Complex Variables, 1995
Vol. 27, pp. 67–96

Quaternionic ψ-hyperholomorphic functions, singular integral operators and boundary value problems. II. Algebras of singular integral operators and Riemann type boundary value problems

M.V. Shapiro

Departmento de Mathemáticas, ESFM del I.P.N., Mexico City, Mexico

N.L. Vasilevski

Departmento de Mathemáticas, CINVESTAV del I.P.N., Mexico City, Mexico

We study some algebras generated by the singular integral operator with the quaternionic Cauchy kernel and the multiplication operators by continuous or piece-wise continuous functions. This allows us to describe the Fredholm theory of hyperholomorphic Riemann boundary value problems whose formulations take into account both the non-commutativity of the quaternionic multiplication and the existence of various classes of hyperholomorphic functions. Certain classes of hyperholomorphic Toeplitz operators are also studied.

This paper is a continuation of our work [SV]. We therefore will use systematically the notations and results from [SV].

Some algebras generated by singular integral operators with the quaternionic Cauchy kernel are studied. These results are applied to the study of several analogues of the Riemann boundary value problem (or the linear conjugation problem) in the theory of functions of (one) complex variable. Settings of these analogues take into consideration both non-commutativity of the quaternionic multiplication and the existence of the various types of hyperholomorphy. It appears that the Fredholm property conditions of such quaternionic Riemann problems depend on both the number of variables and on the choice of pairs of hyperholomorphic function classes.

Hyperholomorphic Toeplitz operators are also introduced and the relationship between these operators and their pre-symbols are investigated.

1 On a class of C^*-algebras

1.1 In the next sections various boundary value problems for hyperholomorphic functions and Töplitz operators will be studied. This study is based on the connection with the singular integral operators (s.i.o.) theory. Therefore different algebras of s.i.o. related to hyperholomorphic functions will be described. These descriptions are based on one general result expounding in this paragraph.

1.2 Let H be a complex Hilbert space, $\mathcal{L}(H)$ and $\mathcal{T}(H)$ be the sets of all linear bounded, respectively, compact operators on H. Let U be a compact topological space and let μ be a faithful representation of the algebra $C(U) \otimes \mathbb{C}^{l\times l} = C(U, \mathbb{C}^{l\times l})$ in H, i.e. the map

$$\mu : C(U, \mathbb{C}^{l\times l}) \longrightarrow \mathcal{L}(H)$$

is an isometric *-isomorphism of C^*-algebras. Denote by $\mathcal{F}$ the image of the algebra $C(U, \mathbb{C}^{l\times l})$ in $\mathcal{L}(H)$. Furthermore let two operators S_1 and S_2 from $\mathcal{L}(H)$ be given with the properties:

1) $S_k^2 - I \in \mathcal{T}(H),\ S_k^* - S_k \in \mathcal{T}(H),\ k \in \{1;2\}$

2) for any $a \in C(U, \mathbb{C}^{l\times l})$ the commutator $[\mu(a), S_k] := \mu(a)S_k - S_k \cdot \mu(a)$ is compact, or equivalently

2′) for any $F \in \mathcal{F} : [F, S_k] \in \mathcal{T}(H)$.

Our aim is to investigate the C^*-algebra $\mathcal{R} = \mathcal{R}(\mathcal{F}; S_1, S_2; \mathcal{T})$ generated by all acting on H operators of the form

$$N = F_0 + F_1 \cdot S_1 + F_2 \cdot S_2 + T,$$

where

$$F_k \in \mathcal{F},\ T \in \mathcal{T}(H).$$

Let

$$\pi : \mathcal{R} \longrightarrow \widehat{\mathcal{R}} := \mathcal{R}/\mathcal{T}(H)$$

be the natural projection. To describe the symbol algebra Sym $\mathcal{R} := \widehat{\mathcal{R}}$ of the algebra $\mathcal{R}$ we use the local principle [Dgl],[Sim]. As one can easily see the algebra

$$Z\ :=\ \pi \cdot \mu\left(\{a \otimes E_l | a \in C(U)\}\right),$$

isomorphic to the algebra $C(U)$, is a central commutative subalgebra of the algebra Sym $\mathcal{R}$. Therefore we will localize with respect to the points t_0 of the compact U—the compact of maximal ideals of algebra Z. For any $t_0 \in U$ let $J(t_0)$ denotes the maximal ideal of Z corresponding to the point t_0 and let $\widehat{J}(t_0) := J(t_0)\widehat{\mathcal{R}}$ denotes the closed two-sided ideal of the algebra $\widehat{\mathcal{R}}$ generated by the ideal $J(t_0)$. Then the algebra $\widehat{\mathcal{R}}(t_0) := \widehat{\mathcal{R}}/\widehat{J}(t_0)$ is a local algebra corresponding to the point $t_0 \in U$. Let $\nu_{t_0} : \widehat{\mathcal{R}} \longrightarrow \widehat{\mathcal{R}}(t_0)$ be the natural projection. Then the mapping

$$\pi_{t_0} : \mathcal{R} \xrightarrow{\pi} \widehat{\mathcal{R}} \xrightarrow{\nu_{t_0}} \widehat{\mathcal{R}}(t_0)$$

identifies locally equivalent in the point t_0 [Sim] elements of the algebra $\mathcal{R}$.

1.3 Let $t_0 \in U$ and consider the local algebra $\widehat{\mathcal{R}}(t_0)$. If $a \in C(U, \mathbb{C}^{l\times l})$ then obviously there is a local equivalence in the point t_0 giveh by

$$F := \mu(a) \sim F(t_0) := \mu(a(t_0)),$$

and moreover

$$\mathcal{F}(t_0) := \{F(t_0) | a \in C(U, \mathbb{C}^{l\times l})\} \cong \mathbb{C}^{l\times l}.$$

Put: $s_1(t_0) := \pi_{t_0}(S_1),\ s_2(t_0) := \pi_{t_0}(S_2)$ and let

$$p_k(t_0) := \frac{1}{2}(e + s_k(t_0)),\ k \in \{1;2\}.$$

We have $p_k^2(t_0) = p_k^*(t_0) = p_k(t_0)$, whence we obtain the isomorphism

$$\begin{aligned}\widehat{\mathcal{R}}(t_0)\ &\cong\ \mathcal{R}(\mathcal{F}(t_0);\, p_1(t_0);\, p_2(t_0)) \cong \\ &\cong\ \mathcal{R}(\mathbb{C}; p_1(t_0),\, p_2(t_0)) \otimes \mathbb{C}^{l\times l},\end{aligned}$$

where $\mathcal{R}(\mathbb{C};\, p_1(t_0),\, p_2(t_0))$ is the algebra generated by the two orthogonal projections (idempotents) $p_1(t_0)$ and $p_2(t_0)$.

Now we adapt the results from [VaSp] containing a description of the algebra generated by two orthogonal projections p_1 and p_2.

Let $\Delta := sp(p_1 - p_2)^2 \subset [0;1]$,let $X = \Delta \backslash \{0;1\}$ and let Y be a subset of the set $\{(i,j)|\ i,j \in \{0;1\}\}$ defined by the conditions

1. $(0,j) \in Y$ if $j \in \Delta$ and $(-1)^j \in sp(p_1(2p_2 - e)p_1)$
2. $(1,j) \in Y$ if $j \in \Delta$ and $(-1)^{j+1} \in sp((e-p_1)(2p_2 - e)(e-p_1))$

where e is the identity.

Note that if $j \in \overline{X}$ then the points $(0;j)$ and $(1;j)$ belong to Y.

Let $\mathfrak{A}$ denotes the set of all pairs $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 \in C(Y)$, $\sigma_2 \in C(\overline{X}; \mathbb{C}^{2\times 2})$, such that the following condition holds: if $j \in \overline{X}$ $(j \in \{0;1\})$ then

$$\lim_{X \ni x \to j} \sigma_2(x) = \begin{pmatrix} \sigma_1(0;j) & 0 \\ 0 & \sigma_1(1;j) \end{pmatrix}.$$

With respect to component-wise operations and to the norm

$$\|\sigma\| := \max\{\max\{|\sigma_1(i,j)||\ (i,j) \in Y\};\ \max\{\|\sigma_2(x)\||\ x \in \overline{X}\}$$

the set $\mathfrak{A}$ is a C^*-algebra. We then have

Theorem 1.1 [VaSp] *The algebra $\mathcal{R}(\mathbb{C}; p_1, p_2)$ is isometrically isomorphic to the algebra $\mathfrak{A}$ and the isomorphism*

$$\nu : \mathcal{R}(\mathbb{C}; p_1, p_2) \longrightarrow \mathfrak{A}$$

is generated by the next mapping of generators of the algebra $\mathcal{R}(\mathbb{C}; p_1, p_2)$:

$$\begin{aligned} \nu(p_1) &= \begin{cases} 1-j, & (i,j) \in Y \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & x \in \overline{X} \end{cases}, \\ \nu(p_2) &= \begin{cases} 1 - |i-j|, & (i,j) \in Y \\ \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix}, & x \in \overline{X}. \end{cases} \end{aligned}$$

Corollary 1.2 *The local algebra $\widehat{\mathcal{R}}(t_0)$ is isometrically isomorphic to the algebra $\mathfrak{A}(t_0) \otimes \mathbb{C}^{l\times l}$. Moreover the homomorphism*

$$\pi_{t_0} : \mathcal{R} \longrightarrow \widehat{\mathcal{R}}(t_0) \cong \mathfrak{A}(t_0) \otimes \mathbb{C}^{l\times l}$$

is generated by the next mapping of generators of the algebra $\mathcal{R}$:

$$\begin{aligned} \pi_{t_0}(\mu(a)) &= \begin{cases} a(t_0) & , \ (i,j) \in Y(t_0), \\ a(t_0) \otimes E_2 & , \ x \in \overline{X}(t_0), \end{cases} \\ \pi_{t_0}(S_1) &= \begin{cases} (1-2i)E_l & , \ (i,j) \in Y(t_0), \\ E_l \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & , \ x \in \overline{X}(t_0), \end{cases} \\ \pi_{t_0}(S_2) &= \begin{cases} (1-2|i-j|)E_2 & , \ (i,j) \in Y(t_0), \\ E_l \otimes \begin{pmatrix} 1-2x; & 2\sqrt{x(1-x)} \\ 2\sqrt{x(1-x)}; & 2x-1 \end{pmatrix} & , \ x \in \overline{X}(t_0). \end{cases} \end{aligned}$$

Hereby E_l is the $l \times l$ unit matrix.

1.4 Introduce

$$\begin{aligned} X &:= \{(t,x)|t \in U,\ x \in X(t)\}, \\ Y &:= \{(t,y)|t \in U,\ y \in Y(t)\}, \end{aligned}$$

and let $\lambda : \overline{X} \cap (U \times \{0;1\}) \longrightarrow Y^2$ be the map defined by

$$\lambda(t,j) = ((t,(0;j));\ (t,(1;j)),\ j \in \{0;1\}.$$

Denote by $\mathfrak{G}$ the set of all pairs $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 \in C(Y; \mathbb{C}^{l \times l})$, $\sigma_2 \in C(\overline{X}; \mathbb{C}^{2l \times 2l})$, such that the following is fulfilled: if $\lambda(x_0) = (y_1, y_2)$ then

$$\lim_{X \ni x \to x_0} \sigma_2(x) = \begin{pmatrix} \sigma_1(y_1) & 0 \\ 0 & \sigma_1(y_2) \end{pmatrix}.$$

With respect to component-wise operations and to the norm

$$\|\sigma\| := \max\left\{\sup\{\|\sigma_1(y)\| \mid y \in Y\};\ \sup\{\|\sigma_2(x)\| \mid x \in \overline{X}\}\right\}$$

the set $\mathfrak{G}$ is a C^*-algebra.

Together with results of [Vsl] and [DH] this leads to the following theorem.

Theorem 1.3 *The symbol algebra Sym $\mathcal{R}$ of the algebra $\mathcal{R} = \mathcal{R}(\mathcal{F}; S_1, S_2; \mathcal{T})$ is isometrically isomorphic to the algebra $\mathfrak{G}$. Moreover the homomorphism*

$$\mathrm{sym} : \mathcal{R} \longrightarrow \mathrm{Sym}\ \ \mathcal{R}$$

is generated by the following mapping of the generators of $\mathcal{R}$:

$$\begin{aligned} \mathrm{sym}\ \ (\mu(a)) &= \begin{cases} a(t) & , \quad (t,y) \in Y, \\ a(t) \otimes E_2 & , \quad (t,x) \in \overline{X}, \end{cases} \\ \mathrm{sym}\ \ S_1 &= \begin{cases} (1-2i)E_l & , \quad (t,(i,j)) \in Y, \\ E_l \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & , \quad (t,x) \in \overline{X}, \end{cases} \\ \mathrm{sym}\ \ S_2 &= \begin{cases} (1-2|i-j|)E_l & ; \quad (t,(i,j)) \in Y, \\ E_l \otimes \begin{pmatrix} 1-2x & ; & 2\sqrt{x(1-x)} \\ 2\sqrt{x(1-x)} & ; & 2x-1 \end{pmatrix} & ; \quad (t,x) \in \overline{X}. \end{cases} \end{aligned}$$

Corollary 1.4 *An arbitrary operator A from $\mathcal{R}$ is Fredholmian if and only if its symbol* sym A *is invertible, that is*

$$\begin{aligned} \det \mathrm{sym}\ A(y) &\neq 0 \ \ on\ \ Y, \\ \det \mathrm{sym}\ A(x) &\neq 0 \ \ on\ \ \overline{X}. \end{aligned}$$

Remark. The set $\mathfrak{M} := \overline{X} \cup Y$ coincides with the spectrum of the C^*-algebra Sym $\mathcal{R}$ [Dx].

1.5 Let H_0 be a real Hilbert space and $H := H_0 \otimes_{\mathbb{R}} \mathbb{C}$ is its complexification. Further let $H_{\mathbb{R}}$ be the space H with multiplication by real constants only. Then of course $H_{\mathbb{R}} \cong H_0 \oplus H_0$. Assume furthermore thet for the operator A from $\mathcal{L}(H)$ the (real) space H_0 is invariant and associate with (complex linear) operator $A : H \to H$ the two real-linear operators

$$\begin{aligned} A_0 &= A|\ H_0 : H_0 \longrightarrow H_0, \\ A_{\mathbb{R}} &= A : H_{\mathbb{R}} \longrightarrow H_{\mathbb{R}}. \end{aligned}$$

Then obviously

Lemma 1.5 *The operators A, A_0, $A_{\mathbb{R}}$ are simutaneously Fredholmian in their corresponding space and if they are Fredholmian then*

$$\mathrm{Ind}\ A_0 = \mathrm{Ind}\ A = \frac{1}{2}\ \mathrm{Ind}\ A_{\mathbb{R}}.$$

1.6 Let Ω be a domain in $\mathbb{R}^m$ with Liapunov boundary $\Gamma = \partial\Omega$. Consider the real linear space $H_0 = L_2(\Gamma; \mathbb{H}) \cong L_2^4(\Gamma; \mathbb{R})$ and its complexification

$$H = L_2(\Gamma; \mathbb{H}(\mathbb{C})) \cong L_2^4(\Gamma; \mathbb{C}).$$

Furthermore introduce the algebra $C(\Gamma; \mathbb{C}^{2\times 2})$ and its representation

$$\mu : C(\Gamma; \mathbb{C}^{2\times 2}) \longrightarrow \mathcal{L}(L_2(\Gamma; \mathbb{H}(\mathbb{C}))),$$

defined by

$$\mu : a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \longmapsto F = M^f,$$

where

$$\begin{aligned} f &= \frac{1}{2}\Big((a_1 + a_4) - i(a_1 - a_4)i_1 + (a_2 - a_3)i_2 + \\ &\quad +i(a_2 + a_3)i_3\Big); \end{aligned}$$

and i_1, i_2, i_3 are the imaginary units in $\mathbb{H}$ and i is the imaginary unit in $\mathbb{C}$.

We have that $f \in C(\Gamma; \mathbb{H}(\mathbb{C}))$. Note also that $f \in C(\Gamma; \mathbb{H})$ if and only if for $a = \mu^{-1}(f)$

$$a_4 = \overline{a}_1; \quad a_3 = -\overline{a}_2,$$

where $\overline{a}_k$ stands for the complex-conjugate to a_k.

In the sequel we take

$$\mathcal{F} := \mu(C(\Gamma; \mathbb{C}^{2\times 2})) = C(\Gamma; \mathbb{H}(\mathbb{C})).$$

Introduce $\mathcal{F}_0 := C(\Gamma; \mathbb{H}) \subset \mathcal{F}$ and suppose that in the space $H = L_2(\Gamma; \mathbb{H}(\mathbb{C}))$ there are given two linear bounded operators S_1 and S_2 subordinate to the requirements:

1. $S_k^2 - I \in \mathcal{T}$, $\quad S_k^* - S_k \in \mathcal{T}$,
2. for any $F \in \mathcal{F}$ $\ [F, S_k] \in \mathcal{T}$,
3. the real linear space $H_0 = L_2(\Gamma; \mathbb{H})$ is invariant for S_1 and S_2.

Now we are ready to give the "quaternionic" formulation of Theorem 1.3 taking into account the specific character of the algebra $\mathcal{F} := \mu(C(\Gamma; \mathbb{C}^{2\times 2}))$.

As in 1.4 put

$$\begin{aligned} X &= \{(t,x) |\ t \in \Gamma,\ \ x \in X(t)\}\,, \\ Y &= \{(t,y) |\ t \in \Gamma,\ \ y \in Y(t)\}\,. \end{aligned}$$

A mapping $\nu : \overline{X} \cap (\Gamma \times \{0;1\}) \to Y^2$ is given by

$$\mu(t,j) = ((t,(0,j)),\ \ (t,(1,j))),\ \ j \in \{0;1\}.$$

Denote by $\mathfrak{G}_0$ the set of all pairs $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 \in C(Y; \mathbb{H})$, $\sigma_2 \in C(\overline{X}, \mathbb{H}) \otimes \mathbb{R}^{2\times 2}$ and such that the following condition is fulfilled: if $\nu(x_0) = (y_1, y_2)$ then

$$\lim_{X \ni x \to x_0} \sigma_2(x) = \begin{pmatrix} \sigma_1(y_1) & 0 \\ 0 & \sigma_1(y_2) \end{pmatrix}.$$

Introduce $\mathfrak{G} := \mathfrak{G}_0 \otimes_{\mathbb{R}} \mathbb{C}$. With respect to component-wise operations and to the norm

$$\|\sigma\| := \max\{\sup\{\|\sigma_1(\cdot)\|\, |Y\};\ \sup\{\|\sigma_2(\cdot)\|\, |\overline{X}\}\}$$

$\mathfrak{G}$ is a C^*-algebra.

Theorem 1.6 *The symbol algebra Sym $\mathcal{R}$ of the algebra $\mathcal{R} = \mathcal{R}(\mathcal{F};\ S_1,\ S_2;\ \mathcal{T})$ is isometrically isomorphic to $\mathfrak{G}$. Moreover the homomorphism*

$$\mathrm{sym} : \mathcal{R} \longrightarrow \mathrm{Sym}\ \mathcal{R}$$

is generated by the following mapping of generators of the algebra R:

$$\begin{aligned} \mathrm{sym}\ M^f &= \begin{cases} f(t) & , \quad (t;y) \in Y, \\ f(t)E_2 & , \quad (t;x) \in \overline{X}, \end{cases} \\ \mathrm{sym}\ S_1 &= \begin{cases} 1-2i & , \quad (t,(i,j)) \in Y, \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & , \quad (t,x) \in \overline{X}, \end{cases} \\ \mathrm{sym}\ S_2 &= \begin{cases} 1-2|i-j| & , \quad (t,(i,j) \in Y, \\ \begin{pmatrix} 1-2x & 2\sqrt{x(1-x)} \\ 2\sqrt{x(1-x)} & 2x-1 \end{pmatrix} & , \quad (t,x) \in \overline{X}. \end{cases} \end{aligned}$$

Corollary 1.7 *An arbitrary operator $A \in \mathcal{R}$ is Fredholmian if and only if its symbol* sym A *is invertible in the algebra $\mathfrak{G}$.*

Now consider the algebra $\mathcal{R}_0 := \mathcal{R}(\mathcal{F}_0;\ S_1,\ S_2;\ \mathcal{F})$. Any operator from this algebra acts both in the space $H = L_2(\Gamma; \mathbb{H}(\mathbb{C}))$ and in the space $H_0 = L_2(\Gamma; \mathbb{H})$. For an operator $A \in \mathcal{R}_0 \subset \mathcal{R}$ its symbol sym A is an element of an algebra $\mathfrak{G}_0$. Taking into account Lemma 1.5 we have

Corollary 1.8 *The symbol algebra* sym $\mathcal{R}_0$ *of the algebra $\mathcal{R}_0$ coincides with algebra $\mathfrak{G}_0$. Any operator $A \in \mathcal{R}_0$ is Fredholmian in $L_2(\Gamma; \mathbb{H})$ if and only if its symbol is invertible (in the algebra $\mathfrak{G}_0$).*

2 Algebra generated by operators of singular integration corresponding to different classes of hyperholomorphy

2.1 In this Section the abstract results of Section 1 are applied to the case when S_1 and S_2 are the singular integration operators considered in [SV].

We will consider $m = 3$ and $m = 4$ at the same time. Let φ and ψ be two structural sets and let ${}^{\varphi}S_\Gamma$ and ${}^{\psi}S_\Gamma$ denote the corresponding operators of singular integration. They act on the spaces $L_2(\Gamma; \mathbb{H}(\mathbb{C}))$ and $L_2(\Gamma; \mathbb{H})$ and the conditions 1), 2), 3) from 1.6 are fulfilled evidently. Furthermore let $\mathcal{F} = C(\Gamma, \mathbb{H}(\mathbb{C}))$, $\mathcal{F}_0 = C(\Gamma, \mathbb{H}) \subset \mathcal{F}$.

We will study the algebra $\mathcal{R} = \mathcal{R}(\mathcal{F};\ {}^{\varphi}S_\Gamma;\ {}^{\psi}S_\Gamma;\ \mathcal{T})$, generated by operators of the form

$$A = F_0 + F_1 \cdot {}^{\varphi}F_\Gamma + F_2 \cdot {}^{\psi}S_\Gamma + T, \tag{2.1}$$

where $F_k = M^{f_k}$ for $f_k \in \mathcal{F}$, $T \in \mathcal{T}$, as well as the algebra $\mathcal{R}_0 = \mathcal{R}(\mathcal{F}_0;\ {}^{\varphi}S_\Gamma,\ {}^{\psi}S_\Gamma;\ \mathcal{T})$ generated by operators of the form (2.1) with $f_k \in \mathcal{F}_0$ (acting on $L_2(\Gamma; \mathbb{H})$ now).

2.2 As in the previous Section, let $p_1(t)$ and $p_2(t)$ be the images of the operators $P_1 = \frac{1}{2}(I +^{\varphi} S_\Gamma)$ and $P_2 = \frac{1}{2}(I + {}^{\psi}S_\Gamma)$ under the homomorphism

$$\pi_t : \mathcal{R} \longrightarrow \widehat{\mathcal{R}}(t), \quad t \in \Gamma.$$

Describe the set $\Delta(t) := \Delta^{\varphi,\psi}_{m,\Gamma}(t) = sp(p_1(t) - p_2(t))^2$. It follows from Theorem 1.1 that

$$\Delta(t) = \{\lambda^2 |\ (\lambda I - (p_1(t) - p_2(t)) \quad \text{is invertible}\}.$$

From [SV, Section 5]

$$\lambda I - (p_1(t) - p_2(t)) =: z = z_1 - z_2 \cdot i \in \mathbb{H}(\mathbb{C}),$$

where $z_1 = \lambda \in \mathbb{R}$,

$$z_2 = \frac{1}{2}\left(\frac{1}{i}(\text{sym } {}^{\psi}S)(t,\xi) - \frac{1}{i}(\text{sym } {}^{\varphi}S)(t,\xi)\right) \in \widetilde{\mathbb{R}^3}.$$

We are interested in those λ for which $z \notin G\mathbb{H}(\mathbb{C})$, that is $z \cdot \overline{z} = 0$, which in its turn is equivalent to

$$|z_1|^2 - |z_2|^2 + 2i \cdot \langle z_1, z_2 \rangle_{\mathbb{R}^4} = 0.$$

But $\langle z_1, z_2 \rangle_{\mathbb{R}^4} = 0$ therefore

$$\Delta(t) = Im_\xi |z_2|^2 = Im_\xi \left(\frac{1}{4}|{}^{\psi}\mu(\xi) - {}^{\varphi}\mu(\xi)|^2\right),$$

where

$${}^{\psi}\mu(\xi) := \frac{1}{i}(\text{sym } {}^{\psi}S)(t,\xi),\ {}^{\varphi}\mu(\xi) := \frac{1}{i}(\text{sym } {}^{\varphi}S)(t,\xi)$$

and $Im_\xi |z_2|^2$ denotes the image of the mapping $\xi \mapsto |z_2|^2$.

Note that for all $\xi \in \mathbb{S}^{m-2}$ we have $|{}^{\psi}\mu(\xi)| = |{}^{\varphi}\mu(\xi)| = 1$ so that

$$\begin{aligned} |{}^{\psi}\mu(\xi) - {}^{\varphi}\mu(\xi)|^2 &= \langle {}^{\psi}\mu(\xi) - {}^{\varphi}\mu(\xi),\ {}^{\psi}\mu(\xi) - {}^{\varphi}\mu(\xi)\rangle_{\mathbb{R}^3} = \\ &= \langle {}^{\psi}\mu(\xi),\ {}^{\psi}\mu(\xi)\rangle + \langle {}^{\varphi}\mu(\xi),\ {}^{\varphi}\mu(\xi)\rangle - 2\langle {}^{\varphi}\mu(\xi),\ {}^{\psi}\mu(\xi)\rangle = \\ &= 2(1 - \langle {}^{\psi}\mu(\xi),\ {}^{\varphi}\mu(\xi)\rangle). \end{aligned}$$

From [SV, Section 5] it follows that the vectors ${}^{\psi}\mu(\xi)$ and ${}^{\varphi}\mu(\xi)$ have the following quaternionic form:

$$\begin{aligned}
\text{for } m=3 \text{ and } \Gamma=\mathbb{R}^2,\quad {}^{\psi}\mu(\xi) &= \psi^2\cdot\xi_1-\psi^1\cdot\xi_2,\\
\text{for } m=4 \text{ and } \Gamma=\mathbb{R}^3,\quad {}^{\psi}\mu(\xi) &= -\sum_{k=1}^{3}\psi^k\cdot\xi_k,\\
\text{for } m=3 \text{ and } \Gamma=\mathbb{S}^2,\quad {}^{\psi}\mu(\xi) &= \sum_{k=1}^{3}\psi^k\cdot\tau_k(t,\xi),\\
\text{for } m=4 \text{ and } \Gamma=\mathbb{S}^3,\quad {}^{\psi}\mu(\xi) &= \overline{t}^{\psi}\cdot h^{\psi}(t,\xi).
\end{aligned}$$

Thus

$$|z_2(\xi)|^2=\frac{1}{4}|{}^{\psi}\mu(\xi)-{}^{\varphi}\mu(\xi)|^2=\frac{1}{2}(1-\cos\beta(\xi)), \tag{2.2}$$

where $\beta(\xi)$ is the angle between the vectors ${}^{\psi}\mu(\xi)$ and ${}^{\varphi}\mu(\xi)$.

Now let $m=3$ and let $\Gamma=\mathbb{S}^2$. Then ${}^{\psi}\mu(\xi)=\sum_{k=1}^{3}\psi^k\cdot\tau_k(t,\xi)$, where $\tau=(\tau_1,\,\tau_2,\,\tau_3)$ is a unit tangent vector to the sphere $\mathbb{S}^2$ in the point t. Denote by U_ψ the matrix where the $k-th$ column coincides with the vector $\vec{\psi^k}$. Then ${}^{\psi}\mu(\xi)=U_\psi\cdot\tau$ and we have

$$\begin{aligned}
\langle{}^{\psi}\mu(\xi),\ {}^{\varphi}\mu(\xi)\rangle &= \langle U_\psi\cdot\tau;\ U_\varphi\cdot\tau\rangle=\\
=\langle\tau,\ U'_\psi\cdot U_\varphi\cdot\tau\rangle &= \cos\beta(\xi),
\end{aligned}$$

where U'_ψ is the transposed matrix.

The matrix $w:=U'_\psi\cdot U_\varphi$ determines a rotation of the space $\mathbb{R}^3$ around the axis defined by the unit vector $l=(l_1,\,l_2,\,l_3)$ through the angle ω where

$$l_1=\frac{w_{32}-w_{23}}{2\sin\omega},\ l_2=\frac{w_{13}-w_{31}}{2\sin\omega},\ l_3=\frac{w_{21}-w_{12}}{2\sin\omega},$$

$$\cos\omega=\frac{1}{2}(Tr\,w-1)=\frac{1}{2}\left(w_{11}+w_{22}+w_{33}-1\right).$$

It is easy to see that

$$W=\left(\langle\psi^p,\,\varphi^q\rangle\right)_{p,q=1}^3=-\left(Re(\psi^p\cdot\varphi^q)\right)_{p,q=1}^3$$

and therefore

$$\begin{aligned}
l_1=\frac{\langle\psi^3,\varphi^2\rangle-\langle\psi^2,\varphi^3\rangle}{2\sin\omega} &= \frac{Re(\psi^2\varphi^3-\psi^3\varphi^2)}{2\sin\omega},\\
l_2=\frac{\langle\psi^1,\varphi^3\rangle-\langle\psi^3,\varphi^1\rangle}{2\sin\omega} &= \frac{Re(\psi^3\varphi^1-\psi^1\varphi^3)}{2\sin\omega},\\
l_3=\frac{\langle\psi^2,\varphi^1\rangle-\langle\psi^1,\varphi^2\rangle}{2\sin\omega} &= \frac{Re(\psi^1\varphi^2-\psi^2\varphi^1)}{2\sin\omega},
\end{aligned}$$

$$\cos\omega=\frac{1}{2}\left(\sum_{k=1}^{3}\langle\psi^k,\varphi^k\rangle-1\right)=-\frac{1}{2}\left(Re\sum_{k=1}^{3}\psi^k\varphi^k+1\right).$$

Denoting by $\alpha(t,\xi)$ the angle between the vectors l and $\tau(t,\xi)$, we have

$$1-\cos\beta(t,\xi)=(1-\cos\omega)\cdot\sin^2\alpha(t,\xi).$$

The set $\{\tau(t,\xi)\}$ coincides with the set of all unit tangent vectors in $t \in \mathbb{S}^2$ and so

$$\begin{aligned} \Delta(t) &= Im_\xi \frac{1}{2}(1-\beta(\xi)) = \\ &= Im_\xi \frac{1}{2}(1-\cos\omega)\cdot \sin^2\alpha(t,\xi) = \\ &= [\cos^2\gamma(t)\cdot \sin^2\frac{\omega}{2}; \sin^2\frac{\omega}{2}], \end{aligned} \tag{2.3}$$

where $\gamma(t)$ is the angle between the vector l and the radius-vector of the point $t \in \mathbb{S}^2$.

Now let $m=4$ and $\Gamma = \mathbb{R}^3$. In this case ${}^{\psi}\mu(\xi) = U_\psi \cdot \xi$ and

$$\begin{aligned} \langle {}^{\psi}\mu(\xi),\ {}^{\varphi}\mu(\xi)\rangle &= \langle U_\psi\cdot\xi,\ U_\varphi\cdot\xi\rangle = \\ = \langle \xi,\ U'_\psi \cdot U_\varphi\cdot\xi\rangle &= \cos\beta(\xi), \end{aligned}$$

where $\beta(\xi)$ is the angle between ξ and $U'_\psi\ U_\varphi\xi$.

Assume first that $w := U'_\psi \cdot U_\varphi \in O_3^+(\mathbb{R})$, that is ψ and φ are cooriented. Then as above the matrix w gives a rotation for which l and ω are defined by the same formulae. We have

$$\Delta(t) = Im_\xi \frac{1}{2}(1-\cos\omega)\cdot\sin^2\alpha(\xi) = [0;\ \sin^2\frac{\omega}{2}], \tag{2.4}$$

where $\alpha(\xi)$ is the angle between l and ξ. If $w \in O_3^-(\mathbb{R})$, that is ψ and φ are contrarily oriented, then we put $\widetilde{w} := -w \in O_3^+(\mathbb{R})$, leading to $\widetilde{l}$, $\widetilde{\omega}$ and $\widetilde{\alpha}(\xi)$. Now

$$1+\cos\beta(\xi) = (1-\cos\widetilde{\omega})\cdot\sin^2\widetilde{\alpha}(\xi),$$

whence

$$\begin{aligned} 1-\cos\beta(\xi) &= 2-(1-\cos\widetilde{\omega})\cdot\sin^2\widetilde{\alpha}(\xi) = \\ &= 2(1-\sin^2\frac{\widetilde{\omega}}{2}\cdot\sin^2\widetilde{\alpha}(\xi)) \end{aligned}$$

and so

$$\Delta(t) = Im_\xi\left(1-\sin^2\frac{\widetilde{\omega}}{2}\cdot\sin^2\widetilde{\alpha}(\xi)\right) = [\cos^2\frac{\widetilde{\omega}}{2}; 1]. \tag{2.5}$$

Now consider the case $m=4$ and $\Gamma = \mathbb{S}^3$. Then we have ${}^{\psi}\mu(\xi) = \overline{t^{\psi}}\cdot h^{\psi}(t,\xi)$. Transforming this expression using formulas (5.19), (5.20) from [SV], we obtain

$${}^{\psi}\mu(\xi) = \sum_{k=1}^{3}\psi^k\cdot\tau_k(t,\xi),$$

where for $\operatorname{sgn}\psi = 1$

$$\begin{aligned} \tau_1(t,\xi) &= t_1\xi_1 - t_0\xi_2 - t_3\xi_3 + \frac{t_2}{1-t_3}\sum_{q=0}^{2} t_q\cdot\xi_{q+1}, \\ \tau_2(t,\xi) &= t_2\xi_1 - t_0\xi_3 + t_3\xi_2 - \frac{t_1}{1-t_3}\sum_{q=0}^{2} t_q\cdot\xi_{q+1}, \\ \tau_3(t,\xi) &= t_1\xi_3 - t_2\xi_2 + t_3\xi_1 - \frac{t_0}{1-t_3}\sum_{q=0}^{2} t_q\cdot\xi_{q+1}, \end{aligned}$$

and for sgn $\psi = -1$

$$\begin{aligned}
\tau_1(t,\xi) &= t_1\xi_1 - t_0\xi_2 + t_3\xi_3 - \frac{t_2}{1-t_3}\sum_{q=0}^{2} t_q \cdot \xi_{q+1},\\
\tau_2(t,\xi) &= t_2\xi_1 - t_0\xi_3 - t_3\xi_2 + \frac{t_1}{1-t_3}\sum_{0}^{2} t_q \cdot \xi_{q+1},\\
\tau_3(t,\xi) &= t_2\xi_2 - t_1\xi_3 + t_3\xi_1 - \frac{t_1}{1-t_3}\sum_{0}^{2} t_q \cdot \xi_{q+1}.
\end{aligned}$$

In order to compute $\Delta(t)$ we have to repeat computations made for $\Gamma = \mathbb{R}^3$. Taking into account that if ξ runs through the whole $\mathbb{R}^3$ then τ runs through $\mathbb{R}^3$ as well, we obtain:

$$\begin{aligned}
\text{for} \quad \operatorname{sgn} \psi = 1, \quad \Delta &= [0, \sin^2 \frac{\omega}{2}],\\
\text{for} \quad \operatorname{sgn} \psi = -1, \quad \Delta &= [\cos^2 \frac{\widetilde{\omega}}{2}; 1],
\end{aligned}$$

where the angles ω and $\widetilde{\omega}$ have the same meaning and can be computed by the same formulas as for $\Gamma = \mathbb{R}^3$.

Finally let $m = 3$ and let $\Gamma = \mathbb{R}^2$. Representing the vector ${}^{\psi}\mu(\xi) = \psi \cdot \xi_1 - \psi^1 \cdot \xi_2$ in the form

$${}^{\psi}\mu(\xi) = \sum_{k=1}^{3} \psi^k \cdot \tau_k,$$

where $\tau_1 = -\xi_2$, $\tau_2 = \xi_1$, $\tau_3 = 0$, it is easy to see, repeating previous reasonings, that

$$\Delta(t) = \left[\sin^2 \alpha_0 \cdot \sin^2 \frac{\omega}{2};\ \sin^2 \frac{\omega}{2}\right],$$

where α_0 is the slope angle of the rotation axis with respect to the plane $\mathbb{R}^2$.

Taking into account the formulas expressing the elements of w in terms of structural quaternions we may summarize in the following way

Lemma 2.1 *Let $\varphi := (\varphi^{4-m}, \ldots, \varphi^3)$ and $\psi := (\psi^{4-m}, \ldots, \psi^3)$ be structural sets. Then the set $\Delta(t) := sp(p_1(t) - p_2(t))^2$ is defined by the formulas:*

1. *if $m = 4$ then for co-oriented ψ and φ*

$$\Delta(t) = \left[0; \frac{3 - \sum\limits_{k=1}^{3} \langle \psi^k, \varphi^k \rangle}{4}\right], \tag{2.6}$$

and for contrarily oriented ψ and φ

$$\Delta(t) = \left[\frac{1 - \sum\limits_{k=1}^{3} \langle \psi^k, \varphi^k \rangle}{4}; 1\right]; \tag{2.7}$$

where theformulas are the same both for $\Gamma = \mathbb{S}^3$ and for $\Gamma = \mathbb{R}^3$;

2. if $m = 3$ then for $\Gamma = \mathbb{S}^2$

$$\Delta(t) = \left[\cos^2\gamma(t)\frac{3-\sum\limits_{k=1}^{3}\langle\psi^k,\varphi^k\rangle}{4}; \frac{3-\sum\limits_{1}^{3}\langle\psi^k,\varphi^k\rangle}{4}\right] \tag{2.8}$$

(here $\gamma(t)$ is the angle between the vector l and the radius-vector of the point t), and for $\Gamma = \mathbb{R}^2$

$$\Delta(t) = \left[\sin^2\alpha_0\frac{3-\sum\limits_{1}^{3}\langle\psi^k,\varphi^k\rangle}{4}; \frac{3-\sum\limits_{1}^{3}\langle\psi^k,\varphi^k\rangle}{4}\right] \tag{2.9}$$

(here α_0 is the slope angle of the rotation axis with respect to the plane $\mathbb{R}^2$).

Remark 2.2 *The set $\Delta(t)$ has the property; if $\Delta(t) \neq \{0\}$ then $\Delta(t) = \overline{\Delta(t)\setminus\{0;1\}}$.*

This property allows to simplify the formulation of some theorems, using only the elements of σ_2 in the description of the corresponding algebras.

Theorem 1.6 and Remark 2.2 now lead to the following result. Let

$$\begin{aligned}\overline{X} &= \overline{X^{\varphi,\psi}_{m,\Gamma}} = \{(t,x)|\ t\in\Gamma,\ x\in\Delta(t)\}\,,\\ \overline{Y} &= \overline{X}\cap(\Gamma\times\{0;1\}).\end{aligned}$$

Theorem 2.3 *The symbol algebra* Sym $\mathcal{R}$ *of the algebra $\mathcal{R} = \mathcal{R}(\mathcal{F};\, {}^{\varphi}S_\Gamma,\ {}^{\psi}S_\Gamma; \mathcal{T})$ is isometrically isomorphic to the algebra of all continuous 2×2-matrices on $\overline{X}$ whose elements are $\mathbb{H}$-valued functions and these matrices become diagonal in the points of Y. Moreover the homomorphism*

$$\mathrm{sym}: \mathcal{R} \longrightarrow \mathrm{Sym}\ \mathcal{R}$$

is generated by the following mapping of generators of the algebra $\mathcal{R}$:

$$\begin{aligned}\mathrm{sym}\ M^f &= f\cdot E_2, \quad \mathrm{sym}\ {}^{\varphi}S_\Gamma = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\\ \mathrm{sym}\ {}^{\psi}S_\Gamma &= \begin{pmatrix}1-2x & 2\sqrt{x(1-x)}\\ 2\sqrt{x(1-x)} & 2x-1\end{pmatrix},\end{aligned} \tag{2.10}$$

where $(t,x)\in\overline{X}$. Finally arbitrary operator A from $\mathcal{R}$ is Fredholmian if and only if its symbol sym A *is invertible.*

Taking into account Corollary 1.8 we obtain

Corollary 2.4 *The symbol algebra* sym $\mathcal{R}_0$ *of the algebra $\mathcal{R}_0 = \mathcal{R}(\mathcal{F}_0;\, {}^{\varphi}S_\Gamma,\, {}^{\psi}S_\Gamma;\, \mathcal{T})$ is isometrically isomorphic to the algebra of all continuous 2×2-matrices on $\overline{X}$ whose elements are $\mathbb{H}$-valued functions and these matrices become diagonal in the points of Y. Moreover the homomorphism*

$$\mathrm{sym}: \mathcal{R}_0 \longrightarrow \mathrm{Sym}\ \mathcal{R}_0$$

is generated by (2.10).

Finally arbitrary operator A from $\mathcal{R}_0$ is Fredholmian if and only if its symbol sym A *is invertible.*

3 Algebra generated by the operator of singular integration and multiplication operators by piece-wise continuous functions

3.1 Let Ω and $\Gamma = \partial\Omega$ denote the usual things, and let $\mathcal{L}$ be a smooth closed submanifold of codimension 1 in Γ bounding a simply connected domain D in Γ.

Denote by $PC(\Gamma,\, \mathcal{L};\, \mathbb{H})$ the algebra of all $\mathbb{H}$-valued functions g continuous in $\Gamma\backslash\mathcal{L}$ and having one-sided limit values

$$\begin{aligned} g_+(t) &= g(t+0) = \lim_{D\ni\tau\to t} g(\tau), \\ g_-(t) &= g(t-0) = \lim_{\Gamma\backslash\overline{D}\ni\tau\to t} g(\tau) \end{aligned}$$

in all points $t_0 \in \mathcal{L}$. Furthermore introduce the algebra

$$PC\left(\Gamma, \mathcal{L}; \mathbb{H}(\mathbb{C})\right) := PC\left(\Gamma, \mathcal{L}; \mathbb{H}\right) \otimes_{\mathbb{R}} \mathbb{C},$$

and the algebra $\mathcal{R} = \mathcal{R}(PC(\Gamma, \mathcal{L}; \mathbb{H}(\mathbb{C})); {}^{\psi}S_{\Gamma}; \mathcal{T})$ generated by all acting on $L_2(\Gamma; \mathbb{H}(\mathbb{C}))$ operators of the form

$$A = M^{g_1} + M^{g_2} \cdot {}^{\psi}S_{\Gamma} + T \tag{3.1}$$

with

$$\{g_1, g_2\} \subset PC(\Gamma, \mathcal{L}; \mathbb{H}(\mathbb{C})),\ \ T \in \mathcal{T}.$$

Let χ denotes the characteristic function of D and put $P_2 := M^{\chi}$. Obviously the operators $S_1 := {}^{\psi}S_{\Gamma}$ and $S_2 := 2P_2 - I$ satisfy the conditions 1), 2), 3) of 1.6. To describe the symbol algebra of the algebra

$$\begin{aligned} \mathcal{R} &= \mathcal{R}(PC(\Gamma,\, \mathcal{L};\, \mathbb{H}(\mathbb{C}));\, {}^{\psi}S_{\Gamma};\, \mathcal{T}) = \\ &= \mathcal{R}(C(\Gamma;\, \mathbb{H}(\mathbb{C}));\, S_1,\, S_2;\, \mathcal{T}) \end{aligned}$$

we use the results from Section 1. As before let $p_1(t)$ and $p_2(t)$ be the images of the operators $P_1 := \frac{1}{2}(I + {}^{\psi}S_{\Gamma})$ and $P_2 = M^{\chi}$ under the homomorphism

$$\pi_t : \mathcal{R} \longrightarrow \widehat{\mathcal{R}}(t), \quad t \in \Gamma.$$

Describe the set $\Delta(t) := \Delta^{\psi}_{m,\Gamma} = sp(p_1(t) - p_2(t))^2$, $t \in \Gamma$. Since $\chi|D = \mathbb{1}_D$ then for $t \in D$ we have $\Delta(t) = \{0; 1\}$ and (see Section 1) $X(t) = \emptyset$, $Y(t) = \{(0,0); (0,1)\}$. Also from $\chi|\Gamma\backslash\overline{D} = \mathbb{0}_{\Gamma\backslash\overline{D}}$ it follows that for $t \in \Gamma\backslash\overline{D}$ $\Delta(t) = \{0; 1\}$, $X(t) = \emptyset$, $Y(t) = \{(1,0); (1,1)\}$.

Now let $t \in \mathcal{L}$ and let us compute the spectrum of a local representative in a point t for the operator $A = (I - P_2) + P_2 \cdot {}^{\psi}S_{\Gamma} \cdot P_2$ using I.B. Simonenko's results [Sim1, Theorem 3.2]. These results are applied just to the operator A for $\Gamma = \mathbb{R}^{m-1}$ and to the operator WAW^* for $\Gamma = \mathbb{S}^{m-1}$ (see [SV, Section 5]).

We have: $sp\ \pi_t(A) = [-1; 1]$.

On the other hand computing $sp\ \pi_t(A)$ with the help of Theorem 1.1 we obtain $\Delta(t) = [0, 1]$ whence

$$X(t) = (0, 1),\ Y(t) = \{(i, j)\} = \{0; 1\} \times \{0; 1\}.$$

Put

$$\begin{aligned} X &= \left\{(t, x) | t \in \mathcal{L},\ x \in (0, 1)\right\}, \\ Y &= \left\{(t, y) | t \in \Gamma;\ y \in Y(t)\right\} = \\ &= \overline{D} \times \{(0,0); (0,1)\} \cup \overline{\Gamma\backslash D} \times \{(1,0); (1,1)\}. \end{aligned}$$

One can describe Y in another (but equivalent of course) way. Let $\widehat{\Gamma}$ denotes the closure of a manifold Γ cut along $\mathcal{L}$. Then Y is the disconnected union of the two copies of $\widehat{\Gamma}$:

$$Y = \widehat{\Gamma}_0 \cup \widehat{\Gamma}_1.$$

In this case the mapping $\nu : \overline{X} \backslash X \to Y^2$ is defined by

$$\begin{aligned} \nu(t,0) &= (t_0+0,\, t_1-0), \\ \nu(t,1) &= (t_0-0,\, t_1+0), \end{aligned}$$

where $(t,j) \in \overline{X}\backslash X$, $t_j \pm 0 \in \widehat{\Gamma}_j$, $j \in \{0;1\}$.

Furthermore let $\mathfrak{G}_0$ be the algebra of all pairs $\sigma = (\sigma_1,\, \sigma_2)$, where $\sigma_1 \in C(Y;\mathbb{H})$, $\sigma_2 \in C(Y;\mathbb{H}) \otimes \mathbb{R}^{2\times 2}$ such that the condition is fulfilled: if $\nu(x_0) = (y_0,\, y_1)$ for $x_0 \in \overline{X}\backslash X$, $y_0 \in \widehat{\Gamma}_0$, $y_1 \in \widehat{\Gamma}_1$ then

$$\lim_{X \ni x \to x_0} \sigma_2(x) = \begin{pmatrix} \sigma_1(y_0) & 0 \\ 0 & \sigma_1(y_1) \end{pmatrix}.$$

The set $\mathfrak{G} = \mathfrak{G}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is a C^*-algebra. The next assertion is a corollary of [SV] Theorem 5.6.

Theorem 3.1 *The symbol algebra* Sym $\mathcal{R}$ *of the algebra*

$$\mathcal{R} = \mathcal{R}(PC(\Gamma,\, \mathcal{L},\, \mathbb{H}(\mathbb{C}));\, {}^{\psi}S_\Gamma;\, \mathcal{T})$$

is isometrically isomorphic to the algebra $\mathfrak{G}$. Moreover the homomorphism

$$\mathrm{sym} \ : \mathcal{R} \longrightarrow \mathrm{Sym}\ \mathcal{R}$$

is generated by the following mapping of generators : for $A = M^{g_1} + M^{g_2} \cdot {}^{\psi}S_\Gamma + T$, $\{g_1,\, g_2\} \subset PC(\Gamma,\, \mathcal{L},\, \mathbb{H}(\mathbb{C}))$,

$$\mathrm{sym}\ A = \begin{cases} c(t) & , \quad t \in \widehat{\Gamma}_0, \\ d(t) & , \quad t \in \widehat{\Gamma}_1, \\ \begin{pmatrix} c(t+0)\cdot(1-x) + c(t-0)\cdot x; & [d(t+0)-d(t-0)]\sqrt{x(1-x)} \\ [c(t+0)-c(t-0)]\sqrt{x(1-x)}; & d(t+0)\cdot x + d(t-0)\cdot(1-x) \end{pmatrix} & , \quad t \in \mathcal{L},\ x \in [0;1]; \end{cases}$$

where $c = g_1 + g_2$, $d = g_1 - g_2$.

Corollary 3.2 *Arbitrary operator $A \in \mathcal{R}$ is Fredholmian if and only if its symbol* sym A *is invertible (in the algebra $\mathfrak{G}$).*

3.2 Consider now the general case of piece-wise continuous coefficients. We mean that for $k \in \{1, \ldots, n\}$ $\mathcal{L}_k$ is a smooth oriented submanifold in Γ of codimension 1 such that $\mathcal{L}_k \cap \mathcal{L}_j = \emptyset$. Each $\mathcal{L}_k$ may be both closed and with (smooth enough) boundary. Put $\mathcal{L} := \cup_{k=1}^{n} \mathcal{L}_k$. Then it is clear what are the meanings of $PC(\Gamma,\, \mathcal{L};\, \mathbb{H}(\mathbb{C}))$, $\mathcal{R} = \mathcal{R}(PC(\Gamma,\, \mathcal{L};\, \mathbb{H}(\mathbb{C})),\, {}^{\psi}S_\Gamma,\, \mathcal{T})$, $\widehat{\Gamma}_0$, $\widehat{\Gamma}_1, \nu$ and etc. As in 3.1 $\Gamma = \mathbb{R}^{m-1}$ or $\Gamma = \mathbb{S}^{m-1}$.

We will obtain the description of the algebra Sym $\mathcal{R}$ by combining the local principle and the results of the previous section.

Theorem 3.3 *The symbol algebra* Sym $\mathcal{R}$ *of the algebra* $\mathcal{R}$ *is isometrically isomorphic to the algebra* $\mathfrak{G}$. *The homomorphism*

$$\text{sym} : \mathcal{R} \longrightarrow \text{Sym } \mathcal{R}$$

is generated by the following mapping of generators for $A = M^{g_1} + M^{g_2} \cdot {}^{\psi}S_{\Gamma} + T$

$$\text{sym } A = \tag{3.2}$$

$$\begin{cases} c(t) & , \quad t \in \widehat{\Gamma}_0 \\ d(t) & , \quad t \in \widehat{\Gamma}_1 \\ \begin{pmatrix} c^+(t)(1-x) + c-(t)x & ; & (d^+(t) - d^-(t))\sqrt{x(1-x)} \\ (c^+(t) - c^-(t))\sqrt{t(1-x)} & ; & d^+(t)x + d^-(t)(1-x) \end{pmatrix} & , \quad (t,x) \in \overline{X}. \end{cases}$$

Corollary 3.4 *Arbitrary operator* $A \in \mathcal{R}$ *is Fredholmian if and only if its symbol* sym A *is invertible.*

Corollary 1.8 allows us to formulate some results for the algebra $\mathcal{R}_0 = \mathcal{R}(PC(\Gamma, \mathcal{L}; \mathbb{H}), {}^{\psi}S_{\Gamma}, \mathcal{T})$ with $\mathbb{H}$-valued coefficients.

Theorem 3.5 *The algebra* Sym $\mathcal{R}_0$ *of the symbols of algebra* $\mathcal{R}_0$ *is isomorphic and isometric to the algebra* $\mathfrak{G}_0$. *The homomorphism*

$$\text{sym } : \mathcal{R}_0 \longrightarrow \text{Sym } \mathcal{R}_0$$

is generated by the following mapping of generators: if $A = M^{g_1} + M^{g_2} \cdot {}^{\psi}S_{\Gamma} + T$, *then* sym A *is given by* (3.2). *Arbitrary operator* $A \in \mathcal{R}_0$ *is Fredholmian if and only if its symbol is invertible.*

3.3 Consider now the case of a manifold with boundary. To simplify we suppose again Γ to be $\mathbb{S}^{m-1}$ or $\mathbb{R}^{m-1}$ and D to be a part of Γ, bounded by $\mathcal{L} = \cup_{k=1}^{n} \mathcal{L}_k$ where $\mathcal{L}_k$ is a smooth submanifold of codimension 1, $\mathcal{L}_k \cap \mathcal{L}_j = \emptyset$. Let χ denotes the characteristic function of $D$and let $P_D := M^{\chi}$ be the projection of $L_2(\Gamma; \mathbb{H})$ onto $L_2(D; \mathbb{H})$ and ${}^{\psi}S_D := P_D^{\psi} S_{\Gamma} | L_2(D; \mathbb{H})$. Further denote by $\mathcal{R}_D = \mathcal{R}(C(\overline{D}; \mathbb{H}); {}^{\psi}S_D; \mathcal{T})$ the algebra generated by acting on $L_2(D, \mathbb{H})$ operators of the form

$$A = M^{g_1} + M^{g_2} \cdot {}^{\psi}S_D + T$$

with $\{g_1, g_2\} \subset C(\overline{D}, \mathbb{H}), T \in \mathcal{T}$.

It is clear that the Fredholmian character of $A \in \mathcal{R}_D$ is the same as that of the operator

$$A' = A \cdot P_D + (I - P_D) \in \mathcal{R}_0 = \mathcal{R}(PC(\Gamma, \mathcal{L}; \mathbb{H}(\mathbb{C})); {}^{\psi}S_{\Gamma}; \mathcal{T}).$$

So we can use the corresponding results for $\mathcal{R}_0$. Let $\overline{D}_0$ and $\overline{D}_1$ be two disjoint copies of $\overline{D}$. In $\overline{D} \cup \overline{D}_1 \cup (\mathcal{L} \times [0;1])$ we identify the points $t \in \overline{D}_j \cap \mathcal{L}$ with the points $(t, j) \in \mathcal{L} \times [0,1], j \in \{0;1\}$. Ddenote by Z the corresponding set.

Applying the automorphism

$$\begin{aligned} \sigma_1(t) &\longmapsto \sigma_1(t), \\ \sigma_2(t,x) &\longmapsto \begin{pmatrix} \sqrt{1-x} & \sqrt{x} \\ -\sqrt{x} & \sqrt{1-x} \end{pmatrix} \cdot \sigma_2(t,x) \cdot \begin{pmatrix} \sqrt{1-x} & -\sqrt{x} \\ \sqrt{x} & \sqrt{1-x} \end{pmatrix} \end{aligned}$$

of $\mathfrak{G}_0$ to describe the algebra Sym $\mathcal{R}_0$ we obtain from the Theorem 3.5

Theorem 3.6 *The symbol algebra* Sym $\mathcal{R}_D$ *of the algebra* $\mathcal{R}_D = \mathcal{R}(C(D;\mathbb{H});\, {}^{\psi}S_D;\, \mathcal{T})$ *is isometrically isomorphic to* $C(Z;\mathbb{H})$*. The homomorphism*

$$\mathrm{sym} : \mathcal{R}_D \longrightarrow \mathrm{Sym}\ \mathcal{R}_D$$

is generated by the following mapping of generators of $\mathcal{R}_D$ *: if* $A = M^{g_1} + M^{g_2} \cdot {}^{\psi}S_D + T$ *then*

$$\mathrm{sym}\ A = \begin{cases} c(t) & , \quad t \in \overline{D}_0, \\ d(t) & , \quad t \in \overline{D}_1, \\ c(t)(1-x) + d(t)x & , \quad (t,x) \in \mathcal{L} \times [0,1], \end{cases}$$

with $c := g_1 + g_2$ *and* $d = g_1 - g_2$*. An arbitrary* $A \in \mathcal{R}_D$ *is Fredholmian if and only if its symbol is invertible.*

4 Analogues of the Riemann boundary value problem for ψ- hyperholomorphic functions

4.1 The Riemann boundary value problem, (or the linear conjugation problem) in the theory of functions of (one) complex variable may be formulated as follows (see [G] for instance). Let Γ be a simple smooth closed curve in the complex plane and let Γ bounds the domain Ω^+ and $\Omega^- := \mathbb{C}\backslash\overline{\Omega}^+$. On Γ there are given two $\mathbb{C}$-valued functions G and g from some function space X. Find functions Φ^+ and Φ^- with the folliwing properties: they are homomorphic in Ω^+ and Ω^- respectively; they are representable by C.i.t.i. with a density from a certain class and they have (everywhere or almost everywhere on Γ) limit values such that on Γ

$$\Phi^+ = G \cdot \Phi^- + g. \tag{4.1}$$

We now consider several quaternionic analogues of the Riemann problem.

4.2 Because of the non-commutativity of the quaternionic multiplication there are two direct analogues.

Problem ${}^{\psi}\mathfrak{R}_r(a,b;X)$ (the right Riemann problem for left-ψ-hyperholomorphic functions): Find a pair of functions $F^{\pm}$ from the classes ${}^{\psi}\mathfrak{A}(\Omega^{\pm};X)$ satisfying the boundary condition on Γ

$$F^+ \cdot a = F^- \cdot b + g. \tag{4.2}$$

Here $g \in X$, a and b are given multiplicators for X. If we change the boundary condition (4.2) to

$$a \cdot F^+ = b \cdot F^- + g, \tag{4.3}$$

then we obtain the problem ${}^{\psi}\mathfrak{R}_l(a,b;X)$ (or the left Riemann problem for left-ψ-hyperholomorphic functions).

Applying the Sokhotski-Plemelj formulas one can reduce the problems ${}^{\psi}\mathfrak{R}_r(a,b;X)$ and ${}^{\psi}\mathfrak{R}_l(a,b;X)$ in a classical way to singular integral equations defined by the operators

$$\begin{aligned} {}^{\psi}R_r &:= M^a \cdot {}^{\psi}P_+ + M^b \cdot {}^{\psi}P_-, \\ {}^{\psi}R_l &:= {}^{a}M \cdot {}^{\psi}P_+ + {}^{b}M \cdot {}^{\psi}P_-. \end{aligned} \tag{4.4}$$

Let us call the problems ${}^{\psi}\mathfrak{R}_r(a,b;X)$ and ${}^{\psi}\mathfrak{R}_l(a,b;X)$ Fredholmian on X, if the corresponding operators ${}^{\psi}R_2$ and ${}^{\psi}R_l$ are Fredholm in X.

4.3 Assume further that $\Gamma = \mathbb{S}^{m-1}$, i.e. $\Omega^+ = \mathbb{B}^+ := Int\,\mathbb{B}^m(0;1)$, $\Omega^- = \mathbb{B}^- := \mathbb{R}^m \backslash \overline{\mathbb{B}^+}$, or $\Gamma = \mathbb{R}^{m-1}$, i.e. $\Omega^\pm := \{(x^0,x^1,x^2,x^3)|\ x^3 \geq 0\}$ for $m=4$ and $\Omega^\pm := \{(x^1,x^2,x^3)|\ x^3 \geq 0\}$ for $m=3$. Furthermore assume that $p=2$. The Riemann problems ${}^\psi\mathfrak{R}_r(a,b;X)$ and ${}^\psi\mathfrak{R}_l(a,b;X)$ we will denote briefly by ${}^\psi\mathfrak{R}_r(a,b)$ and ${}^\psi\mathfrak{R}_l(a,b)$. It should be noted just now that the main reason for our restrictions (on a domain Ω and on boundary properties of unknown functions) is a simplification of the account. All the basic results can be extended for arbitrary $p>1$ and for the spaces H_μ, for instance, as well as for a domain with Liapounov's boundary.

Start with the problem ${}^\psi\mathfrak{R}_r(a,b)$.

Theorem 4.1 *Let $\{a,b\} \subset C(\Gamma;\mathbb{H})$. The problem ${}^\psi\mathfrak{R}_r(a,b)$ is Fredholm in $L_2(\Gamma;\mathbb{H})$ if and only if*

$$a(t) \neq 0, \quad b(t) \neq 0 \tag{4.5}$$

everywhere on Γ.

Proof. It follows from 4.2 that we are interested in Fredholm properties of the operator ${}^\psi R_r$ defined by (4.4). From Section 1 we obtain that the operator ${}^\psi R_r$ is Fredholm simultaneously with invertibility of the operator

$$\mathrm{Sym}\ {}^\psi R_r = M^{a\ \cdot\ sym\,{}^\psi P_+}M + M^{b\ \cdot\ sym\,{}^\psi P_-}M \tag{4.6}$$

in $\mathbb{H}(\mathbb{C})$ (which is considered as $\mathbb{C}$-linear space).

If $\Gamma = \mathbb{R}^{m-1}$, then

$$\mathrm{sym}\ {}^\psi P_\pm = \frac{1}{2}(i_0 \mp i\xi^\psi)$$

where

$$\xi^\psi = \begin{cases} \sum\limits_{k=1}^{3} \psi^k \xi_k & \text{for } m=4, \\ \psi^1\xi_2 - \psi^2\xi_1 & \text{for } m=3. \end{cases}$$

If (4.5) is valid then for every $t \in \Gamma$ there exist $a^{-1}(t)$ and $b^{-1}(t)$. Direct computation shows that the operator

$$M^{a^{-1}\ \cdot\ sym\,{}^\psi P_+}M + M^{b^{-1}\ \cdot\ sym\,{}^\psi P_-}M \tag{4.7}$$

is the inverse operator for (4.6). Vice versa let $a(t)=0$, $b(t)\neq 0$ for instance. Then the equation

$$\mathrm{sym}\ {}^\psi P_- \cdot X \cdot b(t) = 0$$

has a non-trivial solution $X_0 := 1 - i\xi^\psi$, whence (4.6) is not invertible. The rest is quite the same.

Now let $\Gamma = \mathbb{S}^{m-1}$. Then (see [VS, (5.17) and (5.21)])

$$\mathrm{sym}\ {}^\psi P_\pm = \begin{cases} \frac{1}{2}(i_0 \pm i\tau^\psi) & \text{for } m=3, \\ \frac{1}{2}(i_0 \pm i\overline{t}^\psi h^\psi) & \text{for } m=4, \end{cases}$$

where

$$\tau^\psi := \sum_{k=1}^{3} \psi^k \cdot \tau_k.$$

It is easy to check that τ^ψ and $\overline{t}^\psi h^\psi$ are purely imaginary quaternions with length one, that is both of them have properties quite analogous to the properties of ξ^ψ, and so the proof is quite analogous as in the case of $\Gamma = \mathbb{R}^{m-1}$.

Theorem 4.2 *Let $\{a,b\} \subset C(\Gamma;\mathbb{H})$. The problem ${}^{\psi}\mathfrak{R}_l(a,b)$ is Fredholm in $L_2(\Gamma;\mathbb{H})$ if and only if everywhere on Γ*

$$Re\ \left(a(t)\overline{b}(t)\right) \neq 0 \tag{4.8}$$

or equivalently

$$\langle a(t),\ b(t)\rangle_{\mathbb{R}^4} \neq 0.$$

Proof. From Section 1 again we obtain that the operator ${}^{\psi}R_l$ is Fredholmian simultaneously with the invertibility of the complex quaternion

$$z := a(t)\mathrm{sym}\ {}^{\psi}P_+ + b(t)\mathrm{sym}\ {}^{\psi}P_- \tag{4.9}$$

for all t and ξ. As for $z \in \mathbb{H}(\mathbb{C})$, its invertibility is equivalent to the condition

$$z\overline{z} = |z_1|^2 - |z_2|^2 + 2i\langle z_1, z_2\rangle_{\mathbb{R}^4} \neq 0. \tag{4.10}$$

Applying (4.10) to (4.9) we obtain for all t, ξ

$$\langle a, b\rangle_{\mathbb{R}^4} + i\langle a+b, (a-b)\alpha\rangle_{\mathbb{R}^4} \neq 0, \tag{4.11}$$

where $\alpha(t,\xi)$ is a certain purely imaginary quaternion with unit length.

Let $a(t) \neq b(t)$. We have

$$\langle a+b, (a-b)\alpha\rangle_{\mathbb{R}^4} = \langle \frac{\overline{a}-\overline{b}}{|a-b|^2}(a+b),\ \alpha\rangle_{\mathbb{R}^4} =$$

$$= \langle Im\left(\frac{(\overline{a}-\overline{b})(a+b)}{|a-b|^2}\right),\ \alpha\rangle_{\mathbb{R}^3}.$$

According to the properties of $\alpha(t,\xi)$, for any fixed $t \in \Gamma$ a point $\xi \in \mathbb{S}^{m-1}$ exists such that the vector $\overrightarrow{\alpha(t,\xi)}$ is orthogonal to vector

$$\overrightarrow{Im\frac{(\overline{a}-\overline{b})}{|a-b|^2}\cdot(a+b)}.$$

Hence the relation (4.11) is equivalent to (4.8).

Comparing Theorem 4.1 and Theorem 4.2, it follows that the Fredholm conditions for the left Riemann boundary value problem are stronger than for the right one with the same coefficients. Also the last conditions are of the same type as for the classical Riemann problem in the theory of functions of one complex variable. In particular the condition (4.5) is necessary but not sufficient in the case of the left problem. The fact that left-hyperholomorphic functions form a right (but not left) $\mathbb{H}$-module is one of the reason of this.

4.4 The results of Section 3 allow us to obtain a Fredholm criterion for the problem ${}^{\psi}\mathfrak{R}_r(a,b)$ in the case of piece-wise continuous coefficients a and b. Let (as in 3.2) $\mathcal{L} = \cup_{k=1}^{n}\mathcal{L}_k$ be a set of smooth, oriented, mutually disjoint submanifolds on Γ of codimension 1 where each $\mathcal{L}_k$ may be both closed (i.e. without boundary) and with boundary (smooth enough for $m = 4$). We consider the problem ${}^{\psi}\mathfrak{R}_r(a,b)$ with $\{a,b\} \subset PC(\Gamma,\mathcal{L};\mathbb{H})$. The corresponding operator ${}^{\psi}R_r$ (see (4.4)) has the symbol defined by (3.2). Then Theorem 3.5 implies

Theorem 4.3 *Let $\{a,b\} \subset PC(\Gamma,\mathcal{L};\mathbb{H})$. The problem ${}^{\psi}\mathfrak{R}_r(a,b)$ is Fredholm on $L_2(\Gamma;\mathbb{H})$ if and only if:*

1) $a(t) \neq 0,\ b(t) \neq 0\ \forall t \in \Gamma$,

2) the matrix-function

$$(t,x)\in\mathcal{L}\times[0,1]\longmapsto \begin{pmatrix} a^+(t)(1-x)+a^-(t)x & ; & (b^+(t)-b^-(t))\sqrt{x(1-x)} \\ (a^+(t)-a^-(t))\sqrt{x(1-x)} & ; & b^+(t)x+b^-(t)(1-x) \end{pmatrix}$$

is invertible.

Corollary 4.4 *If $b \in PC(\Gamma,\mathcal{L};\mathbb{H})$ then the problem ${}^{\psi}\mathfrak{R}_r(1,b)$ is Fredholm in $L_2(\Gamma,\mathbb{H})$ if and only if:*

1) $b(t) \neq 0 \quad \forall t \in \Gamma$,

2) $b^+(t)x + b^-(t)(1-x) \neq 0 \quad \forall (t,x) \in \mathcal{L}\times[0,1]$.

We conclude this section by considering the right problem with continuous coefficients for the case of a manifold–with–boundary. Using the notations from 3.3 we obtain from Theorem 3.6.

Theorem 4.5 *Let $\{a,b\} \subset C(D;\mathbb{H})$. The problem ${}^{\psi}\mathfrak{R}_r(a,b)$ is Fredholm in $L_2(D,\mathbb{H})$ if and only if:*

1) $a(t) \neq 0,\ b(t) \neq 0 \quad \forall t \in \overline{D}$,

2) $a(t)(1-x) + b(t)x \neq 0 \quad \forall (t,x) \in \mathcal{L}\times[0;1]$.

4.5 Consider now the generalization of the complex Riemann problem taking into account the existence of different classes of hyperholomorphy (we restrict ourselves to the right problem for left-hyperholomorphic functions).

Problem ${}^{\varphi,\psi}\mathfrak{R}_r(a,b,c,d)$. Find two pairs of functions $\Phi^\pm \in {}^{\varphi}\mathfrak{A}(\Omega^\pm)$ and $\Psi^\pm \in {}^{\psi}\mathfrak{A}(\Omega^\pm)$ satisfying the boundary condition on Γ

$$\Phi^+a + \Psi^+b + \Phi^-c + \Psi^-d = g. \tag{4.12}$$

It can be easily seen that this problem is incorrect. Indeed let g_1 be an arbitrary function from $L_2(\Gamma;\mathbb{H})$. Then any solution of the pair of correctly posed problems

$$\begin{aligned} \Phi^+a + \Phi^-c &= g_1 \\ \Psi^+b + \Psi^-d &= g - g_1 \end{aligned}$$

yields a solution of the problem (4.12).

One of the ways to overcome this is to require $\Phi^\pm$ and $\Psi^\pm$ to be representable by the C.i.t.i. with the same density. Assume this requirement if fulfilled. Then by definition the problem ${}^{\varphi,\psi}\mathfrak{R}_r(a,b,c,d)$ is Fredholm if the operator

$${}^{\varphi,\psi}R_r := M^a \cdot {}^{\varphi}P_+ + M^b \cdot {}^{\psi}P_+ - M^c \cdot {}^{\varphi}P_- - M^d \cdot {}^{\psi}P_- \tag{4.13}$$

is Fredholm.

We have from Section 1:

$$\operatorname{sym}\ {}^{\psi}P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{sym}\ {}^{\psi}P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{aligned}\mathrm{sym}\ {}^{\varphi}P_{+} &= \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix},\\ \mathrm{sym}\ {}^{\varphi}P_{-} &= \begin{pmatrix} x & -\sqrt{x(1-x)} \\ -\sqrt{x(1-x)} & 1-x \end{pmatrix},\end{aligned}$$

where $x \in \Delta(t)$. Hence

$$\mathrm{sym}\ {}^{\varphi,\psi}R_r = \begin{pmatrix} a(t)+b(t)-x(a(t)+c(t)) & ; & (a(t)+c(t))\sqrt{x(1-x)} \\ (a(t)+c(t))\sqrt{x(1-x)} & ; & (a(t)+c(t))x-(c(t)+d(t)) \end{pmatrix}. \tag{4.14}$$

Now let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

has elements in any (non-commutative in general) ring with the identity. Then one can easily check that: $\exists A^{-1} \Longleftrightarrow$ at any rate one of the following elements

$$\begin{aligned}\Delta_{11} &:= a_{11}\,a_{22} - a_{11}\,a_{21}\,a_{11}^{-1}\,a_{12},\\ \Delta_{22} &:= a_{22}\,a_{11} - a_{22}\,a_{12}\,a_{22}^{-1}\,a_{21},\\ \Delta_{12} &:= a_{12}\,a_{22}\,a_{12}^{-1}\,a_{11} - a_{12}\,a_{21},\\ \Delta_{21} &:= a_{21}\,a_{11}\,a_{21}^{-1}\,a_{22} - a_{21}\,a_{12},\end{aligned}$$

exists and is invertible. The corresponding inverse matrix then has the form:

$$A^{-1} = \begin{pmatrix} a_{11}^{-1}(1+a_{12}\Delta_{11}^{-1}a_{11}a_{21}a_{11}^{-1}) & ; & -a_{11}^{-1}a_{12}\Delta_{11}^{-1}a_{11} \\ -\Delta_{11}^{-1}a_{11}a_{21}a_{11}^{-1} & ; & \Delta_{11}^{-1}a_{11} \end{pmatrix};$$

$$A^{-1} = \begin{pmatrix} \Delta_{22}^{-1}a_{22} & ; & -\Delta_{22}^{-1}a_{22}a_{12}a_{22}^{-1} \\ -a_{22}^{-1}a_{21}\Delta_{22}^{-1}a_{22} & ; & a_{22}^{-1}(1+a_{21}\Delta_{22}^{-1}a_{22}a_{12}a_{22}^{-1}) \end{pmatrix};$$

$$A^{-1} = \begin{pmatrix} \Delta_{22}^{-1}a_{12}a_{22}a_{12}^{-1} & ; & -\Delta_{12}^{-1}a_{12} \\ a_{12}^{-1}(1-a_{11}\Delta_{12}^{-1}a_{12}a_{22}a_{12}^{-1}) & ; & a_{12}^{-1}a_{11}\Delta_{12}^{-1}a_{12} \end{pmatrix};$$

$$A^{-1} = \begin{pmatrix} a_{21}^{-1}a_{22}\Delta_{21}^{-1}a_{21} & ; & a_{21}^{-1}(1-a_{22}\Delta_{21}^{-1}a_{21}a_{11}a_{21}^{-1}) \\ -\Delta_{21}^{-1}a_{21} & ; & \Delta_{21}^{-1}a_{21}a_{11}a_{21}^{-1} \end{pmatrix}.$$

Applying this result to the matrix (4.14) we have the following

Theorem 4.6 *The boundary value problem ${}^{\varphi,\psi}\mathfrak{R}_r(a,b,c,d)$ is Fredholmian on $L_2(\Gamma;\mathbb{H})$ if and only if for $\forall t \in \Gamma$ and for $\forall x \in \Delta(t)$ at any rate one of the following quaternions*

$$\begin{gathered}[(a(t)+b(t))-x(a(t)+c(t)]\cdot[(a(t)+c(t))x-(c(t)+d(t))]-\\ -[(a(t)+b(t))-x(a(t)+c(t))]\cdot(a(t)+c(t))\cdot\left[\overline{(a(t)+b(t))}-x(\overline{a(t)+c(t)})\right]\cdot\\ \cdot(a(t)+c(t))\cdot\frac{x(1-x)}{|(a(t)+b(t))-x(a(t)+c(t))|^2};\end{gathered}$$

$$\begin{gathered}[(a(t)+c(t))x-(c(t)+d(t))]\cdot[((a(t)+b(t))-x(a(t)+c(t)))-\\ -(a(t)+c(t))\cdot((\overline{(a(t)+c(t))}x-\overline{(c(t)+d(t))})(a(t)+c(t))\frac{x(1-x)}{|(a+c)x-(c+d)|^2}];\end{gathered}$$

$$(a(t)+c(t))\sqrt{x(1-x)}\Big[((a(t)+c(t))x-(c(t)+d(t)))\cdot$$
$$\overline{(a(t)+c(t))}((a(t)+b(t))-x(a(t)+c(t)))\frac{1}{|\omega(t)+c(t)|^2\sqrt{x(1-x)}}-$$
$$-(a(t)+c(t))\sqrt{x(1-x)},\Big];$$

$$(a(t)+c(t))\sqrt{x(1-x)}\Big[((a(t)+b(t))-x(a(t)+c(t)))\cdot$$
$$\cdot\overline{(a(t)+c(t))}((a(t)+c(t))x-(c(t)+d(t)))\frac{1}{|a(t)+c(t)|^2\sqrt{x(1-x)}}-$$
$$-(a(t)+c(t))\sqrt{x(1-x)}\Big]$$

exists and is invertible. Here $\Delta(t)$ is defined in Section 1.

Note that the formulas for A^{-1} given above enable us to construct the regularizator for the Fredholmian operator ${}^{\varphi,\psi}R_r$.

4.6 Now we consider two-element boundary value problems contained in the problem ${}^{\varphi,\psi}\mathfrak{R}_r(a,b,c,d)$. First of all it contains the problem ${}^{\psi}\mathfrak{R}_r(a,c)$ described above. Furthermore we divide the problems for different classes of hyperholomorphy into two groups:

1. the unknown functions are defined in different domains, i.e. the boundary conditions have the form
$$\Phi^+a+\Psi^-b=g \tag{4.15}$$
(problem ${}^{\varphi,\psi}\mathfrak{R}(a,0,0,b)$).
2. the unknown functions are defined in the same domain, i.e. the boundary condition has the form
$$\Phi^+a+\Psi^+b=g \tag{4.16}$$
for a bounded domain (problem ${}^{\varphi,\psi}\mathfrak{R}_r(a,b,0,0)$) or it has the form
$$\Phi^-a+\Psi^-b=g \tag{4.17}$$
for an unbounded domain (problem ${}^{\varphi,\psi}\mathfrak{R}_r(0,0,a,b)$).

We will prove that the Fredholm conditions for each of these problems include two factors: dependence on the number of variables $m\in\{3;4\}$ and dependence on the interrelationship of classes of hyperholomorphy.

For the problem with a boundary condition (4.15)

$$\operatorname{sym}{}^{\varphi,\psi}R_r=\begin{pmatrix} a(t)(1-x) & a(t)\sqrt{x(1-x)} \\ a(t)\sqrt{x(1-x)} & a(t)x-b(t)\end{pmatrix}.$$

It follows from the previous Section that the invertibility of sym ${}^{\varphi,\psi}R_r$ is equivalent to

$$a(t)\neq 0,\quad b(t)\neq 0,\quad x\neq 1$$

for every $t\in\Gamma,\ x\in\Delta(t)$ (that is $1\notin\Delta(t)$). Applying Lemma 2.1 as well we obtain

Theorem 4.7 *The Riemann problem* ${}^{\varphi,\psi}\mathfrak{R}_r(a,0,0,b)$ *with boundary condition*

$$\Phi^+ a + \Psi^- b = g$$

1. *is non-Fredholmian for any a and b, if $m = 4$ and the structural sets φ and ψ are contrarily oriented or if φ and ψ are equally oriented but*

$$\sum_{k=1}^{3} \langle \varphi^k, \psi^k \rangle = -1;$$

2. *is Fredholmian for equally oriented φ and ψ with $\sum_{k=1}^{3} \langle \varphi^k, \psi^k \rangle \neq -1$ if and only if*

$$a(t) \neq 0, \quad b(t) \neq 0 \quad \forall t \in \Gamma.$$

Now consider the problems with boundary conditions (4.16) and (4.17). Their symbols are respectively given by the matrices

$$\begin{pmatrix} a(1-x)+b & a\sqrt{x(1-x)} \\ a\sqrt{x(1-x)} & a \cdot x \end{pmatrix}$$

and

$$\begin{pmatrix} -ax & a\sqrt{x(1-x)} \\ a\sqrt{x(1-x)} & -(a(1-x)+b) \end{pmatrix}.$$

As before the invertibility of each of them is equivalent to

$$a(t) \neq 0, \quad b(t) \neq 0, \quad x \neq 0$$

for every $t \in \Gamma$, $x \in \Delta(t)$ (that is $0 \notin \Delta(t)$ in particular).

Again taking into account Lemma 2.1 we obtain

Theorem 4.8 *THe Riemann problems* ${}^{\varphi,\psi}\mathfrak{R}_r(a,b,0,0)$ *and* ${}^{\varphi,\psi}\mathfrak{R}_r(0,0,a,b)$ *are*

1. *non-Fredholmian for any a and b if*

   I) $m = 4$ and φ and ψ are equally oriented; or

   II) $m = 4$, φ and ψ are contrarily oriented and

$$\sum_{k=1}^{3} \langle \psi^k, \varphi^k \rangle = 1; \quad \textit{or}$$

   III) $m = 3$ and $\Gamma = \mathbb{S}^2$; or

   IV) $m = 4$, $\Gamma = \mathbb{R}^2$ and the rotation axis l lies in the plane Γ (in particular when $\varphi = \psi$).

2. *Fredholmian in the case of non-fulfillment of all conditions from 1) if and only if their coefficients are non-degenerate on Γ.*

4.7 The problems in 4.6 are of interest even in the case of constant coefficients. Thus let coefficients a and b in (4.15)–(4.17) be constant. If we write the boundary condition in the form

$$\Phi a + \Psi b = g \tag{4.18}$$

then it contains (under suitable arrangement of the upper indices $\pm$) all three boundary conditions. We may easily conclude from Section 1 that the Fredholmian property of the problem with boundary condition (4.18) is equivalent to its unconditional and univalent solvability. And in this case its solution is given by the formulas

$$\begin{aligned} \Phi &= \left(P_1 P_2 \sum_{k=0}^{\infty} (P_2 - P_1)^{2k}[g]\right) a^{-1} \\ \Psi &= \left((I - P_2)(I - P_1) \sum_{k=0}^{\infty} (P_2 - P_1)^{2k}[g]\right) b^{-1} \end{aligned} \tag{4.19}$$

with $P_1 := \frac{1}{2}(I \pm {}^{\varphi}S)$, $P_2 = \frac{1}{2}(I \pm {}^{\psi}S)$. The choice of the signs $\pm$ is corresponds to the arrangement of the $\pm$ in (4.18) pointed out above.

In particular for $a = -b = 1$ the formulas (4.19) give representations of a function $g \in L_2(\Gamma; \mathbb{H})$ as a difference of limit values of functions hyperholomorphic in different meanings, i.e. they solve the "jump problem" for different hyperholomorphies. It is worthwhile to note that by varying the pairs of classes of hyperholomorphy we obtain different representations of a function g including its representations by functions hyperholomorphic in the same domain.

5 Hyperholomorphic Toeplitz operators

5.1 Toeplitz operators belong to the classical objects which connect complex analysis and operator theory. Let us recall the main definitions for the classical one-dimensional case.

Let $\mathbb{B} = \mathbb{B}^2(0,1)$ be the unit disk in $\mathbb{C}$ and let $\Gamma = \mathbb{S}^1$ be its boundary. We say that holomorphic in $\mathbb{B}$ function f belongs to the Hardy space $H_p = H_p(\mathbb{B})$ if

$$\|f\|_{H_p}^p := \sup\left\{\int_0^{2\pi} |f(re^{i\theta})|^p d\theta;\ 0 < r < 1\right\} < +\infty. \tag{5.1}$$

The space H_p is isometrically isomorphic to a certain subspace of the space $L_p(\Gamma)$: if $f \in H_p$ then it is mapped into a function from L_p which coincides almost everywhere on Γ with the nontangential limit values of f. The operator $P := \frac{1}{2}(I + S_\Gamma)$ is the Szego projection of $L_p(\Gamma)$ onto H_p, where

$$S_\Gamma[f](t) := \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} d\tau.$$

Let $a \in L_\infty(\Gamma)$, then acting on H_p the Toeplitz operator T_a with a pre-symbol a is defined by the rule

$$T_a[f] := P[af].$$

More generally let $\mathcal{R}$ be any operator algebra on $L_p(\Gamma)$, and let $A \in \mathcal{R}$. Then the Toeplitz operator T_A with the pre-symbol A is defined by the rule

$$T_A[f] := TA[f].$$

One of the essential problems of the Toeplitz operator theory is to describe the algebras generated by Toeplitz operators with pre-symbols from different algebras.

5.2 Relying on the deep analogy between hyperholomorphic function theory and one-dimensional complex analysis we introduce the notion of hyperholomorphic Toeplitz operators. We assume that $\partial\Omega = \Gamma$ denotes $\mathbb{S}^{m-1}$ or $\mathbb{R}^{m-1}$.

First of all note that the definition of the $\mathbb{H}$-module ${}^{\psi}\mathcal{H}_p(\Omega, \mathbb{H})$ introduced in [SV, 3.10] admits an equivalent definition in terms analoguous to (5.1) (see [BDS], [BVA], [Fö]).

It follows from [VS, Section 5] that the operator ${}^{\psi}P_+ = \frac{1}{2}(I + {}^{\psi}S)$ is the projection of the space $L_p(\Gamma; \mathbb{H})$ onto ${}^{\psi}\mathcal{H}_p(\Omega, \mathbb{H})$.

Let M be a map from ${}^{\psi}\mathcal{H}_p$ into L_p such that

$$T_M := {}^{\psi}P_+ M : {}^{\psi}\mathcal{H}_p(\Omega; \mathbb{H}) \to {}^{\psi}\mathcal{H}_p(\Omega; \mathbb{H}) \tag{5.2}$$

is linear. We call T_M a hyperholomorphic Toeplitz operator with pre-symbol M. Let us show that the study of properties of the operator T_M acting on ${}^{\psi}\mathcal{H}_p(\Omega; \mathbb{H})$ can be reduced to study of the corresponding properties of a special singular integral operator acting on $L_p(\partial\Omega; \mathbb{H})$. Indeed put

$$A_M := {}^{\psi}P_+ \cdot M \cdot {}^{\psi}P_+ + {}^{\psi}P_-. \tag{5.3}$$

It is obvious that the operator ${}^{\psi}P_+ \cdot M \cdot {}^{\psi}P_-$ acts invariantly on ${}^{\psi}\mathcal{H}_p$ and that ${}^{\psi}P_+ \cdot M \cdot {}^{\psi}P_+|{}^{\psi}\mathcal{H}_p = T_M$. The operator ${}^{\psi}P_-$ acts invariantly on L_p^- and ${}^{\psi}P_-|L_p^- = I_{L_p^-}$. We thus have that

$$A_M = T_M \oplus I_{L_p^-}. \tag{5.4}$$

It follows immediately from (5.4) that, for instance:

1. T_M is Fredholm on ${}^{\psi}\mathcal{H}_p(\Omega; \mathbb{H}) \iff A_M$ is Fredholm on $L_p(\Gamma; \mathbb{H})$.
2. T_M is invertible on ${}^{\psi}\mathcal{H}_p(\Omega; \mathbb{H}) \iff A_M$ is invertible on $L_p(\Omega; \mathbb{H})$
3. $A_M(L_p) = T_M({}^{\psi}\mathcal{H}_p) \oplus L_p^-$.

Hence we can describe the structure of the hyperholomorphic Toeplitz operator algebra and Fredholm properties of these operators (for several pre-symbol algebras) with the help of the singular integral operator algebras studied above. We denote a (real) algebra of hyperholomorphic Toeplitz operators by $\mathfrak{T}$, and the corresponding (real) algebra of singular integral operators by $\mathfrak{A}$.

5.3 First let $M := M^a$ with $a \in C(\Gamma; \mathbb{H})$. Note that M^a is not a linear operator in the right $\mathbb{H}$-module $L_p(\Gamma; \mathbb{H})$. Describe the structure of algebras $\mathfrak{T}$ and $\mathfrak{A}$ in this case. It follows from [SV, 5.4] that an arbitrary operator A from $\mathfrak{A}$ has the form

$$A = {}^{\psi}P_+ M^a \cdot {}^{\psi}P_+ + \kappa {}^{\psi}P_- + K \tag{5.5}$$

with $\kappa \in \mathbb{R}$, $a \in C(\Gamma; \mathbb{H})$ and K compact .

Lemma 5.1 *If $a \in C(\Gamma; \mathbb{H})$, $b \in L_\infty(\Gamma; \mathbb{H})$, $T_a := T_{M^a}$, then the operators $T_a \cdot T_b - T_{ba}$ and $T_b \cdot T_a - T_{ab}$ are compact on ${}^{\psi}\mathcal{H}_p(\Gamma; \mathbb{H})$.*

The proof follows from a chain of equalities:

$$T_aT_b = {}^\psi P_+M^a \cdot {}^\psi P_+M^b = {}^\psi P_+M^a\; {}^\psi P_+ \cdot M^b\; {}^\psi P_+|{}^\psi\mathcal{H}_P =$$

$$\begin{aligned} &= {}^\psi P_+({}^\psi P_+ \cdot M^a - K_a)M^b \cdot {}^\psi P_+|{}^\psi\mathcal{H}_P = \\ &= \left({}^\psi P_+ \cdot M^{ba} \cdot {}^\psi P_+ - {}^\psi P_+ \cdot K_aM^b\; {}^\psi P_+\right)|{}^\psi\mathcal{H}_P = T_{ba} + \widetilde{K}, \end{aligned}$$

where $K_a := {}^\psi P_+M^a - M^a \cdot {}^\psi P_+$ and $\widetilde{K}$ are compact operators.

In particular if a is a constant function, then $K_a = 0$ and

$$T_a \cdot T_b = T_{ba}, \quad T_b \cdot T_a = T_{ab}. \tag{5.6}$$

Corollary 5.2 *An arbitrary operator $B \in \mathfrak{I}$ is representable in the form*

$$B = T_a + K \tag{5.7}$$

for a certain function $a \in C(\Gamma;\mathbb{H})$ and a compact operator K.

It is clear that Lemma 5.1 and Corollary 5.2 are true for any Liapunov surface Γ, not only for $\mathbb{S}^{m-1}$ or $\mathbb{R}^{m-1}$.

Using Section 1 as well we obtain

Theorem 5.3 *An operator T_a is Fredholmian on ${}^\psi\mathcal{H}_2(\Gamma;\mathbb{H})$ if and only if $0 \notin a(\Gamma)$.*

5.4 Now let $M = {}^aM$ with $a \in C(\Gamma;\mathbb{H})$. Then in general the operator

$${}_aK := {}^\psi P_+ \cdot {}^aM - {}^aM \cdot {}^\psi P_-$$

is not compact, so that the algebras $\mathfrak{I}$ and $\mathfrak{A}$ do not have such a simple structure as in 5.3. But the connection between ${}_aT := T_{a_M}$ and ${}_aA := {}^\psi P_+{}^aM{}^\psi P_+ + {}^\psi P_-$ remains.

Theorem 5.4 *An operator ${}_aT$ is Fredholmian on ${}^\psi\mathcal{H}_2(\Gamma;\mathbb{H})$ if everywhere on Γ*

$$Re\ a(t) \neq 0. \tag{5.8}$$

Proof. We have for ${}_aA$:

$$\begin{aligned} \text{sym}\ {}_aA &= \frac{1}{4}(1 + i\alpha(t,\xi))a(t)(1 + i\alpha(t,\xi)) + \frac{1}{2}(1 - i\alpha(t,\xi)) = \\ &= \frac{1}{4}((a + 2 - \alpha a\alpha) - i(2\alpha - \alpha a + a\alpha)), \end{aligned}$$

where like in Theorem 4.2, $\alpha(t,\xi)$ is a purely imaginary quaternion with unit length and $\alpha(t,\xi)$ depends on Γ and m. As

$$\begin{aligned} \alpha a + a\alpha &= 2Re\ a\alpha - 2\langle\alpha, Im\ a\rangle_{\mathbb{R}^3}, \\ 2\alpha - \alpha a - a\alpha &= 2\left(\langle\alpha, Im\ a\rangle + (1 - Re\ a)\alpha\right), \\ \alpha a\alpha &= -Re\ a - \langle\alpha, Im\ a\rangle\alpha + Im\ a - \langle\alpha, Im\ a\rangle\alpha, \\ a + 2 - \alpha a\alpha &= 2\left(1 + Re\ a + \langle\alpha, Im\ a\rangle\alpha\right), \end{aligned}$$

then

$$\text{sym }_a A = \frac{1}{2}(z_1 - iz_2)$$

with

$$\begin{aligned} z_1 &:= 1 + Re\ a + \langle \alpha, Im\ a \rangle \alpha, \\ z_2 &:= \langle \alpha, Im\ a \rangle + (1 - Re\ a)\alpha. \end{aligned}$$

Now inequality (4.10) takes the form:

$$\frac{1}{4}((1 + Re\ a)^2 + \langle \alpha, Im\ a \rangle^2 - \langle \alpha, Im\ a \rangle^2 - (1 - Re\ a)^2) -$$
$$-2i \langle ((1 + Re\ a)\langle \alpha, Im\ a \rangle + (1 - Re\ a)\langle \alpha, Im\ a \rangle) =$$
$$= Re\ a + i \langle \alpha, Im\ a \rangle \neq 0.$$

It follows from the properties of $\alpha(t, \xi)$ that for any fixed $t \in \Gamma$ there exists $\xi \in \mathbb{S}^{m-1}$ such that the vector $\alpha(t)\xi$ is orthogonal to the vector $Im\ \ a(t)$. Therefore the invertibility of sym $_a A$ in $\mathbb{H}(\mathbb{C})$ is equivalent to (5.8).

5.5 Now consider the hyperholomorphic Toeplitz operator with the pre-symbol $M = M^a \cdot {}^{\varphi}P_+ + M^b \cdot {}^{\varphi}P_-$ where $\{a, b\} \subset C(\Gamma; \mathbb{H})$, and the structural set φ differs from ψ generally saying. Like in 4.5, for the corresponding operator A_M we have

$$\text{sym } A_M = \begin{pmatrix} a(t)(1 - x) + b(t)x & 0 \\ 0 & 1 \end{pmatrix},$$

when $t \in \Gamma$, $x \in \Delta(t)$. Hence the invertibility of sym A_m is equivalent to

$$a(t)(1 - x) + b(t)x \neq 0 \tag{5.9}$$

for every $t \in \Gamma$, $x \in \Delta(t)$. Consequently we proved

Theorem 5.5 *The operator $T_M := {}^{\psi}P_+(M^a \cdot {}^{\varphi}P_+ + M^b \cdot {}^{\varphi}P_-)$ is Fredholmian on ${}^{\psi}\mathcal{H}_2(\Gamma; \mathbb{H})$ if and only if the condition (5.9) holds for every $t \in \Gamma$ and every $x \in \Delta(t)$ where $\Delta(t)$ is defined as in Lemma 2.1.*

5.6 Now consider hyperholomorphic Toeplitz operators with piece-wise continuous pre-symbols. We use the notations from 3.2 and we denote by $\mathfrak{I}$ the algebra generated by the operators T_a with $a \in PC(\Gamma, \mathcal{L}; \mathbb{H})$.

Theorem 5.6 *The algebra* Sym $\mathfrak{I}$ *is isometrically isomorphic to the algebra $C(\overline{X}; \mathbb{H})$. Moreover the homomorphism*

$$\text{sym} : \mathfrak{I} \longrightarrow \text{Sym } \mathfrak{I}$$

is generated by following mapping of the generators of algebra $\mathfrak{I}$: if $a \in PC(\Gamma, \mathcal{L}; \mathbb{H})$ then

$$\text{sym } T_a = \begin{cases} a(t) & , \quad t \in \Gamma \backslash \mathcal{L}, \\ a^+(t)(1 - x) + a^-(t)x & , \quad t \in \mathcal{L},\ x \in [0, 1]. \end{cases}$$

Corollary 5.7 *An arbitrary operator $T \in \mathfrak{I}$ is Fredholm if and only if its symbol* sym T *is non-degenerated everywhere on X.*

References

[BDS] F. Brackx, R. Delanghe, F. Sommen. *Clifford Analysis.* Boston, Pitman, 1982.

[BVA] F. Brackx, N. Van Acker. H_p–spaces of monogenic functions, *Clifford Algebras and their Applications in Mathemetical Physics*, Kluwer Academic Publishers (1989).

[Dgl] R.G. Douglas. *Banach Algebra Techniques in Operator Theory.* Academic Press, 1972.

[DH] J. Dauns, K. Hofmann. *Representation of rings by sections.* Memoirs Amer. Math. Soc., no. 83, 1968.

[Dx] J. Diximer. *Les C^*–Algebras et Leurs Représentations* Gauthier–Villars, 1969.

[Fö] V. Förster. *Kugelfunktionen in Quaternioneb und Clifford – Algebren.* Ph. D. Thesis, Aachen, 1985.

[G] F.D. Gakhov. *Boundary Value Problems.* Nauka, 1977. (Russian).

[Sim] I.B. Simonenko. New general method of investigation of linear operator equations of the type of singular integral equations I. *Izvestia AN SSSR, ser. matem.* **29** (1965). 567–586 (Russian).

[Sim1] I.B. Simonenko. New general method of investigation of linear operator equations of the type of singular integral equations II. *Izvestia AN SSSR, ser. matem.* **29** (1965). 757–782 (Russian).

[SV] M.V. Shapiro, N.L. Vasilevski. Quaternionic ψ-hyperholomorphic functions, singular integral operators and boundary value problems. I. ψ-hyperholomorphic function theory. *Complex Variable*, this volume.

[VaSp] N.L. Vasilievski, I.M. Spitkovski, On an algebra generated by two projectors. *Doklady Akad Nauk UkrSSR, ser. A*, no. 8 (1981). 10–13 (Russian).

[Vsl] N.B. Vasiliev. C^*-algebras with finite-dimensional irreducible representations. *Uspekhi Mat. Nauk* (Russian Math. Surveys), **21** (1966). 135–154.

[XZ] Xu Zhenyuan. *?????.* Ph. D. Thesis, Gent, 1989.