Group Convolution Operators on Step Two Nilpotent Lie Groups

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1 Introduction

The classical approach to the investigation of convolution operators on the real line is based on the use of the Fourier transformation. The usual step under consideration of group convolution operators on the nilpotent Lie groups is to generalize the classical Fourier transformation up to appropriate group Fourier transformation and to use afterwards the machinery of non–commutative harmonic analysis.

The investigation of group convolution operators on step two nilpotent Lie groups and pseudodifferential operators connected with them has sufficiently long history by this time. We mention here the works [1–3,5,7,8,10–12] only, which have extensive bibliography.

But the classical Fourier transformation gives an example of symplectic change of variables in $T^*\mathbb{R}$ as well. Our aim is to introduce a new technique for the investigation of group convolution operators on step two nilpotent Lie groups, originated on the symplectic nature of the classical Fourier transformation. This technique is based on the direct integral construction, Weyl symbolic calculus for pseudodifferential operators and symplectic changes of variables. It allows to consider at the same time not only both the right and the left convolution operators (which is impossible following non–commutative harmonic analysis line), but it turns out to be useful to consider the so-called two–sided convolutions.

The structure of these group convolution operators is described. The boundedness results extend the known ones onto the case of noncontinuous Fourier image of the convolution operator kernel with respect to the "real" variable $\lambda \in \mathbb{R}^m$.

The groups under consideration appear naturally in the complex analysis. They can be identified with the so–called standard CR–manifolds, and in some cases with the distinguished boundary of type II Siegel domains. The simple and more evident than, for example, in [4,6,9] isomorphic description of the Hardy space and of the Szegö projection associated with the type II Siegel domains is given. For the case of the Heisenberg group the algebra generated by Toeplitz operators associated with the corresponding Siegel domain is described as well.

This work is an expanded version of the talk given at the ISAAC Congress, Newark, Delaware, June 3–7, 1997, and presents a condensed survey of the results of [13,14].

2 Main definitions

Denote by $\mathbb{H}_F$ the space $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{C}^n$, $N = m + 2n$, with the following group operation

$$(t_1, z_1) \cdot (t_2, z_2) = (t_1 + t_2 + 2\,\mathrm{Im}\, F(z_1, z_2), z_1 + z_2),$$

where $(t_k, z_k) \in \mathbb{R}^m \times \mathbb{C}^n$, $k = 1, 2$, and $F(z_1, z_2)$ is a Hermitian form defined on $\mathbb{C}^n$ and valued in $\mathbb{C}^m$.

It is known that the group $\mathbb{H}_F$ is a step two nilpotent Lie group and its (two-sided) Haar measure coincides with the Lebesque measure on $\mathbb{R}^N$. For $m = 1$ and $F(z_1, z_2) = z_1 \cdot \overline{z}_2$ $\mathbb{H}_F$ is just the Heisenberg group $\mathbb{H}_n$.

The noncommutativity of the group operation induces (in contrast to the standard summation operation in $\mathbb{R}^N$) the existence of two shift operators: the right and the left shift operators that are defined on $L_2(\mathbb{H}_n)$ by the rule

$$\begin{aligned}(\pi_r(g)\varphi)(h) &= \varphi(hg), \\ (\pi_l(g)\varphi)(h) &= \varphi(g^{-1}h).\end{aligned}$$

Integrating these shift operators with some kernel functions we arrive at the right and the left group convolution operators

$$\begin{aligned}B &= (2\pi)^{-\frac{N}{2}} \int_{\mathbb{H}_F} b(g)\pi_r(g)dg, \\ C &= (2\pi)^{-\frac{N}{2}} \int_{\mathbb{H}_F} c(g)\pi_l(g)dg.\end{aligned}$$

Moreover, it turns out to be useful to consider the so–called two–sided convolution operator

$$K = (2\pi)^{-N} \int_{\mathbb{H}_F \times \mathbb{H}_F} k(g_1, g_2)\pi_r(g_2)\pi_l(g_2)dg_1dg_2.$$

We will use the following notations for the direct and the dual coordinates of the points of $\mathbb{H}_F = \mathbb{R}^N = \mathbb{R}^m \times \mathbb{C}^n = \mathbb{R}^m \times \mathbb{R}^{2n} = \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n$:

$$g = (t, z) = (t, \widetilde{x}) = (t, x, y)$$

and

$$\gamma = (\lambda, \zeta) = (\lambda, \widetilde{\xi}) = (\lambda, \xi, \eta).$$

Here, for example, $z = x + iy \in \mathbb{C}^n$, $\widetilde{x} = (x, y) \in \mathbb{R}^{2n}$, $t \in \mathbb{R}^m$.

3 Pseudodifferential form of convolution operators

The center of $\mathbb{H}_F$ has the form

$$\mathcal{Z} = \{(t, z) \in \mathbb{H}_F : z = 0\}.$$

All the operators $\pi_l(g_0)$, with $g_0 \in \mathcal{Z}$, commute with both the right and the left shift operators. And so they commute with all bounded on $L_2(\mathbb{H}_F) = L_2(\mathbb{R}^N)$ right, left and two–sided convolution operators.

We have

$$(\pi_l(g_0)f)(t, z) = f(t - t_0, z) = (\tau_{t_0} \otimes I)f,$$

where $(\tau_{t_0}\varphi)(t) = \varphi(t - to)$, is acting on the space $L_2(\mathbb{R}^m)$ shift operator, $L_2(\mathbb{H}_F) = L_2(\mathbb{R}^m) \otimes L_2(\mathbb{C}^n)$.

Commuting with $\pi_l(g_0)$ operators B, C and K should commute with all elements of the von Neumann algebra $\mathfrak{A}$ generated by these shift operators π_l. Hence these convolution operators belong to the commutant $\mathfrak{A}'$ of the algebra $\mathfrak{A}$.

We have

$$\mathfrak{A} = F^{-1}L_\infty(\mathbb{R}^m)F \otimes \mathbb{C} \cdot I$$

and

$$\begin{aligned} \mathfrak{A}' &= F^{-1}L_\infty(\mathbb{R}^m)F \otimes \mathfrak{B}(L_2(\mathbb{C}^n)) \\ &= {}^{(}F^{-1} \otimes I)(L_\infty(\mathbb{R}^m) \otimes \mathfrak{B}(L_2(\mathbb{C}^n)))(F \otimes I), \end{aligned}$$

where $\mathfrak{B}(X)$ is the algebra of all bounded linear operators acting on a Hilbert space X.

Lemma 3.1 *Let $K \in \mathfrak{B}(L_2(\mathbb{H}_F))$ be an operator commuting with all the operators $\pi_l(g_0)$, for $g_0 \in \mathcal{Z}$. Then:*

(i) $K \in (F^{-1} \otimes I)(L_\infty(\mathbb{R}^m) \otimes \mathfrak{B}(L_2(\mathbb{C}^n)))(F \otimes I)$;

(ii) there exists an essentially bounded measurable operator–function

$$K(\cdot) : \lambda \in \mathbb{R}^m \longmapsto K(\lambda) \in \mathfrak{B}(L_2(\mathbb{C}^n)),$$

which is determined by the operator K with respect to a zero–measure set and which determines the following acting on the space $L_2(\mathbb{H}_F) = L_2(\mathbb{R}^m) \otimes L_2(\mathbb{C}^n)$ operator:

$$\widetilde{K}f(\lambda, x) = (I \otimes K(\lambda))f(\lambda, x);$$

(iii) $K = (F^{-1} \otimes I)\widetilde{K}(F \otimes I)$;

(iv) $\|K\| = \text{ess-sup}_{\lambda \in \mathbb{R}^m} \|K(\lambda)\|$.

Remark 3.2 *The values of an operator–function $K(\lambda)$ almost everywhere in $\mathbb{R}^m$ are obtained by applying the partial Fourier transformation with respect to the variable t $(t \to \lambda)$ and by fixing then the value of λ i.e.*

$$I \otimes K(\lambda) = ((F \otimes I)K(F^{-1} \otimes I))(\lambda).$$

Let $(F_1, F_2, \ldots, F_m)$ be the coordinate notation of the vector form F. Here F_ν, $\nu = \overline{1,m}$, are the scalar Hermitian forms in $\mathbb{C}^n$:

$$F_\nu(z_1, z_2) = {z_1}' F_\nu \overline{z}_2, \quad \nu = \overline{1,m}.$$

For $\lambda \in \mathbb{R}^m$ denote by $L(\lambda)$ the following scalar Hermitian form

$$L(\lambda)(z_1, z_2) = \lambda \cdot F(z_1, z_2) = \sum_{\nu=1}^{m} \lambda_\nu \cdot F_\nu(z_1, z_2).$$

Introduce the $2n \times 2n$ (real) matrix

$$\Lambda(\lambda) = \begin{pmatrix} \text{Im } L(\lambda) & -\text{Re } L(\lambda) \\ \text{Re } L(\lambda) & \text{Im } L(\lambda) \end{pmatrix}.$$

Calculations show that for almost all $\lambda \in \mathbb{R}^m$

$$\begin{aligned} K(\lambda)\varphi(\widetilde{x}) &= (2\pi)^{-2n} \int_{\mathbb{R}^{4n}} \hat{k}(-\lambda, -\widetilde{\xi} + 2\Lambda\left(\frac{\widetilde{x}+\widetilde{y}}{2}\right), \\ &\qquad \lambda, \widetilde{\xi} + 2\Lambda\left(\frac{\widetilde{x}+\widetilde{y}}{2}\right)) \cdot e^{i(\widetilde{x}-\widetilde{y})\cdot\widetilde{\xi}} \varphi(\widetilde{y})\, d\widetilde{y}\, d\widetilde{\xi}, \end{aligned}$$

where $\hat{k}$ is the (Euclidean) Fourier transformation of k.

Lemma 3.3 *The Weyl symbols of the operators $B(\lambda)$, $C(\lambda)$ and $K(\lambda)$ (corresponding to the right, left and two-sided convolution operators on $\mathbb{H}_F$) have the forms*

$$\begin{aligned} \omega_{B(\lambda)} &= \hat{b}(-\lambda, -\xi + 2\text{Im } L(\lambda)\overline{z}, -\eta + 2\text{Re } L(\lambda)\overline{z}), \\ \omega_{C(\lambda)} &= \hat{c}(\lambda, \xi + 2\text{Im } L(\lambda)\overline{z}, \eta + 2\text{Re } L(\lambda)\overline{z}), \\ \omega_{K(\lambda)} &= \hat{k}(-\lambda, -\xi + 2\text{Im } L(\lambda)\overline{z}, -\eta + 2\text{Re } L(\lambda)\overline{z}, \\ &\qquad \lambda, \xi + 2\text{Im } L(\lambda)\overline{z}, \eta + 2\text{Re } L(\lambda)\overline{z}). \end{aligned}$$

Theorem 3.4 *The right, left and the two–sided convolution operators are pseudodifferential operators on $L_2(\mathbb{H}_F) = L_2(\mathbb{R}^N)$ having the following Weyl symbols:*

$$\begin{aligned}
\omega_B(t, \widetilde{x}, \lambda, \widetilde{\xi}) &= \hat{b}(-\lambda, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}), \\
\omega_C(t, \widetilde{x}, \lambda, \widetilde{\xi}) &= \hat{c}(\lambda, \widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}), \\
\omega_K(t, \widetilde{x}, \lambda, \widetilde{\xi}) &= \hat{k}(-\lambda, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}, \lambda, \widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}).
\end{aligned}$$

4 Structure of operators $K(\lambda)$

Let a point $\lambda \in \mathbb{R}^m \setminus \{0\}$ be fixed. Consider bounded on $L_2(\mathbb{R}^{2n}) = L_2(\mathbb{C}^n)$ operator

$$K(\lambda)\varphi(\widetilde{x}) = (2\pi)^{-2n} \int_{\mathbb{R}^{4n}} e^{i(\widetilde{x}-\widetilde{y})\cdot\widetilde{\xi}} \cdot \omega_{K(\lambda)}(\frac{1}{2}(\widetilde{x}+\widetilde{y}),\, \widetilde{\xi})\, \varphi(\widetilde{y})\, d\widetilde{y}\, d\widetilde{\xi}$$

with the Weyl symbol

$$\omega_{K(\lambda)}(\widetilde{x}, \widetilde{\xi}) = \hat{k}(-\lambda,\, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x};\, \lambda,\, \widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}).$$

The Hermitian matrix $L(\lambda)$ is homogeneous of the first order

$$L(\lambda) = |\lambda| \cdot L\left(\frac{\lambda}{|\lambda|}\right),$$

using some unitary matrix $U = U\left(\frac{\lambda}{|\lambda|}\right)$ it can be reduced to a diagonal form

$$U(\frac{\lambda}{|\lambda|})\, L(\frac{\lambda}{|\lambda|})\, U(\frac{\lambda}{|\lambda|})^* = \operatorname{diag}\left(\varepsilon_1^2, \ldots, \varepsilon_{p,}^2 - \varepsilon_{p+1}^2, \ldots, -\varepsilon_{n-k,}^2 \underbrace{0, \ldots, 0}_{k}\right).$$

Introduce the following symplectic in $T^*(\mathbb{C}^n) = T^*(\mathbb{R}^{2n})$ changes of variables:

$$z = U' z_1, \qquad \zeta = U' \zeta_1,$$

or

$$\begin{aligned}
x = U_1' x_1 - U_2' y_1, &\qquad y = U_2' x_1 + U_1' y_1, \\
\xi = U_1' \xi_1 - U_2' \eta_1, &\qquad \eta = U_2' \xi_1 + U_1' \eta_1,
\end{aligned}$$

where $U = U_1 + iU_2$,

and

for $j = \overline{1,p}$

$$
\begin{aligned}
x_j &= -\frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\eta_{1,j} + \sqrt{|\lambda|}\varepsilon_j x_{1,j},\\
y_j &= \frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\eta_{1,j} + \sqrt{|\lambda|}\varepsilon_j x_{1,j},\\
\xi_j &= \frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\xi_{1,j} + \sqrt{|\lambda|}\varepsilon_j y_{1,j},\\
\eta_j &= \frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\xi_{1,j} - \sqrt{|\lambda|}\varepsilon_j y_{1,j},
\end{aligned}
$$

for $j = \overline{p+1, n-k}$

$$
\begin{aligned}
x_j &= -\frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\eta_{1,j} - \sqrt{|\lambda|}\varepsilon_j x_{1,j},\\
y_j &= \frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\eta_{1,j} - \sqrt{|\lambda|}\varepsilon_j x_{1,j},\\
\xi_j &= -\frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\xi_{1,j} + \sqrt{|\lambda|}\varepsilon_j y_{1,j},\\
\eta_j &= -\frac{1}{2\sqrt{|\lambda|}\varepsilon_j}\xi_{1,j} - \sqrt{|\lambda|}\varepsilon_j y_{1,j},
\end{aligned}
$$

for $j = \overline{n-k+1, n}$

$$
x_j = \frac{1}{2\sqrt{|\lambda|}}\xi_{1,j}, \quad y_j = \frac{1}{2\sqrt{|\lambda|}}\eta_{1,j},
$$

$$
\xi_j = -2\sqrt{|\lambda|}x_{1,j}, \quad \eta_j = -2\sqrt{|\lambda|}y_{1,j}.
$$

For $z = x + iy \in \mathbb{C}^n$, introduce $x = (x', x'')$, $\;y = (y', y'') \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ and denote by $\xi', \eta' \in \mathbb{R}^{n-k}$ the dual coordinates for x' and y'. Now

$$
L_2(\mathbb{C}^n) = L_2(\mathbb{R}^{n-k}) \otimes L_2(\mathbb{R}^{2k}) \otimes L_2(\mathbb{R}^{n-k}),
$$

and the point

$$
(x', x'', y'', y') \in \mathbb{R}^{n-k} \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^{n-k} \times \mathbb{R}^{2k} \times \mathbb{R}^{n-k}
$$

corresponds to the point $z = x + iy \in \mathbb{C}^n$.

Introduce n–dimensional complex vectors $\hat{z}_r$ and $\hat{z}_l$:

$$\hat{z}_{r,j} = \begin{cases} x_j + i\xi_j & = x'_j + i\xi'_j \\ x_j - i\xi_j & = x'_j - i\xi'_j \\ x_j + iy_j & = x''_{j-(n-k)} + iy''_{j-(n-k)} \end{cases}$$

$$\hat{z}_{l,j} = \begin{cases} y_i - i\eta_j & = y'_j - i\eta'_j \\ y_i + i\eta_j & = y'_j + i\eta'_j \\ x_j + iy_j & = x''_{j-(n-k)} + iy''_{j-(n-k)}, \end{cases}$$

and two $n \times n$ complex matrices

$$\begin{aligned} \hat{U}_r &= \hat{U}_r\left(\frac{\lambda}{|\lambda|}\right) = 2i\,U'\mathrm{diag}\,(\varepsilon_1, \ldots, \varepsilon_{n-k}, i, \ldots, i), \\ \hat{U}_l &= \hat{U}_l\left(\frac{\lambda}{|\lambda|}\right) = 2i\,U'\mathrm{diag}\,(\varepsilon_1, \ldots, \varepsilon_{n-k}, -i, \ldots, -i). \end{aligned}$$

Theorem 4.1 *Let $\lambda \in \mathbb{R}^m \setminus \{0\}$. Then there exists an unitary on $L_2(\mathbb{C}^n)$ operator ν such that the Weyl symbol of the operator $\nu\, K(\lambda)\, \nu^{-1}$ is calculated by the formula*

$$\omega_{\nu\, K(\lambda)\, \nu^{-1}} = \hat{k}\left(-\lambda,\, \sqrt{|\lambda|}\hat{U}_r\left(\frac{\lambda}{|\lambda|}\right)\hat{z}_r,\, \lambda,\, \sqrt{|\lambda|}\hat{U}_l\left(\frac{\lambda}{|\lambda|}\right)\hat{z}_l\right).$$

Corollary 4.2 *In the case of the Heisenberg group $\mathbb{H}_n$ we have $k = 0$,*

$$L_2(\mathbb{C}^n) = L_2(\mathbb{R}^n) \otimes L_2(\mathbb{R}^n)$$

and

$$\begin{aligned} \nu\, B(\lambda)\, \nu^{-1} &= B_\lambda \otimes I, \\ \nu\, C(\lambda)\, \nu^{-1} &= I \otimes C_\lambda, \end{aligned}$$

where the Weyl symbols of the acting on $L_2(\mathbb{R}^n)$ operators B_λ and C_λ are calculated by the formulas

$$\begin{aligned} \omega_{B_\lambda}(x,\xi) &= \hat{b}\,(-\lambda,\, -2\sqrt{|\lambda|}x,\, -2(\mathrm{sign}\,\lambda)\sqrt{|\lambda|}\xi), \\ \omega_{C_\lambda}(y,\eta) &= \hat{c}\,(\lambda,\, 2(\mathrm{sign}\,\lambda)\sqrt{|\lambda|}\eta,\, 2\sqrt{|\lambda|}y), \end{aligned}$$

and where

$$\begin{aligned} \omega_B &= \hat{b}\,(-\lambda,\, -\xi - 2\lambda y,\, -\eta + 2\lambda x), \\ \omega_C &= \hat{c}\,(\lambda,\, \xi - 2\lambda y,\, \eta + 2\lambda x) \end{aligned}$$

are the Weyl symbols of the right B and left C convolution operators on the Heisenberg group $\mathbb{H}_n$.

5 Boundedness of convolution operators

Introduce on $\mathbb{H}_F = \mathbb{R}^m \times \mathbb{C}^n = \mathbb{R}^m \times \mathbb{R}^{2n}$ the one–parameter group of the nonisotropic dilations $\delta_\tau, \tau > 0$,

$$\delta_\tau(\tau, \zeta) = (\tau\lambda, \tau^{\frac{1}{2}}\zeta), \quad (\lambda, \zeta) \in \mathbb{R}^m \times \mathbb{C}^n,$$

or

$$\delta_\tau(\lambda, \widetilde{\xi}) = (\tau\lambda, \tau^{\frac{1}{2}}\widetilde{\xi}), \quad (\lambda, \widetilde{\xi}) \in \mathbb{R}^m \times \mathbb{R}^{2n}.$$

A function ψ on $\mathbb{H}_F$ is called a δ_τ–homogeneous function of the degree $s \in \mathbb{R}$ if

$$\psi \circ \delta_\tau = \tau^s \cdot \psi.$$

The values of an arbitrary δ_τ–homogeneous function ψ is determined uniquely by its values on the nonisotropic unit sphere Ω in $\mathbb{H}_F$

$$\Omega = \{(\lambda, \widetilde{\xi}) \in \mathbb{H}_F = \mathbb{R}^m \times \mathbb{R}^{2n} : \quad \|(\lambda, \widetilde{\xi})\| = 1\},$$

where $\|(\lambda, \widetilde{\xi})\|^2 = \sum\limits_{\nu=1}^{m} |\lambda_\nu|^2 + \sum\limits_{k=1}^{n} (|\xi_k|^4 + |\eta_k|^4)$.

For an algebra $\mathcal{A}$ ($\subset L_\infty(\Omega)$) of functions on Ω denote by $H_\delta^s(\mathcal{A})$ the algebra of all δ_τ–homogeneous of the degree s functions in $\mathbb{H}_F$ having $\mathcal{A}$ restrictions onto Ω.

Theorem 5.1 *Let E be a closed subset in $\mathbb{R}^m$ with the measure zero and with the "cone" property:*

$$\forall \lambda \in E \qquad \forall \tau > 0 \qquad \tau\lambda \in E,$$

and let $E_\Omega = \Omega \cap (E \times \mathbb{R}^{2n})$. Further let $\widetilde{b}(-\lambda, \widetilde{\xi})$, $\widetilde{c}(\lambda, \widetilde{\xi})$ be functions from $H_\delta^0(C^\infty(\Omega \backslash E_\Omega))$, and $\widetilde{k}(-\lambda, \widetilde{\xi}, \lambda, \widetilde{\xi}) \in H_\delta^0(C^\infty(\Omega \backslash E_\Omega)) \otimes H_\delta^0(C^\infty(\Omega \backslash E_\Omega))$. Then pseudodifferential operators B, C and K with the Weyl symbols

$$\begin{aligned} \omega_B(\widetilde{x}, \lambda, \widetilde{\xi}) &= \widetilde{b}(-\lambda, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}), \\ \omega_C(\widetilde{x}, \lambda, \widetilde{\xi}) &= \widetilde{c}(\lambda, \widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}) \end{aligned}$$

and

$$\omega_K(\widetilde{x}, \lambda, \widetilde{\xi}) = \widetilde{k}(-\lambda, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}; \lambda, \widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}),$$

respectively, are bounded on $L_2(\mathbb{H}_F)$.

Each this operator, say $A \in \{B, C, K\}$, generates an operator–function

$$A(\cdot) : \quad \mathbb{R}^m \to \mathfrak{B}(L_2(\mathbb{R}^{2n}))$$

which is essentially bounded and measurable in $\mathbb{R}^m$ *and is strongly continuous in* $\mathbb{R}^m \backslash E$. *The operator*

$$\widetilde{A} f(\lambda, \widetilde{\xi}) = (I \otimes A(\lambda)) f(\lambda, \widetilde{\xi})$$

defined by $A(\cdot)$ *is bounded on* $L_2(\mathbb{R}^m \times \mathbb{R}^{2n})$ *and is unitary equivalent to the operator* A:

$$A = (F^{-1} \otimes I)\widetilde{A}(F \otimes I),$$

where F *and* F^{-1} *are direct and inverse Fourier transformations on* $L_2(\mathbb{R}^m)$.

And

$$\|A\| = \sup_{\lambda \in \mathbb{R}^m \backslash E} \|A(\lambda)\|.$$

Remark 5.2 *The Weyl symbol of the operator* $A(\lambda)$ *for* $\lambda \in \mathbb{R}^m \backslash E$ *is calculated by the formula*

$$\omega_{A(\lambda)}(\widetilde{x}, \widetilde{\xi}) = \omega_A(\widetilde{x}, \lambda, \widetilde{\xi}).$$

Remark 5.3 *For this important case, when the functions* $\widetilde{b}$, $\widetilde{c}$ *and* $\widetilde{k}$ *are* δ_τ*–homogeneous of zero order, the formulas of Theorem 4.1 and Corollary 4.2 acquire more simple and nice form*

$$\begin{aligned}
\omega_{B_\lambda}(x, \xi) &= \widetilde{b}\,(-\mathrm{sign}\,\lambda,\ -2x,\ -2(\mathrm{sign}\,\lambda)\xi), \\
\omega_{C_\lambda}(y, \eta) &= \widetilde{c}\,(\mathrm{sign}\,\lambda,\ 2(\mathrm{sign}\,\lambda)\eta,\ 2y), \\
\omega_{\nu\, K(\lambda)\, \nu^{-1}} &= \widetilde{k}\left(-\frac{\lambda}{|\lambda|},\ \hat{U}_r\left(\frac{\lambda}{|\lambda|}\right)\hat{z}_r,\ \frac{\lambda}{|\lambda|},\ \hat{U}_l\left(\frac{\lambda}{|\lambda|}\right)\hat{z}_l\right).
\end{aligned}$$

6 Hardy space and Szegö projection associated with Siegel domains

Introduce the type II Siegel domain

$$D = \{(w, z) \in \mathbb{C}^{m+n} :\ \ \mathrm{Im}\ w - F(z, z) \in V\},$$

its distinguished boundary has the form

$$B = \{(w, z) \in \mathbb{C}^{m+n} :\ \ \mathrm{Im}\ w - F(z, z) = 0\}.$$

The mapping

$$\kappa_0 :\ \ (t, z) \in \mathbb{R}^m \times \mathbb{C}^n \ \ \longmapsto\ \ (t + iF(z, z),\ z) \in B$$

identifies $\mathbb{R}^m \times \mathbb{C}^n = \mathbb{H}_F$ with B, and the corresponding mapping

$$\kappa : \; L_2(B) \;\longrightarrow\; L_2(\mathbb{R}^m \times \mathbb{C}^n) = L_2(\mathbb{H}_F),$$

where $(\kappa f)(t,z) = f(\kappa_0(t,z))$, transfers isometrically the Hardy space $H^2(D)$ onto some subspace $H^2(\mathbb{H}_F) = \kappa(H^2(D))$ of $L_2(\mathbb{H}_F)$.

The space $H^2(\mathbb{H}_F)$ is the closure in $L_2(\mathbb{H}_F)$ of a linear vector space of functions which satisfy to the tangential Cauchy–Riemann equations on $\mathbb{H}_F$:

$$\overline{Z} f_0 = \left(\frac{\partial}{\partial \overline{z}} - i\, \overline{L(\frac{\partial}{\partial t})\, z} \right) f_0 = 0.$$

The orthogonal projection $P : \; L_2(\mathbb{H}_F) \;\longrightarrow\; H^2(\mathbb{H}_F)$ is a right group convolution operator on $\mathbb{H}_F$

$$P = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{H}_F} p(g)\, \pi_r(g)\, dg$$

with the Weyl symbol

$$\omega_P(\widetilde{x}, \lambda, \widetilde{\xi}) = \hat{p}(-\lambda, \, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}),$$

where $\hat{p}$ is a Fourier transformation of p, or (in the complex form)

$$\omega_P(z, \lambda, \zeta) = \hat{p}(-\lambda, \, 2i\, \omega_{\overline{Z}}(z, \lambda, \zeta)),$$

where $\omega_{\overline{Z}}(z, \lambda, \zeta) = \frac{1}{2} i\zeta + \overline{L(\lambda)} z$ is a Weyl symbol of the tangential Cauchy–Riemann operator $\overline{Z}$.

The operator P generates essentially bounded measurable operator–function

$$P(\cdot) : \; \mathbb{R}^m \;\longrightarrow\; \mathfrak{B}(L_2(\mathbb{C}^n))$$

such that

$$P = (F^{-1} \otimes I)\, (I \otimes P(\lambda))\, (F \otimes I),$$

and for all $\lambda \in \overline{\mathbb{R}^m \setminus V^*}$ $P(\lambda) = 0$, for $\lambda \in$ Int V^* the Weyl symbol of the operator $P(\lambda)$ has the form

$$\omega_{P(\lambda)}(z, \zeta) = \hat{p}(-\lambda, \, 2i\, \omega_{\overline{Z}(\lambda)}(z, \zeta)),$$

where $\omega_{\overline{Z}(\lambda)}(z, \zeta)$ is a Weyl symbol of the operator

$$\overline{Z}(\lambda) = \frac{\partial}{\partial \overline{z}} + \overline{L(\lambda)} z,$$

and V^* is the dual cone to the cone V.

The operator $P(\lambda)$ is an orthogonal projection of $L_2(\mathbb{C}^n)$ onto the space $H^2(\lambda)$ which is a closure in $L_2(\mathbb{C}^n)$ of a linear vector space of functions satisfying to the equation $\overline{Z}(\lambda)\varphi = 0$.

We have

$$\nu \circ \overline{Z}(\lambda) \circ \nu^{-1} = \sqrt{|\lambda|}\, E \left(\frac{\partial}{\partial x} + x \right),$$

and the operator ν maps isometrically $H^2(\lambda)$ onto the space $\nu_1(\lambda)\,(H^2(\lambda)) = H_1^2(\lambda)$ which is the closure in $L_2(\mathbb{C}^n)$ of a linear vector space of functions satisfying to the equation

$$\left(\frac{\partial}{\partial x} + x \right) \varphi(x, y) = 0.$$

Its general solution has the form

$$\pi^{-\frac{n}{4}} e^{-\frac{x^2}{2}} \cdot g(y),$$

where $g(y) \in L_2(\mathbb{R}^n)$. The function $l(x) = \pi^{-\frac{n}{4}} e^{-\frac{x^2}{2}} \in L_2(\mathbb{R}^n)$ and $\|l(x)\| = 1$.

Denote by L_0 the one–dimensional subspace of $L_2(\mathbb{R}^n)$ generated by $l(x)$. Then $H_1^2(\lambda) = L_0 \otimes L_2(\mathbb{R}^n)$ and $P_1(\lambda) = P_0 \otimes I$, where

$$(P_0 f)(x) = \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x^2+u^2)} f(u)\, du$$

is the orthogonal projection of $L_2(\mathbb{R}^n)$ onto L_0.

Theorem 6.1 *There exists an unitary operator ν_0 which gives an isometrical isomorphism of the space*

$$L_2(\mathbb{R}^m \times \mathbb{C}^n) = L_2(\mathbb{R}^m) \otimes L_2(\mathbb{R}^n) \otimes L_2(\mathbb{R}^n)$$

under which

(i) the space $H^2(\mathbb{H}_F)$ is mapped onto

$$L_2(V^*) \otimes L_0 \otimes L_2(\mathbb{R}^n);$$

(ii) $\nu P \nu^{-1} = \chi_{V^*}(\lambda) I \otimes P_0 \otimes I$, where $\chi_{V^*}(\lambda)$ is the characteristic function of the dual cone V^*.

Corollary 6.2 *The operator $\nu_D = \nu_0 \circ \kappa$ maps isometrically $L_2(B)$ onto*

$$L_2(\mathbb{R}^m) \otimes L_2(\mathbb{R}^n) \otimes L_2(\mathbb{R}^n)$$

and under this mapping we have

(i) $\nu_D(H^2(D)) = L_2(V^*) \otimes L_0 \otimes L_2(\mathbb{R}^n)$;

(ii) $\nu_D P_D \nu_D^{-1} = \chi_{V^*}(\lambda) I \otimes P_0 \otimes I$.

7 Toeplitz operators on the Heisenberg group

Consider
$$D = \{(w,z) \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Im}\ w - z \cdot \overline{z} > 0\},$$
its distinguished boundary
$$B = \{(w,z) \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Im}\ w - z \cdot \overline{z} = 0\}$$
after identification with $\mathbb{R} \times \mathbb{C}^n$ coincides with the Heisenberg group $\mathbb{H}_n$.

Denote by χ_+ the characteristic function of the "upper half sphere", i.e.
$$\chi_+(\lambda,\xi,\eta) = \begin{cases} 1, & \lambda \geq 0 \\ 0, & \lambda < 0, \end{cases}$$
where $(\lambda,\xi,\eta) \in \Omega$, and by $PC^\infty(\Omega)$ the algebra generated by χ_+ and functions from $C^\infty(\Omega)$.

And let $\mathcal{R} = \mathcal{R}(H^\circ_\delta(PC(\Omega)))$ the C^*–algebra generated by all the operators B and C with
$$\begin{aligned} \omega_B(\widetilde{x},\lambda,\widetilde{\xi}) &= \widetilde{b}(-\lambda, -\widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}), \\ \omega_C(\widetilde{x},\lambda,\widetilde{\xi}) &= \widetilde{c}(\lambda, \widetilde{\xi} + 2\Lambda(\lambda)\widetilde{x}), \end{aligned}$$
where $\widetilde{b}, \widetilde{c} \in H^0_\delta(PC^\infty(\Omega))$.

The algebra $\mathcal{R}$ is generated also by all operators K with
$$\widetilde{k} \in H^0_\delta(PC^\infty(\Omega)) \otimes H^0_\delta(PC^\infty(\Omega)).$$

Denote by $CS(\mathbb{R}^k)$ the algebra of functions in $\mathbb{R}^k$ generated by
$$s_0(x) = \frac{1}{\sqrt{1+x^2}}, \quad s_j = \frac{x_j}{\sqrt{1+x^2}}, \quad j = \overline{1,k}, \quad x \in \mathbb{R}^k,$$
and let $\Psi(CS(\mathbb{R}^{2n}))$ be the C^*–algebra generated by all acting on the space $L_2(\mathbb{R}^n)$ operators with (sufficiently smooth) Weyl symbols $a(x,\xi) \in CS(\mathbb{R}^{2n})$.

Denote by $\mathcal{T}(\mathcal{R})$ the algebra generated by all Toeplitz operators
$$T_K : \quad f \in H^2(\mathbb{H}_n) \longmapsto PKf \in H^2(\mathbb{H}_n)$$
with presymbols K from the algebra $\mathcal{R} = \mathcal{R}(H^0_\delta(PC(\Omega)))$.

Theorem 7.1 *The algebra $\mathcal{T}(\mathcal{R})$ is isometrically isomorphic to the algebra $\Psi(CS(\mathbb{R}^{2n}))$. The isomorphism*

$$\tau : \quad \mathcal{T}(\mathcal{R}) \to \Psi(CS(\mathbb{R}^{2n}))$$

is generated by the following mapping of generators of $\mathcal{T}(\mathcal{R})$:

$$\tau : \quad T_K \longmapsto \hat{K},$$

where the Weyl symbol of the operator $\hat{K} \in \Psi(CS(\mathbb{R}^{2n}))$ is calculated by the formula

$$\omega_{\hat{K}}(y, \eta) = \pi^{-n} \int_{\mathbb{R}^{2n}} e^{-(x^2+\xi^2)} \widetilde{k}(-1, -2x, -2\xi; 1, 2\eta, 2y) dx d\xi,$$

where

$$\omega_K(\widetilde{x}, \lambda, \widetilde{\xi}) = \widetilde{k}(-\lambda, -\xi - 2\lambda y, -\eta + 2\lambda x; \lambda, \xi - 2\lambda y, \eta + 2\lambda x)$$

is the Weyl symbol of the operator $K \in \mathcal{R}(H^0_\delta(PC(\Omega)))$.

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