Operator Theory
Advances and Applications
v. 117, 2000, p. 371-386

POLY–FOCK SPACES*

N. L. VASILEVSKI

Consider the space $L_2(\mathbb{C}^n, d\mu_n)$, where $d\mu_n$ is the Gaussian measure, and its Fock subspace $F^2(\mathbb{C}^n)$ consisting of all analytic (entire) functions in $\mathbb{C}^n$. We introduce the so-called true-poly-Fock spaces, and prove that $L_2(\mathbb{C}^n, d\mu_n)$ is the direct sum of the Fock and all true-poly-Fock spaces.

The structure of these spaces and connections between them are described. The orthogonal (Bargmann type) projections onto true-poly-Fock spaces are given.

1 Introduction

We will use the following standard notations: $z = x + iy = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\overline{z} = (\overline{z}_1, \ldots, \overline{z})$ with the usual notion of the complex conjugation, for $z, w \in \mathbb{C}^n$ $z \cdot w = z_1 w_1 + \ldots + z_n w_n$, $|z|^2 = |z_1|^2 + \ldots + |z_n|^2$ $(= z \cdot \overline{z})$. Denote by $d\mu_n(z)$ the following Gaussian measure over $\mathbb{C}^n$

$$d\mu_n(z) = \pi^{-n} e^{-z \cdot \overline{z}} dv(z),$$

where $dv(z) = dxdy$ is the usual Euclidean volume measure on $\mathbb{C}^n = \mathbb{R}^{2n}$.

Introduce the Hilbert space $L_2(\mathbb{C}^n, d\mu_n)$ of μ_n square-integrable functions on $\mathbb{C}^n$ with the following inner product

$$\langle f, \varphi \rangle = \int_{\mathbb{C}^n} f(z) \, \overline{\varphi(z)} \, d\mu_n(z).$$

The closed subspace $F^2(\mathbb{C}^n)$ of $L_2(\mathbb{C}^n, d\mu_n)$ consisting of all analytic (entire) functions is usually called the Fock [3, 4], or the Segal–Bargmann space [1, 6].

The orthogonal Bargmann projection

$$P_n : L_2(\mathbb{C}^n, d\mu_n) \to F^2(\mathbb{C}^n)$$

*This work was partially supported by CONACYT Project 3114P-E9607, México.

is given by the formula [3]

$$(P_n\varphi)(z) = \int_{\mathbb{C}^n} \varphi(\zeta) e^{\overline{\zeta}\cdot z} d\mu_n(\zeta).$$

Note, that for each natural numbers k and l, with $n = k + l$, we have

$$\begin{aligned} L_2(\mathbb{C}^n, d\mu_n) &= L_2(\mathbb{C}^k, d\mu_k) \otimes L_2(\mathbb{C}^l, d\mu_l), \\ F^2(\mathbb{C}^n) &= F^2(\mathbb{C}^k) \otimes F^2(\mathbb{C}^l), \end{aligned}$$

and

$$P_n = P_k \otimes P_l.$$

Thus it is natural to start with the spaces over one-dimensional complex space $\mathbb{C}$. To have then results for the spaces over $\mathbb{C}^n$ one needs just "to tensor" the one-dimensional results.

In the article we answer the following questions:

- being a subspace of $L_2(\mathbb{C}^n, d\mu_n)$, how much room does the Fock space $F^2(\mathbb{C}^n)$ occupy inside $L_2(\mathbb{C}^n, d\mu_n)$,
- does there exist a direct sum decomposition of $L_2(\mathbb{C}^n, d\mu_n)$ onto analytic type spaces,
- what are the structure of these pieces of decomposition, and what are the connections between them,
- how do the orthogonal projections onto these pieces look like,
- does there exist an unitary operator *simultaneously* reducing all these pieces and corresponding projections to some simple transparent form?

It is fitting to note here that it was M. G. Krein himself, who first called my attention to the beauty of the Fock spaces and Berezin machinery, and kindly provided me with abundant information on these topics.

2 One-dimensional case

Consider the space $L_2(\mathbb{C}, d\mu)$ of square-integrable functions on $\mathbb{C}$ with respect to the Gaussian measure

$$d\mu(z) = \pi^{-1} e^{-z\cdot\overline{z}} dv(z),$$

where $dv(z) = dxdy$ is the Euclidean volume measure on $\mathbb{C} = \mathbb{R}^2$, and its Fock subspace $F^2(\mathbb{C})$, consisting of all analytic in $\mathbb{C}$ functions.

Besides the Fock space $F^2(\mathbb{C})$, which can be defined alternatively as the closure of the set of all smooth functions satisfying the (Cauchy–Riemann) equation

$$\frac{\partial\varphi}{\partial\overline{z}} = 0,$$

introduce the poly-Fock spaces, i.e., given $k \in \mathbb{N}$, the k-Fock space $F_k^2(\mathbb{C})$ is the closure of the set of all smooth functions from $L_2(\mathbb{C}, d\mu)$ satisfying the equation

$$\left(\frac{\partial}{\partial \overline{z}}\right)^k \varphi = 0.$$

Introduce the unitary operator

$$U_1 : L_2(\mathbb{C}, d\mu) \to L_2(\mathbb{R}^2) = L_2(\mathbb{R}^2, dxdy),$$

by the rule

$$(U_1\varphi)(z) = \pi^{-\frac{1}{2}} e^{-\frac{z\cdot\overline{z}}{2}} \varphi(z),$$

or

$$(U_1\varphi)(x,y) = \pi^{-\frac{1}{2}} e^{-\frac{x^2+y^2}{2}} \varphi(x+iy).$$

Then the image $F_k^{(1)} = U_1(F_k^2(\mathbb{C}))$ of the k-Fock space $F_k^2(\mathbb{C})$ is the set of all smooth functions in $L_2(\mathbb{R}^2)$ which satisfy the equation

$$\begin{aligned} D_k^{(1)} f = U_1 \left(\frac{\partial}{\partial \overline{z}}\right)^k U_1^{-1} f &= \left(\frac{\partial}{\partial \overline{z}} + \frac{z}{2}\right)^k f \\ &= \frac{1}{2^k}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + x + iy\right)^k f = 0. \end{aligned}$$

The unitary operator $U_2 = I \otimes F$, where

$$(F\psi)(y) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-i\eta\cdot y} \psi(\eta)\; d\eta$$

is the Fourier transformation, maps isometrically the space

$$L_2(\mathbb{R}^2, dxdy) = L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}, dy)$$

onto itself. The image $F_k^{(2)} = (I \otimes F)(F_k^{(1)})$ of the space $F_k^{(1)}$ under the mapping $I \otimes F$ is the closure of the set of all smooth functions in $L_2(\mathbb{R}^2)$ which satisfy the equation

$$\begin{aligned} D_k^{(2)} f &= (I \otimes F) D_k^{(1)} (I \otimes F^{-1}) f = \\ &= \frac{1}{2^k}\left(\frac{\partial}{\partial x} - y + x - \frac{\partial}{\partial y}\right)^k f = 0. \end{aligned}$$

Finally introduce the isomorphism

$$U_3 = U_3^* = U_3^{-1} : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2),$$

where

$$(U_3 f)(x,y) = f\left(\frac{1}{\sqrt{2}}(x+y), \frac{1}{\sqrt{2}}(x-y)\right).$$

The operator U_3 maps the space $F_k^{(2)}$ onto the space $F_k^{(3)}$, which is the closure of the set of the smooth functions satisfying the equation

$$D_k^{(3)} f = U_3 D_k^{(2)} U_3^{-1} f = 2^{-k/2} \left(\frac{\partial}{\partial y} + y \right)^k f = 0.$$

This last equation can be easily solved. Its general solution has the form

$$e^{-y^2/2} \sum_{j=0}^{k-1} y^j d_j(x),$$

where $d_j(x)$, $j = \overline{0, k-1}$, are function from $L_2(\mathbb{R}, dx)$. Or, rearranging the polinomials on y, we have

$$\sum_{j=0}^{k-1} (2^j j! \sqrt{\pi})^{-1/2} \, H_j(y) \, e^{-y^2/2} \, g_j(x),$$

where

$$H_j(y) = (-1)^j e^{y^2} \left(\frac{d}{dy} \right)^j e^{-y^2} = j! \sum_{m=0}^{[j/2]} \frac{(-1)^m (2y)^{j-2m}}{m! \, (j-2m)!} \tag{2.1}$$

is the Hermite polinomial of degree j (see, for example [2, 7]), and $g_j(x)$, $j = \overline{0, k-1}$, are function from $L_2(\mathbb{R}, dx)$.

The functions

$$h_j(y) = (2^j j! \sqrt{\pi})^{-1/2} \, H_j(y) \, e^{-y^2/2}, \qquad j = 0, 1, 2, ... \tag{2.2}$$

form an orthonormal basis in $L_2(\mathbb{R})$. Denote by H_j the one-dimensional subspace of $L_2(\mathbb{R})$, generated by the function $h_j(y)$, and by Q_j the one-dimensional orthogonal projection of $L_2(\mathbb{R})$ onto H_j, which is given obviously by

$$(Q_j \psi)(y) = h_j(y) \int_{\mathbb{R}} \psi(\eta) \, h_j(\eta) \, d\eta. \tag{2.3}$$

Let

$$H_k^{\oplus} = \bigoplus_{j=o}^{k-1} H_j.$$

Now we have

$$F_k^{(3)} = L_2(\mathbb{R}) \otimes H_k^{\oplus}.$$

The above work leads up to the following theorem.

Theorem 2.1 *The unitary operator $U = U_3 U_2 U_1$ provides an isometric isomorphism of the space $L_2(\mathbb{C}, d\mu)$ onto the space $L_2(\mathbb{R}^2) = L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}, dy)$, under which the k-Fock space $F_k^2(\mathbb{C})$ is mapped onto $L_2(\mathbb{R}) \otimes H_k^{\oplus}$*

$$U : F_k^2(\mathbb{C}) \longrightarrow L_2(\mathbb{R}) \otimes H_k^{\oplus}.$$

It is convenient to introduce the spaces

$$\begin{aligned} F^2_{(k)}(\mathbb{C}) &= F^2_k(\mathbb{C}) \ominus F^2_{k-1}(\mathbb{C}), \qquad \text{for } \ k > 1, \\ F^2_{(1)}(\mathbb{C}) &= F^2_1(\mathbb{C}) = F^2(\mathbb{C}), \qquad \text{for } \ k = 1. \end{aligned}$$

We will call the space $F^2_{(k)}(\mathbb{C})$ the *true-k-Fock space.* Introduce as well the true-k-Bargmann projection $P_{(k)}$ as the orthogonal projection on $L_2(\mathbb{C})$ with image $F^2_{(k)}(\mathbb{C})$.

Corollary 2.2 *Under the isometric isomorphism*

$$U : L_2(\mathbb{C}, d\mu) \longrightarrow L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}, dy)$$

1. *the true-k-Fock space $F^2_{(k)}(\mathbb{C})$ is mapped onto $L_2(\mathbb{R}) \otimes H_{k-1}$*

$$U : F^2_{(k)}(\mathbb{C}) \longrightarrow L_2(\mathbb{R}) \otimes H_{k-1},$$

2. *the true-k-Bargmann projection $P_{(k)}$ is unitary equivalent to the following one*

$$UP_{(k)}U^{-1} = I \otimes Q_{k-1},$$

where the one-dimensional projection Q_{k-1} is given by (2.3).

Corollary 2.3 *For the Fock space $F^2(\mathbb{C})$ we have*

$$\begin{aligned} U : F^2(\mathbb{C}) &\longrightarrow L_2(\mathbb{R}) \otimes H_0, \\ UPU^{-1} &= I \otimes Q_0. \end{aligned}$$

Corollary 2.4 *We have the following decomposition of the space $L_2(\mathbb{C}, d\mu)$*

$$L_2(\mathbb{C}, d\mu) = \bigoplus_{k=1}^{\infty} F^2_{(k)}(\mathbb{C}).$$

All the true-k-Fock spaces $F^2_{(k)}(\mathbb{C})$ are isomorphic, one to each other, and each of them is naturally isomorphic to $L_2(\mathbb{R})$. Describe these isomorphisms. Given $k \in \mathbb{N}$, introduce the isometric imbedding

$$R_k : L_2(\mathbb{R}, dx) \longrightarrow L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}, dy)$$

defined by

$$R_k : g(x) \longmapsto g(x) \cdot h_{k-1}(y),$$

where the function $h_{k-1}(y)$ is given by (2.2). The image of R_k is exactly $L_2(\mathbb{R}) \otimes H_{k-1}$. Then the adjoint operator

$$R^*_k : L_2(\mathbb{R}^2) \longrightarrow L_2(\mathbb{R})$$

is defined by

$$(R_k^* f)(x) = \int_{\mathbb{R}} f(x,y)\, h_{k-1}(y)\, dy.$$

And we have

$$\begin{aligned} R_k^* R_k = I \quad &: \quad L_2(\mathbb{R}, dx) \longrightarrow L_2(\mathbb{R}, dx), \\ R_k R_k^* = I \otimes Q_k \quad &: \quad L_2(\mathbb{R}^2) \longrightarrow L_2(\mathbb{R}) \otimes H_{k-1}. \end{aligned}$$

Now, the operator $\widetilde{R}_k = R_k^* U$ maps the space $L_2(\mathbb{C}, d\mu)$ onto $L_2(\mathbb{R}, dx)$ and the restriction

$$\widetilde{R}_k|_{F^2_{(k)}(\mathbb{C})} : F^2_{(k)}(\mathbb{C}) \longrightarrow L_2(\mathbb{R}, dx)$$

is an isometric isomorphism.

The adjoint operator

$$\widetilde{R}_k^* = U^* R_k : L_2(\mathbb{R}, dx) \longrightarrow F^2_{(k)}(\mathbb{C}) \subset L_2(\mathbb{C}, d\mu)$$

maps isomorphically and isometrically the space $L_2(\mathbb{R}, dx)$ onto the true-k-Fock space $F^2_{(k)}(\mathbb{C})$.

Theorem 2.5 *The isomorphism*

$$\widetilde{R}_k^* : L_2(\mathbb{R}, dx) \longrightarrow F^2_{(k)}(\mathbb{C})$$

is given by the formula

$$(\widetilde{R}_k^* g)(z) = (2^{k-1}(k-1)!\sqrt{\pi})^{-1/2} \int_{\mathbb{R}} e^{-(z^2+\eta^2)/2+\sqrt{2}z\eta}\, H_{k-1}(\frac{z+\overline{z}}{\sqrt{2}} - \eta)\, g(\eta)\, d\eta,$$

where $H_{k-1}(y)$ is the Hermite polinomial (2.1).

Proof. Calculate

$$\begin{aligned} (\widetilde{R}_k^* g)(z) &= (U_1^{-1} U_2^{-1} U_2^{-1} R_k g)(z) \\ &= U_1^{-1}(I \otimes F^{-1}) g(\frac{x+y}{\sqrt{2}}) h_{k-1}(\frac{x-y}{\sqrt{2}}) \\ &= \pi^{1/2} e^{(x^2+y^2)/2}\, (2\pi)^{-1/2} \int_{\mathbb{R}} e^{iy\eta}\, g(\frac{x+\eta}{\sqrt{2}})\, h_{k-1}(\frac{x-\eta}{\sqrt{2}})\, d\eta \\ &= (2^{k-1}(k-1)!\sqrt{\pi})^{-1/2} \int_{\mathbb{R}} e^{x^2/2+y^2/2+iy(\sqrt{2}x-\eta)-(\sqrt{2}x-\eta)^2/2}\, g(\eta)\, H_{k-1}(\sqrt{2}x-\eta)\, d\eta \\ &= (2^{k-1}(k-1)!\sqrt{\pi})^{-1/2} \int_{\mathbb{R}} e^{-(z^2+\eta^2)/2+\sqrt{2}z\eta}\, H_{k-1}(\frac{z+\overline{z}}{\sqrt{2}} - \eta)\, g(\eta)\, d\eta. \end{aligned}$$

□

Corollary 2.6 ([1]) *For the Fock space case the isomorphism*

$$\widetilde{R}^* = \widetilde{R}_1^* : L_2(\mathbb{R}, dx) \longrightarrow F^2(\mathbb{C})$$

is given by the formula

$$(\widetilde{R}^* g)(z) = \pi^{-1/4} \int_{\mathbb{R}} e^{-(z^2+\eta^2)/2+\sqrt{2}z\eta}\, g(\eta)\, d\eta.$$

Remark 2.7 The operators $\widetilde{R}_k$ and $\widetilde{R}^*_k$ provide the following decompositions of the true-k-Bargmann projection $P_{(k)}$ and of the identity operator on $L_2(\mathbb{R})$

$$\begin{aligned} P_{(k)} = \widetilde{R}^*_k \widetilde{R}_k \ &: \ L_2(\mathbb{C}, d\mu) \longrightarrow F^2_{(k)}(\mathbb{C}), \\ I = \widetilde{R}_k \widetilde{R}^*_k \ &: \ L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}). \end{aligned}$$

To describe the isomorphisms between the true-poly-Fock spaces $F^2_{(k)}(\mathbb{C})$ start with the spaces $F_{(k)} = L_2(\mathbb{R}) \otimes H_{k-1}$. In $L_2(\mathbb{R}, dy)$ introduce the creation and annihilation operators

$$a^\dagger = \frac{1}{\sqrt{2}}(-\frac{d}{dy} + y), \qquad a = \frac{1}{\sqrt{2}}(\frac{d}{dy} + y),$$

and, in the space $L_2(\mathbb{R}^2) = L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}, dy)$, the operators

$$\mathbf{a}^\dagger = I \otimes a^\dagger, \qquad \mathbf{a} = I \otimes a.$$

It is known (see, for example, [7]), that on functions $h_k(y)$ of the form (2.2) we have

$$\begin{aligned} a^\dagger h_k(y) &= \sqrt{k+1}\, h_{k+1}(y), \\ a\, h_k(y) &= \sqrt{k}\, h_{k-1}(y). \end{aligned}$$

Thus the operator

$$\frac{1}{\sqrt{k}} \mathbf{a}^\dagger|_{F_{(k)}} : F_{(k)} \longrightarrow F_{(k+1)}$$

is an isometric isomorphism, and the operator

$$\frac{1}{\sqrt{k}} \mathbf{a}|_{F_{(k+1)}} : F_{(k+1)} \longrightarrow F_{(k)}.$$

is its inverse. Iterating these isomorphisms we get the following lemma.

Lemma 2.8 *Given natural numbers $k < l$, the operator*

$$\sqrt{\frac{(k-1)!}{(l-1)!}}\, (\mathbf{a}^\dagger)^{l-k}|_{F_{(k)}} : F_{(k)} \longrightarrow F_{(l)}$$

is an isometric isomorphism, and the operator

$$\sqrt{\frac{(k-1)!}{(l-1)!}}\, \mathbf{a}^{l-k}|_{F_{(l)}} : F_{(l)} \longrightarrow F_{(k)}$$

is its inverse.

Introduce now the operators

$$\mathfrak{a}^\dagger = -\frac{\partial}{\partial z} + \overline{z}, \qquad \mathfrak{a} = \frac{\partial}{\partial \overline{z}} \tag{2.4}$$

in the space $L_2(\mathbb{C}, d\mu)$.

Theorem 2.9 *Given natural numbers $k < l$, the isometric isomorphism between the true-poly-Fock spaces $F^2_{(k)}(\mathbb{C})$ and $F^2_{(l)}(\mathbb{C})$ is given by the operator*

$$\sqrt{\frac{(k-1)!}{(l-1)!}}\,(\mathfrak{a}^\dagger)^{\,l-k}|_{F^2_{(k)}(\mathbb{C})} : F^2_{(k)}(\mathbb{C}) \longrightarrow F^2_{(l)}(\mathbb{C}),$$

and the operator

$$\sqrt{\frac{(k-1)!}{(l-1)!}}\,\mathfrak{a}^{l-k}|_{F^2_{(l)}(\mathbb{C})} : F^2_{(l)}(\mathbb{C}) \longrightarrow F^2_{(k)}(\mathbb{C})$$

gives the inverse isomorphism.

Proof. We have $F^2_{(k)}(\mathbb{C}) = U^{-1}(F_{(k)})$, for each $k \in \mathbb{N}$, and the direct calculation shows that

$$\mathfrak{a}^\dagger = U^{-1}\mathbf{a}^\dagger U, \qquad \mathfrak{a} = U^{-1}\mathbf{a}\,U.$$

To finish the proof apply Lemma 2.8. □

Corollary 2.10 *The true-k-Fock space $F^2_{(k)}(\mathbb{C})$ consists of all the functions of the form*

$$\psi(z) = \psi(z,\overline{z}) = \sum_{m=0}^{k-1}(-1)^m \frac{\sqrt{(k-1)!}}{m!\,(k-1-m)!}\,\overline{z}^{k-1-m}\,\varphi^{(m)}(z), \tag{2.5}$$

where $\varphi(z) \in F^2(\mathbb{C})$, $\varphi^{(m)}$ is the m-th derivative of the (analytic) function φ, and

$$\|\psi\|_{F^2_{(k)}(\mathbb{C})} = \|\varphi\|_{F^2(\mathbb{C})}.$$

3 Basic operators

Besides the operators (2.4) introduce

$$\overline{\mathfrak{a}}^\dagger = -\frac{\partial}{\partial \overline{z}} + z, \qquad \overline{\mathfrak{a}} = \frac{\partial}{\partial z} \tag{3.1}$$

in the space $L_2(\mathbb{C}, d\mu)$.

Lemma 3.1 *Functions from $F^2_{(k)}(\mathbb{C})$ admit the following representation*

$$\begin{aligned}\psi(z,\overline{z}) &= \frac{1}{\sqrt{(k-1)!}}\,\langle f(\zeta), [(\overline{\mathfrak{a}}^\dagger_z)^{k-1}\,e^{\zeta\overline{z}}]\rangle \\ &= \frac{1}{\sqrt{(k-1)!}}\int_{\mathbb{C}} f(\zeta)\,[(-\frac{\partial}{\partial z}+\overline{z})^{k-1}e^{\overline{\zeta}z}]\,d\mu(\zeta),\end{aligned} \tag{3.2}$$

where $f \in L_2(\mathbb{C}, d\mu)$.

PROOF. For each $m \in \mathbb{N}$ and $z \in \mathbb{C}$, the functions $\zeta^m e^{\zeta\overline{z}}$ and $\overline{\zeta}^m e^{\overline{\zeta}z}$ belong to $L_2(\mathbb{C}, d\mu)$. Really

$$\begin{aligned} \|\zeta^m e^{\zeta\overline{z}}\|_{L_2(\mathbb{C},d\mu)} &= \|\overline{\zeta}^m e^{\overline{\zeta}z}\|_{L_2(\mathbb{C},d\mu)} = \int_{\mathbb{C}} |\zeta|^{2m}\, e^{\zeta\overline{z}+\overline{\zeta}z}\, d\mu(\zeta) \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} (\xi^2+\eta^2)^m\, e^{-(\xi^2+\eta^2)+\xi x+\eta y}\, d\xi d\eta < \infty, \end{aligned}$$

where $\zeta = \xi + i\eta$, $z = x + iy$. To calculate the exact value of the norm see, for example, [5] formula 3.462.2.

Prove now that all functions of the form (3.2) belong to $F^2_{(k)}(\mathbb{C})$. We have

$$\begin{aligned} (-\frac{\partial}{\partial z}+\overline{z})^{k-1} e^{\overline{\zeta}z} &= \sum_{m=0}^{k-1} (-1)^m \frac{(k-1)!}{m!\,(k-1-m)!}\overline{z}^{k-1-m}\overline{\zeta}^m e^{\overline{\zeta}z} \\ &= (\overline{z}-\overline{\zeta})^{k-1} e^{\overline{\zeta}z}. \end{aligned}$$

Thus the integral (3.2) exists for all $z \in \mathbb{C}$ as the scalar product of two $L_2(\mathbb{C}, d\mu)$ functions. Further,

$$\begin{aligned} \psi(z,\overline{z}) &= \frac{1}{\sqrt{(k-1)!}} \int_{\mathbb{C}} f(\zeta)\, [(-\frac{\partial}{\partial z}+\overline{z})^{k-1} e^{\overline{\zeta}z}]\, d\mu(\zeta) \\ &= \frac{1}{\sqrt{(k-1)!}} \int_{\mathbb{C}} f(\zeta) \sum_{m=0}^{k-1}(-1)^m \frac{(k-1)!}{m!\,(k-1-m)!}\overline{z}^{k-1-m}\overline{\zeta}^m e^{\overline{\zeta}z}\, d\mu(\zeta) \\ &= \sum_{m=0}^{k-1}(-1)^m \frac{\sqrt{(k-1)!}}{m!\,(k-1-m)!}\overline{z}^{k-1-m}\, \varphi_m(z), \end{aligned}$$

where

$$\varphi_m(z) = \int_{\mathbb{C}} f(\zeta)\, \overline{\zeta}^m e^{\overline{\zeta}z}\, d\mu(\zeta), \qquad m = \overline{0, k-1}.$$

The functions $\varphi_m(z)$ are well defined as the scalar product of two $L_2(\mathbb{C}, d\mu)$ functions, $\varphi_0(z) \in F^2(\mathbb{C})$, and standard arguments show that $\varphi_m(z) = \varphi_0^{(m)}(z)$. Thus by (2.5) the function $\psi(z,\overline{z})$ belongs to $F^2_{(k)}(\mathbb{C})$.

Finally, each function (2.5) can be represent in the form (3.2) with $f = \varphi$

$$\begin{aligned} \psi(z) = \psi(z,\overline{z}) &= \sum_{m=0}^{k-1}(-1)^m \frac{\sqrt{(k-1)!}}{m!\,(k-1-m)!}\, \overline{z}^{k-1-m}\, \varphi^{(m)}(z) \qquad (3.3) \\ &= \frac{1}{\sqrt{(k-1)!}}\, \langle \varphi(\zeta), [(\overline{\mathfrak{a}}_z^\dagger)^{k-1}\, e^{\zeta\overline{z}}]\rangle. \end{aligned}$$

□

For each $k \in \mathbb{N}$ introduce the system of functions

$$\begin{aligned} q_z^{\{k\}}(\zeta) &= \frac{1}{(k-1)!}\, (\mathfrak{a}_\zeta^\dagger)^{k-1}\, (\overline{\mathfrak{a}}_z^\dagger)^{k-1}\, e^{\zeta\overline{z}} \qquad (3.4) \\ &= \frac{1}{(k-1)!} \left(-\frac{\partial}{\partial\zeta}+\overline{\zeta}\right)^{k-1} \left(-\frac{\partial}{\partial\overline{z}}+z\right)^{k-1} e^{\zeta\overline{z}}, \quad \zeta \in \mathbb{C}, \qquad (3.5) \end{aligned}$$

parameterized by points $z \in \mathbb{C}$.

Lemma 3.2 *Functions $q_z^{\{k\}}(\zeta)$ belong to $F^2_{(k)}(\mathbb{C})$, have the form*

$$q_z^{\{k\}}(\zeta) = e^{\zeta\overline{z}} p_{k-1}((z-\zeta)(\overline{\zeta}-\overline{z})), \tag{3.6}$$

for some real coefficient polinomials $p_{k-1}(\lambda)$, and

$$q_\zeta^{\{k\}}(z) = \overline{q_z^{\{k\}}(\zeta)}.$$

PROOF. Function

$$\begin{aligned} g(\zeta) &= \frac{1}{\sqrt{(k-1)!}}\left(-\frac{\partial}{\partial \overline{z}} + z\right)^{k-1} e^{\zeta\overline{z}} \\ &= \sum_{m=0}^{k-1} (-1)^m \frac{\sqrt{(k-1)!}}{m!\,(k-1-m)!} z^{k-1-m}\zeta^m e^{\zeta\overline{z}} = (z-\zeta)^{k-1} e^{\zeta\overline{z}} \end{aligned}$$

is analytic and belongs to $L_2(\mathbb{C}, d\mu)$, and thus belongs to $F^2(\mathbb{C})$. Now by Lemma 2.8 the function

$$q_z^{\{k\}}(\zeta) = \frac{1}{\sqrt{(k-1)!}} (\mathfrak{a}^\dagger)^{k-1} g(\zeta)$$

belongs to $F^2_{(k)}(\mathbb{C})$. The following equalities

$$\left(-\frac{\partial}{\partial\zeta} + \overline{\zeta}\right)\left(-\frac{\partial}{\partial\overline{z}} + z\right) e^{\zeta\overline{z}} = e^{\zeta\overline{z}}\left((z-\zeta)(\overline{\zeta}-\overline{z}) + 1\right),$$

$$\begin{aligned} \left(-\frac{\partial}{\partial\zeta} + \overline{\zeta}\right)\left(-\frac{\partial}{\partial\overline{z}} + z\right) e^{\zeta\overline{z}}(z-\zeta)^m(\overline{\zeta}-\overline{z})^m &= e^{\zeta\overline{z}}\left((z-\zeta)^{m+1}(\overline{\zeta}-\overline{z})^{m+1}\right. \\ &\quad + (2m+1)(z-\zeta)^m(\overline{\zeta}-\overline{z})^m + m^2(z-\zeta)^{m-1}(\overline{\zeta}-\overline{z})^{m-1} \end{aligned}$$

prove the second statement of Lemma. The last statement follows from the representation (3.6). $\square$

Theorem 3.3 *The operator*

$$(P_{(k)}f)(z) = \langle f(\zeta), q_z^{\{k\}}(\zeta)\rangle = \int_{\mathbb{C}} f(\zeta)\, q_\zeta^{\{k\}}(z)\, d\mu(\zeta)$$

is the orthogonal Bargmann projection of $L_2(\mathbb{C}, d\mu)$ onto the true-k-Fock space $F^2_{(k)}(\mathbb{C})$.

PROOF. Consider

$$L_2(\mathbb{C}, d\mu) = F^2_{(k)}(\mathbb{C}) \oplus (F^2_{(k)}(\mathbb{C}))^\perp.$$

By Lemma 3.6 $q_z^{\{k\}}(\zeta) \in F^2_{(k)}(\mathbb{C})$, thus for functions $f \in (F^2_{(k)}(\mathbb{C}))^\perp$ we have obviously

$$(P_{(k)}f)(z) = \langle f(\zeta), q_z^{\{k\}}(\zeta)\rangle = 0,$$

and thus $(F^2_{(k)}(\mathbb{C}))^{\perp} \subset \ker P_{(k)}$.

Consider now functions from $F^2_{(k)}(\mathbb{C})$. Using (2.5), Lemma 2.8, and (3.3), we have

$$\begin{aligned}(P_{(k)}\psi)(z) &= \langle \psi(\zeta), q_z^{\{k\}}(\zeta)\rangle \\ &= \langle \frac{1}{\sqrt{(k-1)!}}(\mathfrak{a}^{\dagger})^{k-1}\varphi(\zeta), \frac{1}{\sqrt{(k-1)!}}(\mathfrak{a}^{\dagger}_{\zeta})^{k-1}\frac{1}{\sqrt{(k-1)!}}(\overline{\mathfrak{a}}^{\dagger}_{z})^{k-1}e^{\zeta\overline{z}}\rangle \\ &= \langle \varphi(\zeta), \frac{1}{\sqrt{(k-1)!}}(\overline{\mathfrak{a}}^{\dagger}_{z})^{k-1}e^{\zeta\overline{z}}\rangle = \psi(z).\end{aligned}$$

Thus $P_{(k)}|_{(F^2_{(k)}(\mathbb{C}))^{\perp}} = 0$, and $P_{(k)}|_{F^2_{(k)}(\mathbb{C})} = I$. □

Corollary 3.4 *The system of functions $\{q_z^{\{k\}}(\zeta)\}_{z\in\mathbb{C}}$, given by (3.4), forms a system of coherent states on the true-k-Fock space $F^2_{(k)}(\mathbb{C})$.*

Return now to the operators (2.4) and (3.1). Operators from each pair

$$\mathfrak{a}^{\dagger} = -\frac{\partial}{\partial z} + \overline{z}, \qquad \mathfrak{a} = \frac{\partial}{\partial \overline{z}},$$

or

$$\overline{\mathfrak{a}}^{\dagger} = -\frac{\partial}{\partial \overline{z}} + z, \qquad \overline{\mathfrak{a}} = \frac{\partial}{\partial z} \tag{3.7}$$

are (formally) adjoint one to another in the space $L_2(\mathbb{C}, d\mu)$, and satisfy the commutation relations for the quantum mechanic creation and annihilation operators. But their action in the space

$$L_2(\mathbb{C}, d\mu) = \bigoplus_{k=1}^{\infty} F^2_{(k)}(\mathbb{C})$$

are quite different. According to Theorem 2.9 the operators from the first pair move the true-poly-Fock spaces, while each true-poly-Fock space is invariant with respect to the operators from the second pair. In the last case corresponding vacuum vector in the true-k-Fock space $F^2_{(k)}(\mathbb{C})$ has the form

$$\Phi_0^{(k)}(z) = \frac{1}{\sqrt{(k-1)!}}\,\overline{z}^{k-1}.$$

Remark 3.5 The analytic nature of the Fock space $F^2(\mathbb{C})$ implies that $\overline{\mathfrak{a}}^{\dagger}|_{F^2(\mathbb{C})} = z$, which coincides with the usual definition of the creation operator in the Fock space. But for true-k-Fock spaces $F^2_{(k)}(\mathbb{C})$ with $k > 1$ one has to consider the operators (3.7).

4 Spaces over $\mathbb{C}^n$

Return now to the spaces over $\mathbb{C}^n$. Both the space $L_2(\mathbb{C}^n, d\mu_n)$ and its Fock subspace $F^2(\mathbb{C}^n)$ are the following n-times tensor products of the corresponding spaces over $\mathbb{C}$

$$\begin{aligned} L_2(\mathbb{C}^n, d\mu_n) &= L_2(\mathbb{C}, d\mu) \otimes ... \otimes L_2(\mathbb{C}, d\mu), \\ F^2(\mathbb{C}^n) &= F^2(\mathbb{C}) \otimes ... \otimes F^2(\mathbb{C}). \end{aligned} \tag{4.1}$$

For a multi-index $k = (k_1, k_2, ..., k_n)$, with $k_j \in \mathbb{N}$, $j = \overline{1,n}$, define as usual $|k| = k_1 + ... + k_n$, $k! = k_1! \cdot ... \cdot k_n!$, and $z^k = z_1^{k_1} \cdot ... \cdot z_n^{k_n}$, for $z = (z_1, ..., z_n) \in \mathbb{C}^n$.

Given multi-index $k = (k_1, k_2, ..., k_n)$, introduce the true-k-Fock space $F^2_{(k)}(\mathbb{C}^n)$ as the following tensor product of the true-poly-Fock spaces over $\mathbb{C}$

$$F^2_{(k)}(\mathbb{C}^n) = \bigotimes_{j=1}^{n} F^2_{(k_j)}(\mathbb{C}),$$

and denote by $P_{(k)}$ the orthogonal (Bargmann) projection of $L_2(\mathbb{C}^n, d\mu_n)$ onto the true-k-Fock space $F^2_{(k)}(\mathbb{C}^n)$.

For the multi-index $1 = (1, 1, ..., 1)$ we have

$$F^2_{(1)}(\mathbb{C}^n) = F^2_{(1)}(\mathbb{C}) \otimes ... \otimes F^2_{(1)}(\mathbb{C}) = F^2(\mathbb{C}) \otimes ... \otimes F^2(\mathbb{C}) = F^2(\mathbb{C}^n).$$

Preserve the notations for the following analogs of the previously used unitary operators:

$$U_1 : L_2(\mathbb{C}^n, d\mu_n) \longrightarrow L_2(\mathbb{R}^{2n}) = L_2(\mathbb{R}^{2n}, dxdy),$$

defined by the rule

$$(U_1\varphi)(z) = \pi^{-\frac{n}{2}} e^{-\frac{z\cdot\overline{z}}{2}} \varphi(z);$$

$$U_2 = I \otimes F : L_2(\mathbb{R}^{2n}, dxdy) = L_2(\mathbb{R}^n, dx) \otimes L_2(\mathbb{R}^n, dy) \longrightarrow L_2(\mathbb{R}^n, dx) \otimes L_2(\mathbb{R}^n, dy),$$

where

$$(F\psi)(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\eta\cdot y} \psi(\eta)\ d\eta;$$

and

$$U_3 = U_3^* = U_3^{-1} : L_2(\mathbb{R}^{2n}) \longrightarrow L_2(\mathbb{R}^{2n}),$$

defined by the rule

$$(U_3 f)(x, y) = f(\frac{1}{\sqrt{2}}(x + y), \frac{1}{\sqrt{2}}(x - y)).$$

Given multi-index $k = (k_1, k_2, ..., k_n)$, introduce the function

$$\widetilde{h}_k(y) = \prod_{j=1}^{n} h_{k_j}(y_j),$$

where the functions $h_{k_j}(y_j)$ are given by (2.2).

The functions $\widetilde{h}_k(y)$, $|k| = \overline{0,\infty}$, form obviously an orthonormal basis in $L_2(\mathbb{R}^n)$. Denote by $\widetilde{H}_k$ the one-dimensional subspace of $L_2(\mathbb{R}^n)$ generated by the function $\widetilde{h}_k(y)$, then the operator

$$\widetilde{Q}_k = \bigotimes_{j=1}^{n} Q_{k_j},$$

where each Q_{k_j} is the one-dimensional projection (2.3) acting on the j^{th} multiple of the decomposition (4.1), is the one-dimensional orthogonal projection of $L_2(\mathbb{R}^n)$ onto $\widetilde{H}_k$, and has the following form

$$(\widetilde{Q}_k\psi)(y) = \widetilde{h}_k(y) \int_{\mathbb{R}^n} \psi(\eta)\, \widetilde{h}_k(\eta)\, d\eta.$$

Theorem 4.1 *The unitary operator $U = U_3U_2U_1$ provides an isometric isomorphism of the space $L_2(\mathbb{C}^n, d\mu_n)$ onto the space $L_2(\mathbb{R}^{2n}) = L_2(\mathbb{R}^n, dx) \otimes L_2(\mathbb{R}^n, dy)$, under which*

1. *the true-k-Fock space $F^2_{(k)}(\mathbb{C})$ is mapped onto $L_2(\mathbb{R}^n) \otimes \widetilde{H}_{k-1}$*

$$U : F^2_{(k)}(\mathbb{C}^n) \longrightarrow L_2(\mathbb{R}^n) \otimes \widetilde{H}_{k-1},$$

2. *the true-k-Bargmann projection $P_{(k)}$ is unitary equivalent to the following one*

$$UP_{(k)}U^{-1} = I \otimes \widetilde{Q}_{k-1},$$

where $k-1 = (k_1-1, k_2-1, ..., k_n-1)$.

The following direct sum decomposition of the space $L_2(\mathbb{C}^n, d\mu_n)$ holds

$$L_2(\mathbb{C}^n, d\mu_n) = \bigoplus_{|k|=n}^{\infty} F^2_{(k)}(\mathbb{C}^n).$$

Introduce now multidimensional analogs of the operators (2.4) and (3.1), i.e., the two pairs of the operators

$$\mathfrak{a}^{\dagger} = (-\frac{\partial}{\partial z_1} + \overline{z}_1, -\frac{\partial}{\partial z_2} + \overline{z}_2, ..., -\frac{\partial}{\partial z_n} + \overline{z}_n), \qquad \mathfrak{a} = (\frac{\partial}{\partial \overline{z}_1}, \frac{\partial}{\partial \overline{z}_2}, ..., \frac{\partial}{\partial \overline{z}_n})$$

and

$$\overline{\mathfrak{a}}^{\dagger} = (-\frac{\partial}{\partial \overline{z}_1} + z_1, -\frac{\partial}{\partial \overline{z}_2} + z_2, ..., -\frac{\partial}{\partial \overline{z}_n} + z_n), \qquad \overline{\mathfrak{a}} = (\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, ..., \frac{\partial}{\partial z_n}),$$

acting in the space $L_2(\mathbb{C}^n, d\mu_n)$.

Given multi-index $k = (k_1, k_2, ..., k_n)$, the system of functions

$$q_z^{\{k\}}(\zeta) = \frac{1}{(k-1)!}\, (\mathfrak{a}^{\dagger}_{\zeta})^{k-1}\, (\overline{\mathfrak{a}}^{\dagger}_{z})^{k-1}\, e^{\zeta\cdot\overline{z}}, \qquad \zeta \in \mathbb{C}^n,$$

parameterized by points $z \in \mathbb{C}^n$, forms a system of coherent states in the true-k-Fock space $F^2_{(k)}(\mathbb{C}^n)$. The expressions $(\mathfrak{a}^{\dagger}_{\zeta})^{k-1}$ and $(\overline{\mathfrak{a}}^{\dagger}_{z})^{k-1}$ are understood in the usual multi-index sense, i.e., for example,

$$(\mathfrak{a}^{\dagger}_{\zeta})^{k-1} = \bigotimes_{j=1}^{n} (-\frac{\partial}{\partial \overline{z}_j} + z_j)^{k_j-1}.$$

Theorem 4.2 *All true-poly-Fock spaces are isomorphic, one to each other (and each of them is isomorphic to $L_2(\mathbb{R}^n)$). The isometric isomorphism between the Fock space $F^2(\mathbb{C}^n)$ and the true-k-Fock space $F^2_{(k)}(\mathbb{C}^n)$ is given by the operator*

$$\frac{1}{\sqrt{(k-1)!}}(\mathfrak{a}^\dagger)^k|_{F^2(\mathbb{C}^n)} : F^2(\mathbb{C}^n) \longrightarrow F^2_{(k)}(\mathbb{C}^n),$$

and the operator

$$\frac{1}{\sqrt{(k-1)!}}\,\mathfrak{a}^k|_{F^2_{(k)}(\mathbb{C}^n)} : F^2_{(k)}(\mathbb{C}^n) \longrightarrow F^2(\mathbb{C}^n)$$

gives the inverse isomorphism.

The true-k-Fock space $F^2_{(k)}(\mathbb{C}^n)$ consists of all the functions of the form

$$\psi(z) = \psi(z, \overline{z}) = \sum_{|m|\le k-1} (-1)^{|m|} \frac{\sqrt{(k-1)!}}{m!\,(k-1-m)!}\,\overline{z}^{\,k-1-m}\,\partial^m \varphi(z),$$

where $\varphi(z) \in F^2(\mathbb{C})$,

$$\partial^m \varphi = \frac{\partial^{|m|}\varphi}{\partial z_1^{m_1}\partial z_2^{m_2}...\partial z_n^{m_n}},$$

and

$$\|\psi\|_{F^2_{(k)}(\mathbb{C}^n)} = \|\varphi\|_{F^2(\mathbb{C}^n)}.$$

The operator

$$(P_{(k)}f)(z) = \langle f(\zeta),\, q_z^{\{k\}}(\zeta)\rangle = \int_{\mathbb{C}^n} f(\zeta)\, q_\zeta^{\{k\}}(z)\, d\mu_n(\zeta)$$

is the orthogonal Bargmann projection of $L_2(\mathbb{C}^n, d\mu_n)$ onto the true-k-Fock space $F^2_{(k)}(\mathbb{C}^n)$.

References

[1] V. Bargmann. On a Hilbert space of analytic functions. *Comm. Pure Appl. Math.*, 3:215–228, 1961.

[2] Harry Bateman and Arthur Erdélyi. *Higher transcendental functions, Vol. 2.* McGraw-Hill, 1954.

[3] F. A. Berezin. Covariant and contravariant symbols of operators. *Math. USSR Izvestia*, 6:1117–1151, 1972.

[4] V. A. Fock. Konfigurationsraum und zweite Quantelung. *Z. Phys.*, 75:622–647, 1932.

[5] I. S. Gradshteyn and I. M. Ryzhik. *Tables of Integrals, Series, and Products.* Academic Press, New York, 1980.

[6] I. E. Segal. *Lectures at the Summer Seminar on Appl. Math.* Boulder, Colorado, 1960.

[7] Sundaram Thangavelu. *Lectures on Hermitte and Laguerre expansions.* Princeton University Press, Preiceton, New Jersey, 1993.

Departamento de Matemáticas
CINVESTAV del I.P.N.
Apartado Postal 14-740
07000 México, D.F.
México
e-mail: nvasilev@math.cinvestav.mx

AMS Classification: 46E20, 46E22, 81S05