}^{st}(\gamma^{-1}(\Omega)). \end{array} \tag{4.11}$$

4.5 Every structural set, including the standard one, defines C.i.t.i. and singular integration operator. In this section we show the connection between these important objects for the standard and for arbitrary structural sets.

Let us start with the case $m = 3$. Let ψ be a structural 3-tuple and let ψ^0 be (as in 4.4) such quaternion that a base $\{\vec{\psi}^0, \ldots, \vec{\psi}^3\}$ is equally oriented with the canonical base in $\mathbb{R}^4_y$. As above $h = \overline{\psi}^0$, $\varphi^k = h \cdot \psi^k$; a is the matrix (4.8), and γ_a is the corresponding linear change of variables.

Direct computation shows that

$$\begin{array}{rcl} {}^{\psi}D_x & = & {}^{\overline{h}}M \cdot W_{\gamma_a}^{-1} \cdot {}^{st}D_y \cdot W_{\gamma_a}, \\ \sigma^{(2)}_{\psi,x} & = & \sigma^{(2)}_{\psi,ay} = \overline{h} \cdot \sigma^{(2)}_{st,y}, \end{array} \tag{4.12}$$

$$\mathcal{K}_{\psi}(ay) = \mathcal{K}_{st}(y) \cdot h,$$

where $y \in \Xi$. Hence we have for C.i.t.i. and for the singular integration operator that

$${}^{\psi}K_{\partial\Omega} = W_{\gamma_a}^{-1} \cdot {}^{st}K_{\partial\Xi} \cdot W_{\gamma_a}, \tag{4.13}$$

$${}^{\psi}S_{\partial\Omega} = W_{\gamma_a}^{-1} \cdot {}^{st}S_{\partial\Xi} \cdot W_{\gamma_a}. \tag{4.14}$$

In the case $m = 4$ the matrix a is from $O_3(\mathbb{R})$ in general. Thus

$${}^{\psi}S_{\partial\Omega} = \operatorname{sgn}(\psi) \cdot W_{\gamma_{\widetilde{a}}}^{-1} \cdot {}^{st}S_{\partial\Xi} \cdot W_{\gamma_{\widetilde{a}}},$$

where $\widetilde{a}$ is the matrix (4.10).

5 Properties of the operator of the singular integration

5.1 The operator ${}^{\psi}S$ introduced in 3.5 plays an exceptionally important role in various questions. That is why we shall study its properties more systematically and thoroughly.

Let us first note that the operator ${}^{\psi}S$ can be extended onto sets of $\mathbb{H}(\mathbb{C})$-valued functions. Moreover in many cases it is more natural to consider it in the complexified spaces $L_p(\Gamma; \mathbb{H}(\mathbb{C})) := L_p(\Gamma, \mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$ and

$$H_{\mu}(\Gamma, \mathbb{H}(\mathbb{C})) := H_{\mu}(\Gamma, \mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}, \ \ p > 1, \ \ 0 < \mu < 1.$$

Let X denotes any of these functional sets where X is considered as a right Banach module. Then $\mathcal{L}(X)$ will denote the set of all (right) linear bounded operators acting in X, and $\mathcal{T}(X)$ is the set of all compact in X operators.

5.2 From the Sokhotski-Plemelj formulas the following property may be easily derived: *If Γ is a smooth hypersurface without boundary in $\mathbb{R}^m$, then ${}^{\psi}S$ belongs to any of the sets $\mathcal{L}(L_p(\Gamma;\mathbb{H})$, $\mathcal{L}(L_p(\Gamma;\mathbb{H}(\mathbb{C}))$, $\mathcal{L}(H_\mu(\Gamma;\mathbb{H}))$, $\mathcal{L}(H_\mu(\Gamma;\mathbb{H}(\mathbb{C}))$ for $p \in (1;+\infty)$, $\mu \in (0;1)$. Morewover ${}^{\psi}S$ is an involution, i.e.*

$$ {}^{\psi}S^2 = I. \tag{5.1}$$

This assertion is also preserved for L_p if

$$\Gamma = \mathbb{R}^{m-1} := \left\{(0, x^{5-m}, \ldots, x^3)\right\} \subset \mathbb{R}^m.$$

5.3 Denote

$$ {}^{\psi}P_\pm := \frac{1}{2}(I \pm {}^{\psi}S). \tag{5.2}$$

In accordance with 5.2

$$ {}^{\psi}P_\pm^2 = {}^{\psi}P_\pm; \ {}^{\psi}P_+ \cdot {}^{\psi}P_- = {}^{\psi}P_- \cdot {}^{\psi}P_+ = 0, \tag{5.3}$$

so that ${}^{\psi}P_\pm$ are mutually orthogonal projections both on L_p and on H_μ. Denote

$$ {}^{\psi}P_\pm(L_p) =: {}^{\psi}L_p^\pm; \ \ {}^{\psi}P_\pm(H_\mu) =: {}^{\psi}H_\mu^\pm. \tag{5.4}$$

From 3.9 it follows that, for instance ${}^{\psi}L_p^\pm(\Gamma;\mathbb{H})$ is the space of traces on Γ of functions from ${}^{\psi}\mathfrak{A}_p(\Omega^\pm)$ (see 3.9). Analogously for H_μ.

Thus

$$\begin{aligned} L_p(\Gamma;\mathbb{H}) &= {}^{\psi}L_p^+(\Gamma;\mathbb{H}) \oplus {}^{\psi}L_p^-(\Gamma,\mathbb{H}), \\ H_\mu(\Gamma;\mathbb{H}) &= {}^{\psi}H_\mu^+(\Gamma;\mathbb{H}) \oplus {}^{\psi}H_\mu^-(\Gamma,\mathbb{H}). \end{aligned} \tag{5.5}$$

5.4 Let $a \in H_\mu(\Gamma;\mathbb{H}(\mathbb{C}))$ and consider the composition

$$M^a \cdot {}^{\psi}S : f \in L_p(\Gamma : \mathbb{H}(\mathbb{C})) \longmapsto {}^{\psi}S[f] \cdot a.$$

We have:

$$M^a \cdot {}^{\psi}S[f](t) = {}^{\psi}S[f](t) \cdot a(t) =$$

$$\begin{aligned} &= 2\int_\Gamma \mathcal{K}_\psi(\tau - t) \cdot \sigma_{\psi,\tau} \cdot f(\tau) \cdot (a(t) - a(\tau)) + 2\int_\Gamma \mathcal{K}_\psi(\tau - t)\sigma_{\psi,\tau} f(\tau) a(\tau) = \\ &=: T[f](t) + {}^{\psi}S[f \cdot a](t) = T[f](t) + ({}^{\psi}S \cdot M^a)[f](t). \end{aligned}$$

The kernel of operator T has a weak singularity, whence $T \in \mathcal{T}(L_p(\Gamma;\mathbb{H}(\mathbb{C})))$. Also it follows from the above that $T \in \mathcal{T}(L_p(\Gamma;\mathbb{H}))$ for $a \in H_\mu(\Gamma;\mathbb{H})$. These results can be easily extended to the case of continuous functions using the density of H_μ in C. Moreover one can show that if $a \in H_\mu$ then $T \in \mathcal{T}(H_\mu)$.

Thus we have shown that *the commutator*

$$[M^a; {}^{\psi}S] := M^a \cdot {}^{\psi}S - {}^{\psi}S \cdot M^a$$

is compact in the considered spaces.

It is clear that the same holds for the commutators of M^a with the projections ${}^{\psi}P_\pm$.

5.5 Consider the algebra $\mathcal{R}$ generated by the operators: ${}^{\psi}S$, M^a and T, where $a \in C(\Gamma; \mathbb{H}(\mathbb{C}))$ in the case L_p and $a \in H_\mu(\Gamma; \mathbb{H}(\mathbb{C}))$ in the case H_μ and $T \in \mathcal{T}$. It follows from 5.4 that an arbitrary operator A from $\mathcal{R}$ is respresentable in the form

$$\begin{aligned} A &= M^a \cdot {}^{\psi}P_+ + M^b \cdot {}^{\psi}P_- + T = \\ &= M^c + M^d \cdot {}^{\psi}S + T. \end{aligned}$$

We emphasize in particular that the right (not left) multiplication is essential, and that for the operator ${}^aM$ the result will not be true. This is the effect of left-ψ-hyperholomorphic functions forming a right but not a left $\mathbb{H}$-module.

5.6 We now consider the important situation when $\Gamma = \mathbb{S}^{m-1}$ or $\Gamma = \mathbb{R}^{m-1}$ and the operators act on $L_2(\Gamma; \mathbb{H}(\mathbb{C}))$. We begin with the hyperplane $\mathbb{R}^{m-1}$, in which case we need the well-known Riesz operators R_k defined by

$$R_k[f](t) := \frac{2}{|\mathbb{S}^{m-1}|} \int_{\mathbb{R}^{m-1}} \frac{\tau_k - t_k}{|\tau - t|^m} f(\tau) d\tau, \tag{5.6}$$

$$k \in \{1, \ldots, m-1\}.$$

Let $m = 4$. Then $\Gamma = \mathbb{R}^3 = \{(0, x^1, x^2, x^3)\} \subset \mathbb{R}^4$. We have

$$\begin{aligned} {}^{\psi}S[f](t) &= \frac{2}{|\mathbb{S}^3|} \int_{\mathbb{R}^3} \frac{1}{|\tau - t|^4} \sum_{k=1}^{3} \overline{\psi}^k \cdot (\tau^k - t^k) f(\tau) d\tau = \\ &= \sum_{k=1}^{3} \overline{\psi}^k \cdot \frac{1}{\pi^2} \int_{\mathbb{R}^3} \frac{\tau^k - t^k}{|\tau - t|^4} f(\tau) d\tau, \end{aligned}$$

hence

$${}^{\psi}S_{\mathbb{R}^3} = \sum_{k=1}^{3} \overline{\psi}^k R_k = -\sum_{k=1}^{3} \psi^k \cdot R_k. \tag{5.7}$$

Now let $m = 3$, $\Gamma = \mathbb{R}^2 = \{(x^1, x^2, 0)\} \subset \mathbb{R}^3$.

Then we have analogously

$$\begin{aligned} {}^{\psi}S[f](t) &= \frac{2}{|\mathbb{S}^2|} \int_{\mathbb{R}^2} \frac{1}{|\tau - t|^3} \sum_{k=1}^{3} \overline{\psi}^k (\tau^k - t^k) \psi^3 f(\tau) d\tau = \\ &= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|\tau - t|^3} \left(\psi^1 \psi^3 (\tau^1 - t^1) + \psi^2 \psi^3 (\tau^2 - t^2) \right) f(\tau) d\tau = \\ &= \psi^2 \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\tau^1 - t^1}{|\tau - t|^3} f(\tau) d\tau - \psi^1 \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\tau^2 - t^2}{|\tau - t|^3} f(\tau) d\tau, \end{aligned}$$

and hence

$${}^{\psi}S_{\mathbb{R}^2} = \psi^2 R_1 - \psi^1 R_2. \tag{5.7'}$$

5.7 Let us now consider the direct and the inverse Fourier transforms of functions from $L_2(\mathbb{R}^{m-1}; \mathbb{H}(\mathbb{C}))$:

$$\begin{aligned} F[f](\xi) &:= (2\pi)^{-(m-1)/2} \int_{\mathbb{R}^{m-1}} e^{-i\langle x, \xi \rangle} \cdot f(x)\, dx, \\ F^{-1}[f](\xi) &:= (2\pi)^{-(m-1)/2} \int_{\mathbb{R}^{m-1}} e^{i\langle x, \xi \rangle} \cdot f(x)\, dx. \end{aligned} \tag{5.8}$$

From the properties of Fourier transform of complex-valued functions it follows that the expressions (5.8) are well–defined and that F and F^{-1} are isometric isomorphisms of L_2. Moreover for the Riesz operators (5.6) we have

$$R_k = F^{-1} i \frac{\xi^k}{|\xi|} F \tag{5.9}$$

or

$$F R_k F^{-1} = i \frac{\xi^k}{|\xi|}, \tag{5.10}$$

where $\xi = (\xi^1, \ldots, \xi^{m-1})$. From this and 5.6 we obtain

$$\begin{array}{rcl} {}^{\psi}S_{\mathbb{R}^3} & = & -iF^{-1} \cdot \sum\limits_{k=1}^{3} \psi^k \cdot \frac{\xi^k}{|\xi|} F, \\ {}^{\psi}S_{\mathbb{R}^2} & = & iF^{-1} \left(\psi^2 \cdot \frac{\xi^1}{|\xi|} - \psi^1 \cdot \frac{\xi^2}{|\xi|} \right) F. \end{array} \tag{5.11}$$

As is well known the function

$$\text{sym } (R_k) : \xi \in \mathbb{S}^{m-2} \longmapsto i\xi^k \in \mathbb{C}$$

is called the symbol of the operator R_k. Therefore we introduce *the symbol of the operator* ${}^{\psi}S_{\mathbb{R}^{m-1}}$ by

$$\begin{array}{rcl} \text{sym } ({}^{\psi}S_{\mathbb{R}^3}) : \xi \in \mathbb{S}^2 & \longmapsto & -i \sum\limits_{k=1}^{3} \psi^k \xi^k, \\ \text{sym } ({}^{\psi}S_{\mathbb{R}^2}) : \xi \in S^1 & \longmapsto & i \left(\psi^2 \cdot \xi^1 - \psi^1 \cdot \xi^2 \right). \end{array} \tag{5.12}$$

5.8 Now let $\Gamma = \mathbb{S}^{m-1}$. Recall that (see Section 3)

$$\begin{array}{rl} {}^{\psi}S[f](t) = & \frac{2}{|\mathbb{S}^{m-1}|} \cdot \int\limits_{\mathbb{S}^{m-1}} \frac{1}{|\tau - t|^m} \sum\limits_{k=4-m}^{3} \overline{\psi}^k \cdot (\tau^k - t^k) \cdot \\ & \cdot \sum\limits_{p=4-m}^{3} (-1)^{p+m} \psi^p d\widehat{\tau} \cdot f(\tau). \end{array} \tag{5.13}$$

For the sphere $\mathbb{S}^{m-1}$ we have that $(-1)^{p+m} d\widehat{\tau}^p = \tau^p \cdot ds_{m-1}$, whence (5.13) takes the form

$$\begin{array}{l} {}^{\psi}S[f](t) = \dfrac{2}{|\mathbb{S}^{m-1}|} \displaystyle\int\limits_{\mathbb{S}^{m-1}} \frac{1}{|\tau - t|^m} \sum_{k=4-m}^{3} \overline{\psi}^k \cdot (\tau^k - t^k) \cdot \\ \qquad \cdot \displaystyle\sum_{p=4-m}^{3} \tau^p \cdot \psi^p f(\tau) ds_{m-1}(\tau). \end{array} \tag{5.13$'$}$$

Transforming (5.13$'$) we obtain that

$$\begin{array}{l} {}^{\psi}S[f](t) = (-1)^m \dfrac{2}{|\mathbb{S}^{m-1}|} \displaystyle\int\limits_{\mathbb{S}^{m-1}} \sum_{k=4-m}^{3} \frac{\tau^k - t^k}{|\tau - t|^m} (-1)^k d\widehat{\tau}^k f(\tau) + \\ \displaystyle\sum_{4-m \le k < p \le 3} \overline{\psi}^k \cdot \psi^p \frac{2 \cdot (-1)^m}{|\mathbb{S}^{m-1}|} \int\limits_{\mathbb{S}^{m-1}} \frac{(\tau^k - t^k)(-1)^p \cdot d\widehat{\tau}^p - (\tau^p - t^p)(-1)^k d\widehat{\tau}^k}{|\tau - t|^m} f(\tau) = \end{array}$$

$$=: (-1)^m I_0 + (-1)^m \sum_{4-m\leq k<m\leq 3} \overline{\psi}^k \psi^p I_{kp}.$$

Notice that the following formula has been used for the computations: if ψ is a structural set, $\{a_k, b_p\} \subset \mathbb{R}$ then

$$\begin{aligned} \sum_{k=4-m}^{3} a_k \cdot \overline{\psi}^k \cdot \sum_{p=4-m}^{3} b_p \psi^p &= \\ = \sum_{k=4-m}^{3} a_k \cdot b_k &+ \sum_{4-m\leq k<p\leq 3} (a_k b_p - a_p b_k)\, \overline{\psi}^k \cdot \psi^p. \end{aligned} \tag{5.14}$$

In order to define the symbol of ${}^{\psi}S$ it will be convinient for us to use the stereographic projection of the sphere $\mathbb{S}^{m-1}$ onto $\mathbb{R}^{m-1}$ and the change of variables operators generated by this projection. Let us recall that that the stereographic map is given by

$$\begin{aligned} w : t = (t^{4-m}, \ldots, t^3) \in \mathbb{S}^{m-1} &\longmapsto x = (x^1, \ldots, x^{m-1}) = \\ &=: \left(\frac{t^{4-m}}{1-t^3}, \ldots, \frac{t^2}{1-t^3} \right) \in \mathbb{R}^{m-1}, \end{aligned}$$

and that thecorresponding change of variables operators are given by

$$\begin{aligned} W[f](x) &:= \left(\frac{2}{|x|^2+1} \right)^{\frac{m-1}{2}} \cdot f(w^{-1}(x)), \\ W^{-1}[g](t) &:= \left(\frac{|w(t)|^2+1}{2} \right)^{\frac{m-1}{2}} \cdot g(w(t)). \end{aligned}$$

The operator W is an isometric isomorphism of $L_2(S^{m-1})$ onto $L_2(\mathbb{R}^{m-1})$. Let us consider the operator $W \cdot {}^{\psi}S \cdot W^{-1}$ acting on $L_2(\mathbb{R}^{m-1}, \mathbb{H}(\mathbb{C}))$. Making the change of variables $\tau = w^{-1}(y)$ in the integral $W \cdot {}^{\psi}S \cdot W^{-1}[f]$, direct computations give:

$$\begin{aligned} |\tau - t|^m &= \frac{2^m \cdot |y-x|^m}{(|y|^2+1)^{\frac{m}{2}} \cdot (|x|^2+1)^{\frac{m}{2}}}; \\ \tau^3 - t^3 &= 2\frac{|y|^2 - |x|^2}{(|x|^2+1)\cdot(|y|^2+1)}; \\ d\widehat{\tau}^3 &= \frac{2^{m-1}(1-|y|^2)}{(1+|y|^2)^m} dy; \end{aligned}$$

while for $k \in \{4-m, \ldots, 2\}$

$$\begin{aligned} \tau^k - t^k &= 2\frac{y^{k+m-3}\cdot(|x|^2+1) - x^{k+m-3}(|y|^2+1)}{(|x|^2+1)\cdot(|y|^2+1)}, \\ d\widehat{\tau}^k &= (-1)^k \frac{2^m \cdot y^{k+m-3} \cdot dy}{(1+|y|^2)^m}. \end{aligned}$$

Let us now compute $W I_0 W^{-1}$. We have

$$W I_0 W^{-1}[f](x) = \frac{2}{|\mathbb{S}^{m-1}|(|x|^2+1)^{\frac{m-1}{2}}} \int_{\mathbb{R}^{m-1}} \frac{(|y|^2+1)^{\frac{m}{2}} \cdot (|x|^2+1)^{\frac{m}{2}}}{2^m \cdot |y-x|^m}.$$

$$\cdot\left(\sum_{k=4-m}^{2} 2^{m+1}\cdot\frac{y^{k+m-3}\cdot(|x|^2+1)-x^{k+m-3}\cdot(|y|^2+1)}{(|x|^2+1)\cdot(|y|^2+1)^{m+1}}\cdot y^{k+m-3}-\right.$$

$$\left.-2^m\cdot\frac{(|y|^2-|x|^2)\cdot(1-|y|^2)}{(|x|^2+1)\cdot(|y|^2+1)^{m+1}}\right)(|y|^2+1)^{\frac{m-1}{2}}f(y)dy=$$

$$=\frac{2}{|\mathbb{S}^{m-1}|\cdot\sqrt{|x|^2+1}}\int\limits_{\mathbb{R}^{m-1}}\frac{f(y)dy}{(|y|^2+1)^{\frac{3}{2}}\cdot|y-x|^m}\cdot$$

$$\cdot\Big(2\cdot|y|^2\cdot(|x|^2+1)-2\langle x,y\rangle_{\mathbb{R}^{m-1}}(|y|^2+1)-(|y|^2-|x|^2)\cdot(1-|y|^2)\Big)=$$

$$=\frac{2}{|\mathbb{S}^{m-1}|\cdot\sqrt{|x|^2+1}}\int\limits_{\mathbb{R}^{m-1}}\frac{f(y)dy}{\sqrt{|y|^2+1}\cdot|y-x|^{m-2}}.$$

Hence, the operator WI_0W^{-1} is compact on $L_2(\mathbb{R}^{m-1})$.

Now we compute $WI_{kp}W^{-1}$ for $p\leq 2$. We have

$$WI_{kp}W^{-1}[f](x)=\frac{4}{|\mathbb{S}^{m-1}|\cdot(|x|^2+1)^{\frac{m-1}{2}}}\cdot$$

$$\cdot\int\limits_{\mathbb{R}^{m-1}}\frac{(|y|^2+1)^{\frac{m}{2}}\cdot(|x|^2+1)^{\frac{m}{2}}}{2^m\cdot|y-x|^m}\cdot\Big(\frac{y^{k+m-3}\cdot(|x|^2+1)-x^{k+m-3}\cdot(|y|^2+1)}{(|x|^2+1)\cdot(|y|^2+1)^{m+1}}\cdot 2^m\cdot y^{p+m-3}-$$

$$-\frac{y^{p+m-3}\cdot(|x|^2+1)-x^{p+m-3}\cdot(|y|^2+1)}{(|x|^2+1)\cdot(|y|^2+1)^{m+1}}\cdot 2^m\cdot y^{k+m-3}\Big)\cdot(|y|^2+1)^{\frac{m-1}{2}}f(y)dy=$$

$$=\frac{4}{|\mathbb{S}^{m-1}\cdot\sqrt{|x|^2+1}}\int\limits_{\mathbb{R}^{m-1}}\frac{f(y)dy}{(|y|^2+1)^{\frac{1}{2}}\cdot|y-x|^m}\cdot\Big(x^{p+m-3}\cdot y^{k+m-3}-x^{k+m-3}\cdot y^{p+m-3}\Big).$$

Thus

$$WI_{kp}W^{-1}=2\left(\frac{x^{p+m-3}}{|x|^2+1}\cdot R_{k+m-3}-\frac{x^{k+m-3}}{|x|^2+1}R_{p+m-3}\right)+T_{kp},$$

where R_j denotes $j-th$ Riesz operator and T_{kp} is a compact operator.

Now let $p=3$. Computing $WI_{k3}W^{-1}$ for $k\in\{4-m,\dots,2\}$, we have

$$W\cdot I_{k3}\cdot W^{-1}[f](x)=\frac{2}{|\mathbb{S}^{m-1}|\cdot(|x|^2+1)^{\frac{m-1}{2}}}\int\limits_{\mathbb{R}^{m-1}}\frac{(|y|^2+1)^{\frac{m}{2}}\cdot(|x|^2+1)^{\frac{m}{2}}}{2^m\cdot|y-x|^m}\cdot$$

$$\cdot\Big(2^m\cdot\frac{y^{k+m-3}\cdot(|x|^2+1)-x^{k+m-3}\cdot(|y|^2+1)}{(|x|^2+1)\cdot(|y|^2+1)}\cdot(-1)\cdot\frac{1-|y|^2}{(1+|y|^2)^m}-$$

$$-2^{1+m}\frac{(|y|^2-|x|^2)\cdot y^{k+m-3}}{(|x|^2+1)\cdot(|y|^2+1)^{m+1}}\Big)\cdot(|y|^2+1)^{\frac{m-1}{2}}\cdot f(y)dy=$$

$$=\frac{-2}{|\mathbb{S}^{m-1}|\cdot\sqrt{|x|^2+1}}\int\limits_{\mathbb{R}^{m-1}}\frac{f(y)dy}{(|y|^2+1)^{\frac{1}{2}}\cdot|y-x|^m}\cdot$$

$$\cdot\Big(\Big(y^{k+m-3}-x^{k+m-3}\Big)+\Big(|y|^2\cdot x^{k+m-3}-|x|^2\cdot y^{k+m-3}\Big)\Big)=$$

$$= \frac{-2}{|\mathbb{S}^{m-1}|\cdot\sqrt{|x|^2+1}} \int\limits_{\mathbb{R}^{m-1}} \frac{f(y)dy}{(|y|^2+1)\cdot|y-x|^m}\cdot$$

$$\cdot\left(\left(y^{k+m-3}-x^{k+m-3}\right)\cdot(1-|x|^2)+\sum_{q=1}^{m-1} x^{k+m-3}\cdot(y^q-x^q)\cdot(y^q+x^q)\right).$$

Thus

$$WI_{k3}W^{-1}=\frac{|x|^2-1}{|x|^2+1}R_{k+m-3}-\sum_{q=1}^{m-1}\frac{2\cdot x^q\cdot x^{k+m-3}}{|x|^2+1}R_q+T_{k3},$$

where T_{k3} is a compact operator.

We obtain finally

$$W\cdot{}^{\psi}S\cdot W^{-1}=(-1)^m\cdot 2\cdot\sum_{4-m\le k<p\le 2}\overline{\psi}^k\cdot\psi^p\cdot\Big(\frac{x^{p+m-3}}{|x|^2+1}R_{k+m-3}-$$

$$-\frac{x^{k+m-3}}{|x|^2+1}R_{p+m-3}\Big)+(-1)^m\sum_{k=4-m}^{2}\overline{\psi}^k\cdot\psi^3\cdot\Big(\frac{|x|^2-1}{|x|^2+1}R_{k+m-3}- \tag{5.15}$$

$$-\sum_{q=1}^{m-1}\frac{2x^q\cdot x^{k+m-3}}{|x|^2+1}\cdot R_q\Big)+T,$$

where T is a compact operator.

Now let us write down the symbol of the operator (5.15). We have

$$\mathrm{sym}\,{}^{\psi}S:=\mathrm{sym}\ \ W\cdot{}^{\psi}S\cdot W^{-1}=$$

$$=(-1)^m\cdot i\cdot\Big(2\sum_{4-m\le k<p\le 2}\overline{\psi}^k\cdot\psi^p\left(\frac{x^{p+m-3}}{|x|^2+1}\cdot\xi_{k+m-3}-\frac{x^{k+m-3}}{|x|^2+1}\cdot\xi_{p+m-3}\right)+ \tag{5.16}$$

$$+\sum_{k=4-m}^{2}\overline{\psi}^k\cdot\psi^3\left(\frac{|x|^2-1}{|x|^2+1}\cdot\xi_{k+m-3}-\sum_{q=1}^{m-1}\frac{2\cdot x^q\cdot x^{k+m-3}}{|x|^2+1}\xi_q\right)\Big),$$

where $(x,\xi)\in\mathbb{R}^{m-1}\times\mathbb{S}^{m-2}$.

We again distinguish between $m=3$ and $m=4$. Firstly let $m=3$. Recall that in this case $\mathrm{sgn}\ \psi=1$, $\psi^1\cdot\psi^2=\psi^3$, $\psi^k\cdot\psi^k=-1$. Then we have from (5.16) that

$$\mathrm{sym}\ {}^{\psi}S=i\sum_{k=1}^{3}\psi^k\cdot\tau_k(x,\xi), \tag{5.17}$$

where

$$\begin{aligned}
\tau_1(x,\xi) &:= -\frac{((x^2)^2-(x^1)^2+1)\xi_2+2x^1\cdot x^2\cdot\xi_1}{|x|^2+1},\\
\tau_2(x,\xi) &:= \frac{((x^1)^2-(x^2)^2+1)\xi_1+2x^1\cdot x^2\cdot\xi_2}{|x|^2+1},\\
\tau_3(x,\xi) &:= 2\frac{x^2\cdot\xi_1-x^1\cdot\xi_2}{|x|^2+1}.
\end{aligned} \tag{5.18}$$

Let us note that for all $\xi \in \mathbb{S}^1$ the vector $\tau(x,\xi) := (\tau_1, \tau_2, \tau_3)$ may be interpreted as a unit tangent vector to the sphere $\mathbb{S}^2$ at the point $t = w^{-1}(x)$.

Now consider the case $m = 4$ and $\operatorname{sgn}\psi = 1$. Then (5.16) gives:

$$\operatorname{sym}{}^{\psi}S = \frac{i}{|x|^2+1}\left(\psi^1 \cdot \left(2x^2\xi_1 - 2x^1\xi_2 - (|x|^2-1)\xi_3 + 2x^3 \cdot \sum_{q=1}^{3} x^q \xi_q\right) + \right.$$

$$+ \quad \psi^2 \cdot \left(2x^3\xi_1 - 2x^1\xi_3 + (|x|^2-1)\xi_2 - 2x^2 \sum_{q=1}^{3} x^q \cdot \xi_q\right) +$$

$$\left. + \quad \psi^3 \cdot \left(2x^3\xi_1 - 2x^3\xi_2 + (|x|^2-1)\xi_1 - 2x^1 \sum_{q=1}^{3} x^1 \cdot \xi_q\right)\right),$$

where $(x,\xi) \in \mathbb{R}^3 \times \mathbb{S}^2$. Returning from the variable $x \in \mathbb{R}^3$ to the initial variable $t \in \mathbb{S}^3$ we obtain

$$\begin{aligned} \operatorname{sym}{}^{\psi}S \quad &= \quad i\left(\psi^1 \cdot \left(t_1\xi_1 - t_0\xi_2 - t_3\xi_3 + \frac{t_2}{1-t_3}\sum_{q=0}^{2} t_q \cdot \xi_{q+1}\right) + \right. \\ &+ \quad \psi^2 \cdot \left(t_2\xi_1 - t_0\xi_3 + t_3\xi_2 - \frac{t_1}{1-t_3}\sum_{q=0}^{2} t_q \cdot \xi_{q+1}\right) + \\ &\left. + \quad \psi^3 \cdot \left(t_1\xi_3 - t_2\xi_2 + t_3\xi_1 - \frac{t_0}{1-t_3}\sum_{q=0}^{2} t_q \cdot \xi_{q+1}\right)\right), \end{aligned} \tag{5.19}$$

where $\xi \in \mathbb{S}^2$ as before and where t already belongs to $\mathbb{S}^2$.

Now let $m = 4$ and let $\operatorname{sgn}\psi = -1$. Then we obtain from (5.16):

$$\begin{aligned} \operatorname{sym}{}^{\psi}S \quad &= \quad \frac{i}{|x|^2+1}\left(\psi^1 \cdot \left(2x^2\xi_1 - 2x^1\xi_2 + (|x|^2-1)\xi_3 - 2x^3\sum_{q=1}^{3} x^q\xi_q\right) + \right. \\ &+ \quad \psi^2 \cdot \left(2x^3\xi_1 - 2x^1\xi_3 - (|x|^2-1)\xi_2 + 2x^2\sum_{q=1}^{3} x^q\xi_q\right) + \\ &\left. + \quad \psi^3 \cdot \left(2x^3\xi_2 - 2x^2\xi_3 - \left(|x|^2-1\right)\xi_1 - 2x^1\sum_{q=1}^{3} x^q\xi_q\right)\right), \end{aligned}$$

where $(x,\xi) \in \mathbb{R}^3 \times \mathbb{S}^2$. Returning again from the variable $x \in \mathbb{R}^3$ to the initial variable $t \in \mathbb{S}^3$ we obtain

$$\begin{aligned} \operatorname{sym}{}^{\psi}S \quad &= \quad i\left(\psi^1 \left(t_1\xi_1 - t_0\xi_2 + t_3\xi_3 - \frac{t_2}{1-t_3}\sum_{0}^{2} t_q\xi_{q+1}\right) + \right. \\ &+ \quad \psi^2 \left(t_2\xi_1 - t_0\xi_3 - t_3\xi_2 + \frac{t_1}{1-t_3}\sum_{0}^{2} t_q\xi_{q+1}\right) + \\ &+ \quad \left(\psi^3 \left(t_2\xi_2 - t_1\xi_3 + t_3\xi_1 - \frac{t_0}{1-t_3}\sum_{0}^{2} t_1\xi_{q+1}\right)\right), \end{aligned} \tag{5.20}$$

where again $\xi \in \mathbb{S}^2$ and $t \in \mathbb{S}^2$ already.

It will be convenient for further purposes to introduce the following notations. Let $\psi = (i_0, \psi^1, \psi^2, \psi^3)$ be an arbitrary structural set (i.e. sgn ψ may be equal both $+1$ and -1), and let $t = (t_0,\, t_1,\, t_2,\, t_3) \in \mathbb{S}^3$. Define for each $t \in \mathbb{S}^3$ and $\xi \in \mathbb{S}^2$ the vector $\eta(t,\xi) = (\eta_0,\, \eta_1,\, \eta_2,\, \eta_3) \in \mathbb{R}^4$ by

$$\begin{aligned}
\eta_0(t,\xi) &:= \xi_1 - \frac{t_0}{1-t_3} \cdot \eta_3(t,\xi),\\
\eta_1(t,\xi) &:= \xi_2 - \frac{t_1}{1-t_3} \cdot \eta_3(t,\xi),\\
\eta_2(t,\xi) &:= \xi_3 - \frac{t_2}{1-t_3} \cdot \eta_3(t,\xi),\\
\eta_3(t,\xi) &:= t_0\xi_1 + t_1\xi_2 + t_2\xi_3.
\end{aligned}$$

Direct computations show that for every fixed $t \in \mathbb{S}^3$ and $\xi \in \mathbb{S}^2$ the vector $\eta(t,\xi)$ is orthogonal to the vector t and $|\eta| = 1$. Moreover for any fixed $t \in \mathbb{S}^3$ the point ξ is an arbitrary point of $\mathbb{S}^2$, consequently the set $\{\eta(t,\xi)\}$ coincides with the set of all unit tangent vectors to the sphere $\mathbb{S}^3$ at the point t. For $t \in \mathbb{S}^3$ and $h(t,\xi) \in \mathbb{R}^4$ we introduce

$$\begin{aligned}
t^\psi &:= \sum_{k=0}^{3} \psi^k \cdot t_k,\\
\eta^\psi &:= \sum_{k=0}^{3} \psi^k \cdot \eta_k.
\end{aligned}$$

Then as it can be easily seen the formulas (5.19)–(5.20) may be written in the form

$$\text{sym }\ {}^\psi S = -i \cdot \overline{t}^\psi \cdot \eta^\psi(t,\xi). \tag{5.21}$$

5.9 We finish this section with the proof of the following property: the operators ${}^\psi P_\pm$ (introduced in 5.3) are orthoprojections on $L_2(\Gamma;\mathbb{H})$, i.e. ${}^\psi P_\pm^* = {}^\psi P_\pm$, where $\Gamma = \mathbb{R}^{m-1}$ or $\Gamma = \mathbb{S}^{m-1}$, and $L_2(\Gamma;\mathbb{H})$ is understood as a right $\mathbb{H}$-module. For the definition of a self-adjoint operator we refer to 3.11–3.12.

It is sufficient to show that ${}^\psi S^* = {}^\psi S$. For the sake of simplicity we restrict ourselves to the case $\Gamma = \mathbb{R}^3$. We have:

$$\begin{aligned}
\langle {}^\psi S[f], g\rangle &= \int_{\mathbb{R}^3} \overline{[f](t)} \cdot g(t)dt =\\
&= \sum_{k=1}^{3} \int_{\mathbb{R}^3} \overline{\psi^k \cdot R_k[f](t)} \cdot g(t)dt =\\
&= \sum_{k=1}^{3} \int_{\mathbb{R}^3} R_k\overline{[f]}(t) \cdot \psi^k \cdot g(t)dt =\\
&= \sum_{k=1}^{3} \int_{\mathbb{R}^3_t} \left(\int_{\mathbb{R}^3_\tau} \frac{\tau_k - t_k}{|\tau - t|^4} \overline{f(\tau)} d\tau \right) \psi^k g(t)dt =\\
&= \sum_{k=1}^{3} \int_{\mathbb{R}^3_\tau} \overline{f(\tau)} \psi^k d\tau \int_{\mathbb{R}^3_\tau} \frac{\tau_k - t_k}{|\tau - t|^4} g(t)dt =
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{3} \int_{\mathbb{R}^3} \overline{f(\tau)} \cdot \overline{\psi^k} \cdot R_k[g](\tau) d\tau = \\
&= \langle f, \, {}^{\psi}S[g] \rangle,
\end{aligned}
$$

whence ${}^{\psi}S^* =^{\psi} S$.

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