Integral Equation and
Operator Theory
31, 1998, pp. 113–132

C^*-ALGEBRAS GENERATED BY ORTHOGONAL PROJECTIONS AND THEIR APPLICATIONS*

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Dedicated to the memory of M. G. Krein.

We study the following problem: Given a Hilbert space H and a set of orthogonal projections P, Q_1, ..., Q_n on it, with the conditions $Q_j \cdot Q_k = \delta_{j,k} Q_k, \quad j,k = \overline{1,n}$, $\sum_{k=1}^{n} Q_k = I$, describe the C^*-algebra $\mathcal{C}^*(P, Q_1, ..., Q_n)$ generated by these projections.

Applications to Naimark dilation theorems and to Toeplitz operators associated with the Heisenberg group are given.

1 Preliminaries

1.1 Let a Hilbert space H is given in two different ways as a direct sum of its subspaces

$$H = L \oplus L^{\perp} = M \oplus M^{\perp}.$$

Denote by P the orthogonal projection on H with image L, and by Q the orthogonal projection on H with image M. The subspace

$$H_0 = (L \cap M) \oplus (L^{\perp} \cap M) \oplus (L \cap M^{\perp}) \oplus (L^{\perp} \cap M^{\perp})$$

is invariant with respect to the projections, and both the restrictions of the projections onto H_0 and the (commutative) C^*–algebra generated by these restrictions have the obvious structure. Thus it is natural to exclude the subspace H_0 from the analysis and assume from the very beginning that H_0 is trivial.

Due to P. Halmos [10], the projections P and Q are in *generic position* if the space H_0 is trivial, or, equivalently, they satisfy the conditions

$$\operatorname{Im} P \cap \operatorname{Im} Q = \{0\}, \qquad \ker P \cap \operatorname{Im} Q = \{0\},$$
$$\operatorname{Im} P \cap \ker Q = \{0\}, \qquad \ker P \cap \ker Q = \{0\}.$$

*This work was partially supported by CONACYT Project 3114P-E9608, México.

In this case all the spaces L, $L^{\perp}$, M and $M^{\perp}$ have the same dimension.

The structure of the C^*-algebra with identity generated by two *generic position* orthogonal projections is well understood (see, for example, [8, 10, 12, 21]). The only information one needs to describe this algebra is the spectrum $\mathrm{sp}\,(P-Q)^2$, which *a priori* is a compact subset of $[0,1]$.

Theorem *The C^*–algebra generated by P and Q is isomorphic and isometric to the algebra of all 2×2 matrix–valued functions continuous on $\mathrm{sp}\,(P-Q)^2$ and diagonal at the points of $\{0,1\}\cap \mathrm{sp}\,(P-Q)^2$. This isomorphism is generated by the following mapping:*

$$P \longmapsto \left(\begin{array}{cc} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{array}\right), \tag{1.1}$$

$$Q \longmapsto \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right), \tag{1.2}$$

where $x \in \mathrm{sp}\,(P-Q)^2 \subset [0,1]$.

1.2 The situation becomes unpredictable in the general case of the C^*-algebra generated by a finite set of orthogonal projections (and even in the case of just three projections!).

Theorem (Ch. Davis [5], 1955) *Let H be a separable Hilbert space. There exist three orthogonal projections P_1, P_2 and P_3 on it such that the von Neumann algebra $\mathcal{W}^*(P_1,P_2,P_3)$ generated by them coincides with the algebra $\mathcal{L}(H)$ of all bounded linear operators acting on H.*

Theorem (V. Sunder [17], 1988) *Let H be a separable Hilbert space. Then for each n there exist n orthogonal projections P_1, ..., P_n on it such that*

(i) $\mathcal{W}^*(P_1,...,P_n) = \mathcal{L}(H)$,

(ii) $\mathcal{W}^*(S) \subset \mathcal{L}(H)$, *whenever* $S \subset \{P_1,...,P_n\}$.

1.3 Consider now a special, important for applications case when all the projections but one mutually commute, or (which is equivalent but with a bit more simple setting) when all of these projections but one are mutually orthogonal. Start with three (independent) projections.

Let a Hilbert space H is given in two different ways as a direct sum of its subspaces

$$H = L \oplus L^{\perp} = M_1 \oplus M_2 \oplus M_3,$$

P is the orthogonal projection on H with image L, and Q_k are the orthogonal projection on H with image M_k, $k=\overline{1,3}$.

Example. (E. Ramirez de Arellano, N. Vasilevski [14], 1996).

Let ℓ_1 and ℓ_2 be two arcs on the unit circle $\mathbb{T}$ having or no having common boundary point.

The operators $Q_1 = \chi_{\ell_1}(t)I$ and $Q_2 = \chi_{\ell_2}(t)I$, where $\chi_{\ell_k}(t)$ is the characteristic function of ℓ_k, $k=1,2$, are orthogonal projections on $L_2(\mathbb{T})$ with

$$Q_1\cdot Q_2 = Q_2\cdot Q_1 = 0.$$

Denote by $\mathcal{R} = \mathcal{C}^*(P, Q_1, Q_2)$ the C^*–algebra with identity generated by the orthogonal projections

$$P = \frac{1}{2}(I + S_{\mathbb{T}}), \quad Q_1 \text{ and } Q_2,$$

where $S_{\mathbb{T}}$ is the one-dimensional singular integral operator with the Cauchy kernel, i.e. P is the Szegö projection onto the Hardy space associated with the unit disk.

Theorem *The identical representation*

$$id: \quad \mathcal{R} \longrightarrow \mathcal{L}(L_2(\mathbb{T}))$$

of the algebra $\mathcal{R}$ is irreducible. It contains the ideal $\mathcal{K}$ of all compact operators, but the quotient algebras $\mathcal{R}/\mathcal{K}$ (for the cases, when the arcs ℓ_1 and ℓ_2 have or do not have common boundary point) are quite different, as well as the system of their finite–dimensional irreducible representations.

Note that in both these cases $\mathcal{W}^*(P, Q_1, Q_2) = \mathcal{L}(L_2(\mathbb{T}))$.

EXAMPLE. The Choi Algebra (M.–D. Choi [1], 1979).

In a Hilbert space $H = H_0 \oplus H_1$ with $H_1 = H_\alpha \oplus H_\beta$ and $\dim H_0 = \dim H_1 = \dim H_\alpha = \dim H_\beta$ consider two *unitary* operators U and V such that

$$U : \begin{cases} H_0 \to H_1 \\ H_1 \to H_0 \end{cases}, \quad V : \begin{cases} H_0 \to H_\alpha \\ H_\alpha \to H_\beta \\ H_\beta \to H_0 \end{cases},$$

and $U^2 = I$, $V^3 = I$.

Introduce the orthogonal projections

$$P = 1/2(I + U)$$

and

$$Q_k = 1/3(I + e^{2\pi i(k-1)/3}V + e^{4\pi i(k-1)/3}V^2), \quad k = \overline{1,3}.$$

We have

$$Q_j \cdot Q_k = \delta_{j,k} Q_j, \quad Q_1 + Q_2 + Q_3 = I,$$

and the C^*-algebra $\mathcal{C}^*(U, V)$ generated by U and V coincides with the C^*-algebra with identity $\mathcal{C}^*(P, Q_1, Q_2)$ generated by the projections P, Q_1 and Q_2.

Theorem *The algebra $\mathcal{C}^*(U, V)$ (and thus $\mathcal{C}^*(P, Q_1, Q_2)$) is a simple and algebraically unique C^*-algebra.*

1.4 The aim of this paper is to shed light on the general situation in the above mentioned specific case. We study the following problem:

Consider a Hilbert space H and a finite set of orthogonal projections P, Q_1, ..., Q_n on it, with the conditions

$$Q_j \cdot Q_k = \delta_{j,k} Q_k, \quad j, k = \overline{1,n}, \quad Q_1 + \ldots + Q_n = I,$$
$$\operatorname{Im} P \cap \ker Q_k = \{0\}, \quad \operatorname{Im} Q_k \cap \ker P = \{0\}, \quad k = \overline{1,n}.$$

Describe the C^*-algebra $\mathcal{R} = \mathcal{C}^*(P, Q_1, ..., Q_n)$ generated by these projections.

It turns out that the structure of the algebra $\mathcal{R}$ is determined by the mutual properties of some n positive injective contractions C_k with $\sum_{k=1}^n C_k = I$, and thus by the structure of the C^*-algebra generated by them. The principal difference between the case of two projections and the general case of a finite set of projection is now completely understandable: for $n = 2$ we have only *one* contraction and the spectral theorem leads directly to the representation (1.1), (1.2); for $n \geq 2$ we have to study the C^*-algebra generated by a finite set of arbitrary noncommuting positive injective contractions.

The study of the algebra $\mathcal{R} = \mathcal{C}^*(P, Q_1, ..., Q_n)$ is closely connected with the description of the C^* free product $\mathbb{C}^2 * \mathbb{C}^n$, and, after the change of the generators P, Q_1, ..., Q_n for $u = 2P - 1$, $v = \sum_{k=1}^n e^{2\pi(k-1)/n} Q_k$, with the description of the (full) C^*-algebra of the free product $\mathbb{Z}_2 * \mathbb{Z}_n$ of two cyclic groups of order two and n respectively. Note that the Choi algebra [1] is isomorphic to the reduced C^*-algebra of this group for $n = 3$ (see [16] for the reduced algebra for arbitrary finite n). For the case $n = 3$ one can think of the algebra $\mathcal{R}$ as the C^*-algebra with identity generated by two selfadjoint elements P and $Q = Q_1 - Q_2$ with the conditions

$$P^2 = P, \quad Q^3 = Q.$$

2 Canonical form of all-but-one orthogonal projections

2.1 Let a Hilbert space H is given in two different ways as a direct sum of its subspaces

$$H = L \oplus L^\perp = M_1 \oplus ... \oplus M_n.$$

Denote by P the orthogonal projection on H with image L, and by Q_k the orthogonal projection on H with image M_k, $k = \overline{1, n}$. We have obviously

$$Q_j \cdot Q_k = \delta_{jk} Q_k, \quad j, k = \overline{1, n}, \tag{2.1}$$

$$I = Q_1 + ... + Q_n. \tag{2.2}$$

We will call the system P, Q_1, ..., Q_n of projections (selfadjoint idempotents in a C^*-algebra) *all-but-one* if they satisfy the above conditions (2.1), (2.2).

The subspace

$$\bigoplus_{k=1}^{n} (L \cap M_k) \bigoplus_{k=1}^{n} (L^\perp \cap M_k) \tag{2.3}$$

is invariant with respect to all projections, and both the restrictions of the projections onto (2.3) and the (commutative) C^*–algebra generated by these restrictions have the obvious structure. Thus we assume that the subspace (2.3) is trivial, and that

$$\dim L = \dim L^\perp = \dim M_1 = ... = \dim M_n = \infty.$$

Corresponding to the decomposition

$$H = \bigoplus_{k=1}^{n} M_k.$$

the projections P and Q_k, $k=\overline{1,n}$ have the following matrix form:

$$\begin{aligned} P &= (Q_j P Q_k)_{j,k=1}^n, \\ Q_k &= \mathrm{diag}\,(0,\ldots,0,\underset{k-place}{I},0,\ldots,0), \quad k=\overline{1,n}. \end{aligned} \tag{2.4}$$

For each $k=\overline{1,n}$ introduce an isometry $U_k : L \to M_k$, and the operator $D_k = PQ_kU_k : L \to L$. Then $D_k^* = U_k^* Q_k P|_L : L \to L$. For the unitary operator

$$U = \mathrm{diag}\,(U_1,\ldots,U_n): \ L^n \longrightarrow H = \bigoplus_{k=1}^n M_k.$$

the projections U^*PU and U^*Q_kU, $k=\overline{1,n}$, acting on the space L^n have the following matrix form:

$$\begin{aligned} U^*PU &= (D_j^* \cdot D_k)_{j,k=1}^n, \\ U^*Q_kU &= \mathrm{diag}\,(0,\ldots,0,\underset{k-place}{I},0,\ldots,0), \quad k=\overline{1,n}, \end{aligned}$$

2.2 Lemma *The operators D_k, $k=\overline{1,n}$, are injective and*

$$\sum_{k=1}^n D_k D_k^* = I.$$

PROOF. The first statement follows from $L^\perp \cap M_k = \{0\}$, $k=\overline{1,n}$. Further,

$$\sum_{k=1}^n D_k D_k^* = \sum_{k=1}^n PQ_kU_k \cdot U_k^* Q_k P|_L = P\left(\sum_{k=1}^n Q_k\right) P|_L = P|_L = I.$$

□

2.3 Theorem *For an appropriate choice of the isometry U_k the operator D_k can be*

(i) normal,

(ii) selfadjoint,

(iii) positive,

if and only if $\mathrm{Im}\,P \cap \ker Q_k = \{0\}$.

PROOF. Let $\mathrm{Im}\,P \cap \ker Q_k \neq \{0\}$, it is sufficient to prove that D_k can not be normal. We have $\ker D_k^* = \mathrm{Im}\,P \cap \ker Q_k \neq \{0\}$ and thus $\ker D_k^* \neq \ker D_k = \{0\}$, so the operator D_k can not be normal.

Conversely, let $\mathrm{Im}\,P \cap \ker Q_k = \{0\}$, we will prove that the operator D_k can be positive. It is easy to see that the operator $F_k = Q_k|_L : L \to M_k$ is injective and with the dense range. Thus it has the polar decomposition $F_k = V_k \widetilde{D}_k$ with a unitary operator

$V_k : L \to M_k$ and a positive operator $\widetilde{D}_k : L \to L$. Put $U_k = V_k$, then $D_k = PQ_kV_k = F_k^* V_k = \widetilde{D}_k$. $\square$

2.4 The theorem motivates the following definition. We say that all-but-one projections P, Q_1, ..., Q_n are in *generic position* if they satisfy the conditions

$$\operatorname{Im} P \cap \ker Q_k = \{0\}, \quad \operatorname{Im} Q_k \cap \ker P = \{0\}, \quad k = \overline{1, n}. \tag{2.5}$$

If $n = 2$, i.e. we have only two (independent) projections P and $Q = Q_1$, the notion of generic position is equivalent to one of Halmos [10] and means exactly that the space (2.3) is trivial. Note, that this is the only case when the triviality of the space (2.3) permits to have all the D_k positive, for $n > 2$ our notion of generic position is stronger and implies in particular the triviality of the space (2.3).

2.5 Assuming that all-but-one projections P, Q_1, ..., Q_n are in generic position select all the isometries U_k in such a form that all D_k are positive, introduce $C_k = D_k^2$, $k = \overline{1, n}$. Then all C_k are positive and injective,

$$\sum_{k=1}^{n} C_k = I,$$

and for the projections U^*PU, U^*Q_kU, $k = \overline{1, n}$, acting on the space L^n we have

$$\begin{aligned} U^*PU &= (C_j^{1/2} \cdot C_k^{1/2})_{j,k=1}^n, \\ U^*Q_kU &= \operatorname{diag}(0, \ldots, 0, \underset{k-place}{I}, 0, \ldots, 0), \quad k = \overline{1, n}. \end{aligned}$$

The following theorem summarizes the above results.

2.6 Theorem *Each set of generic position orthogonal projections P, Q_1, ..., Q_n on a Hilbert space H with the properties*

$$\begin{aligned} Q_j \cdot Q_k &= \delta_{jk} \cdot Q_k, \quad j, k = \overline{1, n}, \\ I &= Q_1 + \ldots + Q_n \end{aligned} \tag{2.6}$$

admits the following canonical representation:

There exists a Hilbert space L ($= \operatorname{Im} P$), an unitary operator U from the Hilbert space $L^n = L \times \ldots \times L$ onto H, and n positive injective operators C_1, ..., C_n on L with $C_1 + \ldots + C_n = I$, such that the operators U^*PU and U^*Q_kU, $k = \overline{1, n}$, acting on L^n are, in matrix form, given as

$$\begin{aligned} U^*PU &= (C_j^{1/2} \cdot C_k^{1/2})_{j,k=1}^n, \\ U^*Q_kU &= \operatorname{diag}\,(0, \ldots, 0, \underset{k-place}{I}, 0, \ldots, 0), \quad k = \overline{1, n}. \end{aligned} \tag{2.7}$$

Conversely, each set of n positive injective operators C_1, ..., C_n acting on a Hilbert space L, with $C_1 + \ldots + C_n = I$ and each unitary operator U from L^n onto a Hilbert space H define by formulas (2.7) a set of generic position orthogonal projections P, Q_1, ..., Q_n on a Hilbert space H with the properties (2.6).

2.7 Theorem *Let $\{P', Q_1', ..., Q_n'\}$ and $\{P'', Q_1'', ..., Q_n''\}$ be two systems of all-but-one generic position orthogonal projections on the Hilbert spaces H' and H'' respectively, and let $\{C_1', ..., C_n'\}$*

and $\{C_1'', ..., C_n''\}$ *be the corresponding systems of positive contractions. Then the systems of projections are unitary equivalent, i.e. there exists an unitary operator* $W : H' \to H''$ *such that* $P' = W^*P''W$, $Q_k' = W^*Q_k''W$, $k = \overline{1,n}$, *if and only if the systems of contractions are unitary equivalent, i.e. there exists an unitary operator* $V : \operatorname{Im} P' \to \operatorname{Im} P''$ *such that* $C_k' = V^*C_k''V$, $k = \overline{1,n}$.

PROOF. The part "if" is obvious. Now assume that the system of projections are given in their canonical forms (2.7). Then the conditions $Q_k' = W^*Q_k''W$, $k = \overline{1,n}$, imply that $W = \operatorname{diag}(W_1, W_2, ..., W_n)$, where $W_k : \operatorname{Im} P' \to \operatorname{Im} P''$, $k = \overline{1,n}$, are unitary operators. The condition $P' = W^*P''W$ implies that

$$(C_j')^{1/2}(C_k')^{1/2} = W_j^*(C_j'')^{1/2}(C_k'')^{1/2}W_k, \quad j,k = \overline{1,n},$$

and, in particular, that $C_k' = W_k^*C_k''W_k$ and $(C_k')^{1/2} = W_k^*(C_k'')^{1/2}W_k$, $k = \overline{1,n}$. Thus

$$(C_j')^{1/2}(C_k')^{1/2} = W_j^*(C_j'')^{1/2}W_j\, W_j^*(C_k'')^{1/2}W_k = (C_j')^{1/2}\, W_j^*(C_k'')^{1/2}W_k,$$

and the injectivity of C_j' gives

$$(C_k')^{1/2} = W_j^*(C_k'')^{1/2}W_k.$$

In the same manner

$$(C_k')^{1/2} = ((C_k')^{1/2})^* = W_k^*(C_k'')^{1/2}W_j = W_k^*(C_k'')^{1/2}W_kW_k^*W_j = (C_k')^{1/2}W_k^*W_j$$

gives $W_k^*W_j = I$, or $W_1 = ... = W_n$ $(= V)$. □

3 Naimark dilation theorems

3.1 The understanding of mutual location of all-but-one orthogonal projections yields absolutely elementary and completely constructive proofs of the Naimark dilation theorems. These theorems were proved and discussed, for example, in [2, 6, 18]. In [6] Ch. Davis gave the elementary proof for the finite indices case when $n = 2$ (based in fact on two–projection theory) and used a rather sophisticated doubling procedure for $n = 2^m$.

Start with the finite indices case.

3.2 Theorem (finite case) *Let* A_k, $k = \overline{1,n}$, *be a finite set of positive operators on a Hilbert space* H *with* $\sum_{k=1}^n A_k = I$. *Then there exists an isomorphic injection* $\jmath$ *of* H *into a larger Hilbert space* K *and there exist commuting projections* E_k, $k = \overline{1,n}$, *on* K *with* $\sum_{k=1}^n E_k = I$ *such that* $A_k = \jmath^* E_k \jmath$.

PROOF. Introduce $K = H^n$ and the isomorphic injection $\jmath : H \longrightarrow K$ as follows

$$\jmath : x \longmapsto (A_1^{1/2}x, ..., A_n^{1/2}x),$$

then obviously $\jmath^* : K \longrightarrow H$ is defined by

$$\jmath^* : (x_1, ..., x_n) \longmapsto A_1^{1/2}x_1 + ... + A_n^{1/2}x_n.$$

Finally define the desired projections $E_k : H^n \longrightarrow H^n$ as

$$E_k = 0 \oplus ... \oplus 0 \oplus \underset{k-place}{I} \oplus 0 \oplus ... \oplus 0. \qquad \square$$

The proof admits an obvious generalization to the case of countable set of indices.

3.3 Theorem (countable case) *Let A_k, $k \in \mathbb{N}$, be a countable set of positive operators on a Hilbert space H with $\sum_{k=1}^{\infty} A_k = I$. Then there exists an isomorphic injection $\jmath$ of H into a larger Hilbert space K and there exist commuting projections E_k, $k \in \mathbb{N}$, on K with $\sum_{k=1}^{\infty} E_k = I$ such that $A_k = \jmath^* E_k \jmath$.*

Proof. Introduce $K = H \otimes l_2$ and the isomorphic injection $\jmath : H \longrightarrow K$ as follows

$$\jmath : x \longmapsto \{A_k^{1/2} x\}_{k=1}^{\infty},$$

then obviously $\jmath^* : K \longrightarrow H$ is defined by

$$\jmath^* : (x_1, ..., x_n) \longmapsto \sum_{k=1}^{\infty} A_k^{1/2} x_k$$

Finally define the projections $E_k : H \otimes l_2 \longrightarrow H \otimes l_2$ as

$$E_k : \{x_1, x_2, ...\} \longmapsto \{0, ..., 0, x_k, 0, ...\}. \qquad \square$$

3.4 We say [18], that the $\{B_\lambda\}$, $\lambda \in (0, 1]$ is a positive operator valued distribution if for each λ B_λ is a bounded self-adjoint operator on a Hilbert space H, such that $B_\lambda \leq B_\mu$ for $\lambda < \mu$, $B_\lambda = B_{\lambda+0}$ (in a strong sense), $B_{+0} = 0$ and $B_1 = I$.

Let $a(\lambda)$, $\lambda \in (0, 1]$, be a real valued piece-wise continuous, continuous from the left function. Denote by $0 < t_1 < ... < t_{n-1} < 1$ its jump points, and add $t_0 = 0$, $t_n = 1$. Define

$$\int_{(0,1]} a(\lambda)\, dB_\lambda = \sum_{k=1}^{n} \int_{(t_{k-1}, t_k]} a(\lambda)\, dB_\lambda,$$

where the integrals over $(t_{k-1}, t_k]$ are defined as limits of usual Stieltjes–Riemann integral sums.

3.5 Theorem (continuous case) *Let $\{B_\lambda\}$, $\lambda \in (0, 1]$, be a positive operator valued distribution on a Hilbert space H. Then there exists an isomorphic injection $\jmath$ of H into a larger Hilbert space K and there exists a projection valued distribution (spectral family) $\{E_\lambda\}$, $\lambda \in (0, 1]$, on K, such that $B_\lambda = \jmath^* E_\lambda \jmath$.*

Proof. The well-known problems to recovering a monotonic function from its derivative do not permit following directly the line of previous proofs, but it is still possible to use the same idea.

To construct a Hilbert space K introduce a linear (non-closed) space K_0 as the set of all finite linear combinations

$$x = x(\lambda) = \sum_{k=0}^{n=n(x)} a_k(\lambda) x_k,$$

where $a_k(\lambda)$, $\lambda \in (0,1]$, are piece-wise continuous, continuous from the left functions, $x_k \in H$, and the following form (semi-scalar product) on K_0

$$\begin{aligned}\langle x, y\rangle &= \int_{(0,1]} (dB_\lambda x(\lambda), y(\lambda)) \\ &= \sum_j \sum_k \left(\int_{(0,1]} a_j(\lambda)\overline{b_k(\lambda)} dB_\lambda\, x_j, y_k\right),\end{aligned}$$

where $(\cdot,\cdot)$ is the scalar product in H.

Now the Hilbert space K is the completion of the quotient space $K_0/\ker\langle\cdot,\cdot\rangle$, where

$$\ker\langle\cdot,\cdot\rangle = \{x \in K_0 : \langle x, x\rangle = 0\}.$$

As usual, we will identify a class in the quotient space with (any) its representative. The mapping

$$\jmath : \ x \in H \longmapsto x(\lambda) \equiv x \in K$$

is obviously an isometric injection, and the adjoint operator is given by

$$\jmath^* : \ x(\lambda) \in K \longmapsto \int_{(0,1]} dB_\lambda x(\lambda) \in H.$$

Finally define the spectral family $\{E_\lambda\}$ as $E_\lambda = \chi_{(0,\lambda]} I$, where $\chi_{(0,\lambda]}$ is the characteristic function of $(0,\lambda]$. □

4 C^*–algebra generated by all-but-one projections.

4.1 The C^*–algebra $\mathcal{R} = \mathcal{C}^*(P, Q_1, \ldots, Q_n)$ generated by the generic position orthogonal projections P and Q_k, $k = \overline{1,n}$, having the property (2.6) is naturally isomorphic to the algebra $U^*\mathcal{R}U$. We will identify these algebras and thus will assume that the C^*–algebra $\mathcal{R}$ is generated by the following projections:

$$\begin{aligned}P &= (C_j^{1/2} \cdot C_k^{1/2})_{j,k=1}^n, \\ Q_k &= \operatorname{diag}(0, \ldots, 0, \underset{k-place}{I}, 0, \ldots, 0), \quad k = \overline{1,n},\end{aligned}$$

where C_k, $k = \overline{1,n}$ are positive injective contractions with the property $\sum\limits_{k=1}^n C_k = I$.

4.2 The difficulties begin when $n \geq 3$. In the case of two orthogonal projections we have in fact only one independent contraction, $C = C_2$, then $C_1 = I - C$, and thus using the spectral theorem we come to the known representation of two projections in the generic position:

$$\begin{aligned}P &= \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix}, \\ Q &= Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\end{aligned}$$

where $x \in \mathrm{sp}\, P(I-Q)|_{\mathrm{Im}\, P} = \mathrm{sp}\,(P-Q)^2 \subset [0,1]$.

In the case $n \geq 3$ the description of the C^*–algebra $\mathcal{R}$ depends essentially on the mutual properties of the contractions C_k, and thus on the structure of the C^*–algebra $\mathcal{R}_C = \mathcal{C}^*(C_1, \dots, C_n)$ generated by them.

4.3 Lemma *The set (algebra) of all elements of the form*

$$A = \mathrm{diag}\,(\alpha_1, \dots, \alpha_n) I + (C_j^{1/2} A_{j,k} C_k^{1/2})_{j,k=1}^n, \tag{4.1}$$

where $\alpha_k \in \mathbb{C}$, $A_{j,k} \in \mathcal{R}_C$, is dense in the algebra $\mathcal{R}$.

Proof. It is sufficient to prove that all elements of the form (4.1) where $A_{j,k}$ are noncommutative polynomials of the variables C_1, ..., C_{n-1} belong to $\mathcal{R}$. Denote for a moment by $[A]_{j,k}$ the element of $\mathcal{R}$ with the only nonzero entry j,k equal to A. Now the proof follows directly from the equalities

$$\begin{aligned}
[\, C_j^{1/2} C_k^{1/2} \,]_{j,k} &= Q_j P Q_k, \\
[\, C_j^{1/2} C_m C_k^{1/2} \,]_{j,k} &= [\, C_j^{1/2} C_m^{1/2} \,]_{j,m} [\, C_m^{1/2} C_k^{1/2} \,]_{m,k}, \\
[\, C_j^{1/2} A_1 A_2 C_k^{1/2} \,]_{j,k} &= \sum_{m=1}^n [\, C_j^{1/2} A_1 C_m^{1/2} \,]_{j,m} [\, C_m^{1/2} A_2 C_k^{1/2} \,]_{m,k}, \\
(C_j^{1/2} A_{j,k} C_k^{1/2})_{j,k=1}^n &= \sum_{j,k=1}^n [\, C_j^{1/2} A_{j,k} C_k^{1/2} \,]_{j,k}. \qquad \square
\end{aligned}$$

4.4 Corollary *If all the operators C_k are invertible, than the algebra $\mathcal{R}$ is *-isomorphic to $\mathcal{R}_C \otimes \mathrm{Mat}_n(\mathbb{C})$.*

4.5 Corollary *Let $\mathcal{A}$ be a finitely generated C^*-algebra with identity, and let n_0 be a minimal number of selfadjoint elements generating $\mathcal{A}$. Then for each $n > n_0$ there exists a continuous family of not unitary equivalent systems of all-but-one selfadjoint idempotents (generic position orthogonal projections) p, q_1, ..., q_n, such that for each of these systems we have*

$$\mathcal{A} \otimes \mathrm{Mat}_n(\mathbb{C}) \cong \mathcal{C}^*(p, q_1, \dots q_n).$$

Proof. Denote by s_1, ..., s_{n_0} the selfadjoint elements generating $\mathcal{A}$, and fix any $n > n_0$. Now we need a family of systems of positive inversible elements c_1, ..., c_n of the algebra $\mathcal{A}$, such that

$$\sum_{k=1}^n c_k = e \quad \text{and} \quad \mathcal{C}^*(c_1, \dots, c_n) = \mathcal{A}.$$

Of course, there are many possibilities to construct such a family. For example one may put for $\alpha \in [0,1]$

$$\begin{aligned}
c_k = c_k(\alpha) &= \frac{1}{4n_0}(e + \frac{1+\alpha}{\|s_k\|} s_k), \quad k = \overline{1, n_0}, \\
c_k = c_k(\alpha) &= \frac{1}{n-n_0}(e - \sum_{j=1}^{n_0} c_j), \quad k = \overline{n_0+1, n}.
\end{aligned}$$

Finally, the family of the following systems of selfadjoint idempotents (projections) satisfies all the necessary properties

$$\begin{aligned} p = p(\alpha) &= (c_j^{1/2}(\alpha) \cdot c_k^{1/2}(\alpha))_{j,k=1}^n, \\ q_k = q_k(\alpha) &= \operatorname{diag}(0, \ldots, 0, \underset{k-place}{e}, 0, \ldots, 0), \quad k = \overline{1, n}, \end{aligned}$$

where $\alpha \in [0,1]$. □

4.6 EXAMPLE. Cuntz algebras are generated by all-but-one selfadjoint idempotents.

Recall that the Cuntz algebra [3] $\mathcal{O}_n$ is a *simple* (and algebraically unique) C^*-algebra generated by n isometries S_1, ..., S_n whose ranges are orthogonal complements:

$$S_1 S_1^* + \ldots + S_n S_n^* = I.$$

We have $\mathcal{O}_n \otimes \mathrm{Mat}_n(\mathbb{C}) \cong \mathcal{O}_n$ (see [1] for $n = 2$), and thus $\mathcal{O}_n \cong \mathcal{O}_n \otimes \mathrm{Mat}_{n^k}(\mathbb{C})$, $k \in \mathbb{N}$.

4.7 EXAMPLE. Family of all-but-one generic position orthogonal projections, generating non-isomorphic simple C^*-algebras.

Apply Corollary 4.5 to the irrational rotation algebra [15] $\mathcal{A}_\alpha$, α is an irrational number. This algebra is simple and is generated by two unitary operators

$$(U\varphi)(t) = t\varphi(t) \quad \text{and} \quad (V\varphi)(t) = \varphi(e^{i2\pi\alpha}t),$$

acting on $L_2(\mathbb{T})$. By [15], for irrational $\alpha, \beta \in [0, 1/2]$ the algebras $\mathcal{A}_\alpha \otimes \mathrm{Mat}_n(\mathbb{C})$ and $\mathcal{A}_\beta \otimes \mathrm{Mat}_m(\mathbb{C})$ are isomorphic if and only if $\alpha = \beta$ and $m = n$.

4.8 EXAMPLE. Non type I algebras generated by all-but-one generic position orthogonal projections.

Apply Corollary 4.5 to the Bergman Toeplitz operator algebras [4] $\mathcal{C}^*(\Omega_{\delta_1,\delta_2})$, with irrational $\ln(\delta_2)/\ln(\delta_1)$. These algebras are non type I and are generated by multiplication operators $z_1 I$, $z_2 I$, acting on the Bergman space $\mathcal{A}^2(\Omega_{\delta_1,\delta_2}, \mu)$, where

$$\Omega_{\delta_1,\delta_2} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < \delta_1 \text{ and } |z_2| < 1, \text{ or } |z_1| < 1 \text{ and } |z_2| < \delta_2\}.$$

4.9 Theorem *Each irreducible representation $\rho : \mathcal{R} \to \mathcal{L}(Y)$ of the algebra $\mathcal{R}$ has the following structure:*

There exist an irreducible representation $\pi : \mathcal{R}_C \to \mathcal{L}(X)$ of the algebra $\mathcal{R}_C$ and a set of n orthogonal projections $P^{(j)} : X \to Y_j$, $j = \overline{1, n}$, with the property

$$P^{(j)} \pi(C_j^{1/2} C_k^{1/2}) = \pi(C_j^{1/2} C_k^{1/2}) P^{(k)}, \quad j, k = \overline{1, n}, \tag{4.2}$$

such that

$$Y = \bigoplus_{j=1}^{n} \operatorname{Im} P^{(j)} \subset X^n$$

and

$$\rho : (A_{j,k})_{j,k=1}^n \in \mathcal{R} \longmapsto (\pi(A_{j,k}))_{j,k=1}^n |_Y \tag{4.3}$$

PROOF. The algebra $\mathcal{R}$ is a C^*–subalgebra of $\mathcal{R}_C \otimes \mathrm{Mat}_n(\mathbb{C})$, each irreducible representation of which is of the form

$$\pi^{\#} = \pi \otimes id : \mathcal{R}_C \otimes \mathrm{Mat}_n(\mathbb{C}) \to \mathcal{L}(X^n).$$

where $\pi : \mathcal{R}_C \to \mathcal{L}(X)$ is an irreducible representation of $\mathcal{R}_C$ and id is the identical representation of $\mathrm{Mat}_n(\mathbb{C})$.

Thus [7] for each irreducible representation $\rho : \mathcal{R} \to \mathcal{L}(Y)$ of the algebra $\mathcal{R}$ there exist an irreducible representation $\pi : \mathcal{R}_C \to \mathcal{L}(X)$ and a subspace Y of the Hilbert space X^n such that

$$\begin{aligned} \rho : (A_{j,k})_{j,k=1}^n \in \mathcal{R} \quad &\longmapsto \quad \pi^{\#}((A_{j,k})_{j,k=1}^n) = \\ &= \quad (\pi(A_{j,k})^n)_{j,k=1}|_Y \in \mathcal{L}(Y). \end{aligned}$$

The orthogonal projection $P_Y : X^n \to Y$ should commute with all elements of $\pi^{\#}(\mathcal{R})$. Commutation with $\pi^{\#}(Q_j)$, $j = \overline{1,n}$ leads to

$$P_Y = \bigoplus_{j=1}^n P^{(j)},$$

where $P^{(j)}$ is the orthogonal projections of X onto its subspace Y_j, $j = \overline{1,n}$. Thus

$$Y = \bigoplus_{j=1}^n \mathrm{Im}\, P^{(j)} = \bigoplus_{j=1}^n Y_j.$$

Finally, the commuting of P_Y with all the elements of the form $\pi^{\#}(Q_j P Q_k)$, $j,k = \overline{1,n}$ leads to

$$P^{(j)}\pi(C_j^{1/2}C_k^{1/2}) = \pi(C_j^{1/2}C_k^{1/2})P^{(k)}. \quad \square$$

4.10 Remark. The condition (4.2) is necessary and sufficient for the mapping (4.3) to be a representation.

5 Pairwise commuting contractions C_k

5.1 In this section we consider the special case when all contractions C_k, $k = \overline{1,n}$, are pairwise commuting, and thus the C^*–algebra $\mathcal{R}_C$ generated by them is commutative. This is the simplest and more direct generalization of the two projection situation. This case is itself interesting and its understanding is important for the general case when the commutator ideal of the algebra $\mathcal{R}_C$ is proper.

5.2 Denote by $\Delta = \Delta(C_1, \ldots, C_n) \subset \mathbb{R}^n$ the joint spectrum of the operators $C_1, \ldots, C_n$. Then the C^*–algebra $\mathcal{R}_C$ is naturally isomorphic to $C(\Delta)$, and the coordinate function t_k, $k = \overline{1,n}$, $(t = (t_1, \ldots, t_n) \in \Delta \subset \mathbb{R}^n)$ is the image of C_k under this isomorphism.

Denote by $\mathfrak{S}(\Delta)$ the C^*–algebra generated by the following $n \times n$ matrix–valued functions continuous on Δ:

$$\begin{aligned} p(t) &= (\sqrt{t_j}\sqrt{t_k})_{j,k=1}^n, & (5.1)\\ q_k(t) &= \operatorname{diag}(0,\ldots,0,\underset{k-place}{1},0,\ldots,0), & (5.2) \end{aligned}$$

where $t = (t_1,\ldots,t_n) \in \Delta$.

5.3 To describe the algebra $\mathfrak{S}(\Delta)$ introduce first some kind of "maximal" algebra. Let

$$\Delta_{n-1} = \{t = (t_1,\ldots,t_n) \in \mathbb{R}^n : t_k \geq 0, \quad k = \overline{1,n}, \quad \sum_{k=1}^n t_k = 1\}$$

be the standard $(n-1)$ - dimensional simplex. Thus $\mathfrak{S}(\Delta_{n-1})$ is the C^*–algebra generated by the elements (5.1), (5.2) with $t = (t_1,\ldots,t_n) \in \Delta_{n-1}$.

5.4 Theorem *The algebra $\mathfrak{S}(\Delta_{n-1})$ coincides with the C^*–algebra of all $n\times n$ matrix–valued functions continuous on Δ_{n-1} and having the following properties:*

(i) they are diagonal at all vertices $(1,0,\ldots,0)$, $(0,1,\ldots,0)$, ..., $(0,\ldots,0,1)$ of Δ_{n-1};

(ii) fix a k–tuple $[k] = (i_1,\ldots,i_k)$ of indices $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, and let $[n]\backslash[k]$ be the complementary $(n-k)$–tuple of indices; then for each k–tuple, $1 \leq k \leq n-1$

$$\sigma_{ij}(t) = \sigma_{ji}(t) \equiv 0 \quad \text{for} \quad i \in [k], \quad j \neq i$$

and $\sigma_{ii}(t) = \sigma_i = const$ on $(n-k-1)$–face $t_{i_1} = t_{i_2} = \ldots = t_{i_k} = 0$ of Δ_{n-1}.

All, up to unitary equivalence, irreducible representations of the algebra $\mathfrak{S}(\Delta_{n-1})$ are as follows:

(i) the family of n–dimensional representations parameterized by the points $t \in \operatorname{Int}\Delta_{n-1}$ and defined by the rule

$$\pi_t : \sigma \in \mathfrak{S}(\Delta_{n-1}) \longmapsto \sigma(t) \in \operatorname{Mat}_n(\mathbb{C});$$

(ii) C_n^k families $\{\pi_t^{[k]}\}$ of $(n-k)$–dimensional representations, $k = \overline{1,n-1}$, parameterized by the interior parts of the $(n-k-1)$–face $t_{i_1} = t_{i_2} = \ldots = t_{i_k} = 0$, $i_j \in [k]$ of Δ_{n-1}; then representations (for the corresponding points t) are defined by the rule

$$\pi_t^{[k]} : \sigma \in \mathfrak{S}(\Delta_{n-1}) \mapsto (\sigma_{i,j}(t))_{i,j\in[n]\backslash[k]} \in \operatorname{Mat}_{n-k}(\mathbb{C})$$

(iii) $2n$ one–dimensional representations defined as follows:

$$\begin{aligned} \pi'_{(j)} : \sigma \in \mathfrak{S}(\Delta_{n-1}) &\longmapsto \sigma_{jj}(0,\ldots,0,\underset{j-place}{1},0,\ldots,0) \in \mathbb{C}\\ \pi''_{(j)} : \sigma \in \mathfrak{S}(\Delta_{n-1}) &\longmapsto \sigma_{jj}(\tau_1,\ldots,\tau_{j-1},0,\tau_{j+1},\ldots,\tau_n) \in \mathbb{C}, \end{aligned}$$

where $t = (\tau_1,\ldots,\tau_{j-1},0,\tau_{j+1},\ldots,\tau_n) \in \partial\Delta_{n-1}$.

PROOF. Follows from the corresponding proofs of Lemma 4.7 and Theorem 4.8 in [13] and is based on Stone–Weierstrass theorem for CCR–algebras [7]. $\square$

5.5 Theorem *The algebra $\mathfrak{S}(\Delta)$, where $\Delta = \Delta(C, ..., C_n)$, is the restriction of $\mathfrak{S}(\Delta_{n-1})$ onto the set Δ, i.e.*

$$\mathfrak{S}(\Delta) = \{\sigma|_\Delta : \sigma \in \mathfrak{S}(\Delta_{n-1})\}.$$

PROOF. See [13] for the case $n = 3$. $\square$

5.6 Remark. The set $\Delta = \Delta(C_1, ..., C_n)$ can be an arbitrary compact subset of the simplex Δ_{n-1}.

6 Szegö projection on the Heisenberg group and n-valued functions

6.1 The aim of this sections is to study certain C^*-algebras generated by the Segö projection associated with the unit ball in $\mathbb{C}^m$ and piecewise constant n-valued functions on the unit sphere S^{2m-1}. By technical reasons it is convenient to pass from the unit ball and sphere to their generalized upper half-plane realizations.

The fine and unpredictable point here is that, contrary to the case of 2-valued functions, the structure of the algebras depend essentially on the class of n-valued functions under consideration. This reflects, on the one hand side, the different behavior of the Segö projection with respect to the complex and real directions on the distinguished boundary, and, on the other hand side, the principal difference between the cases of two and more projections.

6.2 Via the Cayley transform the unit ball in $\mathbb{C}^m$ can be identified with the following type II Siegel domain [11, 19]

$$D = \{(w, z) \in \mathbb{C} \times \mathbb{C}^m : \operatorname{Im} w - z \cdot \overline{z} > 0\}.$$

Its distinguished boundary (the image of the unit sphere)

$$B = \{(w, z) \in \mathbb{C} \times \mathbb{C}^m : \operatorname{Im} w - z \cdot \overline{z} = 0\}$$

has a natural Lie group structure, which after identification of B with $\mathbb{R} \times \mathbb{C}^m$ with the help of mapping

$$\kappa_0 : (t, z) \in \mathbb{R} \times \mathbb{C}^m \longmapsto (t + iz \cdot \overline{z},\, z) \in B$$

coincides with the Heisenberg group $\mathbb{H}_m$: the space $\mathbb{R} \times \mathbb{C}^m$ endowed with the following group operation

$$(t_1, z_1) \cdot (t_2, z_2) = (t_1 + t_2 + 2 \operatorname{Im} z_1 \cdot \overline{z}_2,\, z_1 + z_2),$$

where $(t_k, z_k) \in \mathbb{H}_m = \mathbb{R} \times \mathbb{C}^m$, $k = 1, 2$.

The mapping

$$\kappa : \quad L_2(B) \longrightarrow L_2(\mathbb{R} \times \mathbb{C}^m) = L_2(\mathbb{H}_m),$$

where $(\kappa f)(t, z) = f(\kappa_0(t, z))$ transfers isometrically the Hardy space $H^2(D)$ ($\subset L_2(B)$) onto the subspace $H^2(\mathbb{H}_m) = \kappa(H^2(D))$ of the space $L_2(\mathbb{H}_m)$.

The Szegö projection P of the space $L_2(\mathbb{H}_m)$ onto $H^2(\mathbb{H}_m)$ is a right group convolution operator having the following (pseudodifferential) Weyl symbol (for details, see [20, 22])

$$\begin{aligned}\omega_P(\lambda,\widetilde{x},\widetilde{\xi}) &= \chi_+(\lambda)\cdot 2^n\cdot e^{-\frac{|\zeta-2i\lambda z|^2}{4|\lambda|}} = \\ &= \chi_+(\lambda)\cdot 2^n\cdot e^{-\frac{(\xi+2\lambda y)^2+(\eta-2\lambda x)^2}{4|\lambda|}},\end{aligned}$$

where $\widetilde{x}=(x,y),\ \widetilde{\xi}=(\xi,\eta)\in\mathbb{R}^{2m},\quad z=x+iy,\ \zeta=\xi+i\eta\in\mathbb{C}^n,\ \chi_+$ is the characteristic function of the (open) positive half-line.

6.3 Theorem *There exists an unitary operator* $U:\ L_2(\mathbb{H}_m)=L_2(\mathbb{R})\otimes L_2(\mathbb{R}^m)\otimes L_2(\mathbb{R}^m)\longrightarrow L_2(\mathbb{R})\otimes L_2(\mathbb{R}^m)\otimes L_2(\mathbb{R}^m)$ *such that* $U^*PU=\chi_+(\lambda)I\otimes P_0\otimes I$, *where*

$$(P_0f)(x)=\pi^{-\frac{m}{2}}\int_{\mathbb{R}^m}e^{-\frac{1}{2}(x^2+u^2)}f(u)du$$

is the one–dimensional orthogonal projection of $L_2(\mathbb{R}^m)$ *onto the one–dimensional space generated by* $l_0(x)=\pi^{-\frac{m}{4}}e^{-\frac{x^2}{2}}$.

Sketch of the proof (For details, see [20, 22]). On the space $L_2(\mathbb{H}_m)=L_2(\mathbb{R})\otimes L_2(\mathbb{C}^m)$ introduce the operator $F\otimes I$, where F is the Fourier transform on $L_2(\mathbb{R})$. Then we have

$$(F\otimes I)P(F^{-1}\otimes I)=\chi_+(\lambda)I\otimes P(\lambda),$$

where $P(\lambda)=0$ for $\lambda\le 0$, and for positive λ the Weyl symbol of the projection $P(\lambda)$ is given by

$$\omega_{P(\lambda)}(\widetilde{x},\widetilde{\xi})=2^n\cdot e^{-\frac{(\xi+2\lambda y)^2+(\eta-2\lambda x)^2}{4|\lambda|}}.$$

Fix now a positive λ, and introduce the operator $F\otimes I$, where the Fourier transform F on $L_2(\mathbb{R}^m)$ is defined by

$$(Ff)(x)=(2\pi)^{-\frac{m}{2}}\int_{\mathbb{R}^m}e^{-i\xi\cdot x}f(\xi)d\xi,$$

and the unitary operator

$$(W_\lambda f)(x,y)=f(\frac{x}{2\sqrt{\lambda}}+\sqrt{\lambda}y,\ \frac{x}{2\sqrt{\lambda}}-\sqrt{\lambda}y),$$

both acting on $L_2(\mathbb{C}^m)=L_2(\mathbb{R}^m)\otimes L_2(\mathbb{R}^m)$. Then for the unitary operator

$$U(\lambda)=\begin{cases}(F\otimes I)W_\lambda, & \lambda>0\\ I, & \lambda\le 0\end{cases}$$

we have

$$(I\otimes U^*(\lambda))(\chi_+(\lambda)I\otimes P(\lambda))(I\otimes U(\lambda))=\chi_+(\lambda)I\otimes P_0\otimes I,$$

where

$$(P_0f)(x)=\pi^{-\frac{m}{2}}\int_{\mathbb{R}^m}e^{-\frac{1}{2}(x^2+u^2)}f(u)du$$

is the one–dimensional orthogonal projection of $L_2(\mathbb{R}^m)$ onto the one–dimensional space generated by $l_0(x)=\pi^{-\frac{m}{4}}e^{-\frac{x^2}{2}}$.

The operator U of the theorem is given by $U = (F^{-1} \otimes I)(I \otimes U(\lambda))$. $\square$

6.4 Let Y_k, $k = \overline{1,n}$, be n disjoint measurable sets in $\mathbb{R}^m = \mathbb{R}^m_y$ with $\cup_{k=1}^n Y_k = \mathbb{R}^m$, and let

$$Y_k' = \mathbb{R} \times \mathbb{R}^m \times Y_k, \quad k = \overline{1,n},$$

be the corresponding sets in $\mathbb{H}_m = \mathbb{R} \times \mathbb{R}^m_x \times \mathbb{R}^m_y$.

Introduce the characteristic functions χ_k of the sets Y_k' and the orthogonal projections $Q_k = \chi_k I$ acting on $L_2(\mathbb{H}_m)$, here $k = \overline{1,n}$. We have obviously

$$Q_j \cdot Q_k = \delta_{jk} Q_k, \quad j,k = \overline{1,n},$$

$$I = Q_1 + ... + Q_n.$$

Our aim is to describe the C^*-algebra $\mathcal{R}_Y = \mathcal{R}(Y_1, ..., Y_n)$ generated by the Segö projection P and the projections Q_1, ..., Q_n acting on $L_2(\mathbb{H}_m) = L_2(\mathbb{R}) \otimes L_2(\mathbb{C}^m)$.

The algebra $\mathcal{R}_Y$ is unitary equivalent to the algebra $U^*\mathcal{R}_Y U$ which is generated by the following projections:

$$\begin{aligned} U^*PU &= \chi_+(\lambda) I \otimes P_0 \otimes I, \\ U^*Q_kU &= \left\{ \begin{array}{ll} \chi_{2\sqrt{\lambda}Y_k+y}(x) I, & \lambda > 0 \\ \chi_k I, & \lambda \le 0 \end{array} \right., \qquad k = \overline{1,n}. \end{aligned}$$

Thus the algebra $U^*\mathcal{R}_Y U$ splits onto the direct integral of the algebras $\mathcal{R}(\lambda, y)$, where $(\lambda, y) \in \mathbb{R} \times \mathbb{R}^m$. For each non-positive λ the algebra $\mathcal{R}(\lambda, y)$ is isomorphic to $\mathbb{C}^n$, and this isomorphism is generated by the following mapping:

$$\begin{aligned} P(\lambda, y) &\longmapsto 0, \\ Q_k(\lambda, y) &\longmapsto (0, ..., 0, \underset{k-place}{1}, 0, ...0), \quad k = \overline{1,n}. \end{aligned}$$

For each positive λ the algebra $\mathcal{R}(\lambda, y)$ is generated by the orthogonal projections

$$P(\lambda, y) = P_0, \quad Q_k(\lambda, y) = \chi_{2\sqrt{\lambda}Y_k+y}(x) I, \quad k = \overline{1,n},$$

acting on $L_2(\mathbb{R}^m)$.

6.5 Denote by $\mathfrak{S}(Y_1, ..., Y_n)$ the C^*–algebra generated by the following $n \times n$ matrix–valued functions

$$\begin{aligned} p(\lambda, y) &= (\sqrt{\gamma_j(\lambda, y)} \sqrt{\gamma_k(\lambda, y)})_{j,k=1}^n, \\ q_k(\lambda, y) &= \operatorname{diag}(0, ..., 0, \underset{k-place}{1}, 0, ..., 0), \quad k = \overline{1,n}, \end{aligned}$$

where $\lambda > 0$, $y \in \mathbb{R}^m$ and

$$\gamma_k(\lambda, y) = \|\chi_{2\sqrt{\lambda}Y_k+y}(x) \pi^{-\frac{m}{4}} e^{-\frac{x^2}{2}}\|^2_{L_2(\mathbb{R}^m)} = \int_{2\sqrt{\lambda}Y_k+y} |l_0(x)|^2 dx, \quad k = \overline{1,n}.$$

Introduce the set

$$\Delta = \Delta(Y_1, ..., Y_n) =$$

$$= \operatorname{clos}\{(t_1, ..., t_n) \in \Delta_{n-1} : \exists \lambda > 0,\ \exists y \in \mathbb{R}^n \ \text{ with } \ t_1 = \gamma_1(\lambda, y),\ ...,\ t_n = \gamma_n(\lambda, y)\},$$

and let $\mathfrak{S}(\Delta)$ be the restriction of the algebra $\mathfrak{S}(\Delta_{n-1})$ to the set Δ, i.e.,

$$\mathfrak{S}(\Delta) = \{\sigma|_{\Delta} : \sigma \in \mathfrak{S}(\Delta_{n-1})\}.$$

The functions $\gamma_k(\lambda, y)$, $k = \overline{1, n}$, are continuous and thus Δ is a connected compact subset of Δ_{n-1}.

6.6 Theorem *The algebra $\mathfrak{S}(Y_1, ..., Y_n)$ is isomorphic to the algebra $\mathfrak{S}(\Delta)$.*

PROOF. The set $\Delta = \Delta(Y_1, ..., Y_n)$ is the joint spectrum of the corresponding elements $C_k = \gamma_k(\lambda, y)$, $k = \overline{1, n}$. □

6.7 Corollary *The algebra $\mathcal{R}(Y_1, ..., Y_n)$ is *-isomorphic to a subalgebra of the algebra $\mathbb{C}^n \oplus \mathfrak{S}(\Delta)$. The isomorphic inclusion*

$$\nu_Y : \mathcal{R}(Y_1, ..., Y_n) \longrightarrow \mathbb{C}^n \oplus \mathfrak{S}(\Delta)$$

is generated by the following mapping of the projections:

$$\begin{array}{llll} \nu_Y : & P & \longmapsto & ((0, ..., 0),\ p), \\ \nu_Y : & Q_k & \longmapsto & ((0, ..., 0, \underset{k-place}{1}, 0, ..., 0),\ q_k), \end{array}$$

where

$$p = p(t) = (\sqrt{t_j}\sqrt{t_k})_{j,k=1}^n,$$

$$q_k = q_k(t) = \operatorname{diag}(0, ..., 0, \underset{k-place}{1}, 0, ..., 0),$$

$k = \overline{1, n}$, $t = (t_1, ..., t_n) \in \Delta = \Delta(Y_1, ..., Y_n)$.

6.8 Denote by $PC(Y_1, ..., Y_n)$ the algebra of all piecewise constant functions on $\mathbb{H}_m$ of the form

$$a = \alpha_1 \chi_1 + ... + \alpha_n \chi_n,$$

where $\alpha_1, ..., \alpha_n \in \mathbb{C}$, i.e., the algebra of all n–valued functions constant on the sets Y_k', $k = \overline{1, n}$.

Now the algebra $\mathcal{R}(Y_1, ..., Y_n)$ can be introduced as the algebra generated by the Segö projection P and piecewise constant functions from $PC(Y_1, ..., Y_n)$, i.e., by all operators of the form

$$A = aP + b(I - P),$$

with $a = \sum_1^n \alpha_k \chi_k, b = \sum_1^n \beta_k \chi_k \in PC(Y_1, ..., Y_n)$.

6.9 Denote by $\mathcal{T}(PC(Y_1, ..., Y_n))$ the C^*–algebra generated by all the Toeplitz operators

$$T_a : \ \varphi \longmapsto Pa\varphi$$

with symbols $a = \sum_1^n \alpha_k \chi_k \in PC(Y_1, ..., Y_n)$, acting on the Hardy space $H^2(\mathbb{H}_m)$

Corollary 6.7, together with standard considerations, lead directly to the following theorem.

6.10 Theorem *The algebra $\mathcal{T}(PC(Y_1,...,Y_n))$ is isomorphic and isometric to the algebra $C(\Delta(Y_1,...,Y_n))$. The isomorphism*

$$\tau:\ \mathcal{T}(PC(Y_1,...,Y_n)) \longrightarrow C(\Delta(Y_1,...,Y_n))$$

is generated by the following mapping

$$\tau:\ T_a \longmapsto \alpha_1 t_1 + ... + \alpha_n t_n,$$

where $(t_1,...,t_n) \in \Delta = \Delta(Y_1,...,Y_n)$.

6.11 Corollary *The spectrum of the Toeplitz operator T_a is calculated by the formula*

$$\operatorname{sp} T_a = \{\alpha_1 t_1 + ... + \alpha_n t_n :\ (t_1,...,t_n) \in \Delta(Y_1,...,Y_n)\}.$$

6.12 Let now T_k, $k=\overline{1,n}$, be n disjoint measurable sets in $\mathbb{R}$ with $\cup_{k=1}^n T_k = \mathbb{R}$, and let

$$T_k' = T_k \times \mathbb{R}^m \times \mathbb{R}^m, \quad k=\overline{1,n},$$

be the corresponding sets in $\mathbb{H}_m = \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m$.

Introduce the characteristic functions χ_k of the sets T_k' and the orthogonal projections $Q_k = \chi_k I$ acting on $L_2(\mathbb{H}_m)$, here $k=\overline{1,n}$. Again we have

$$Q_j \cdot Q_k = \delta_{jk} Q_k, \quad j,k=\overline{1,n},$$

$$I = Q_1 + ... + Q_n.$$

Now we are going to study the C^*-algebra $\mathcal{R}_T = \mathcal{R}(T_1,...,T_n)$ generated by the Segö projection P and the projections Q_1, ..., Q_n acting on $L_2(\mathbb{H}_m) = L_2(\mathbb{R})\otimes L_2(\mathbb{R}^m)\otimes L_2(\mathbb{R}^m)$.

The algebra $\mathcal{R}_T$ is unitary equivalent to the algebra $(F^{-1}\otimes I\otimes I)U^*\mathcal{R}_T U(F\otimes I\otimes I)$ which is generated by the following projections:

$$\begin{aligned}(F^{-1}\otimes I\otimes I)U^* P U(F\otimes I\otimes I) &= P_+\otimes P_0\otimes I, &(6.1)\\ (F^{-1}\otimes I\otimes I)U^* Q_k U(F\otimes I\otimes I) &= \chi_{T_k}(t) I\otimes I\otimes I, \quad k=\overline{1,n}, &(6.2)\end{aligned}$$

where P_+ is the Segö projection of $L_2(\mathbb{R})$ onto the Hardy space $H^2(\mathbb{R})$ associated with the upper half-plane, and $\chi_{T_k}(t)$ are the characteristic functions of the sets T_k, $k=\overline{1,n}$.

Denote by $\widetilde{\mathcal{R}}_T = \mathcal{C}^*(P_+, \chi_{T_1}, ..., \chi_{T_n})$ the C^*-algebra generated by the projections P_+, $\chi_{T_k}(t)I$, $k=\overline{1,n}$, acting on the space $L_2(\mathbb{R})$. Then the equalities (6.1), (6.2) lead directly to the following theorem.

6.13 Theorem *The algebra $\mathcal{R}(T_1,...,T_n)$ is *-isomorphic to a subalgebra of the algebra $\widetilde{\mathcal{R}}_T\oplus\mathbb{C}^n$. The isomorphic inclusion*

$$\nu_T : \mathcal{R}(T_1,...,T_n) \longrightarrow \widetilde{\mathcal{R}}_T\oplus\mathbb{C}^n$$

is generated by the following mapping of the projections:

$$\begin{aligned}\nu_T:\ \ P &\longmapsto (P_+,\, (0,...,0)),\\ \nu_T:\ \ Q_k &\longmapsto (\chi_{T_k}(t)I,\, (0,...,0,\underset{k-place}{1},0,...,0)), \quad k=\overline{1,n}.\end{aligned}$$

6.14 Remark. The structure of the algebra $\widetilde{\mathcal{R}}_T$ is still unknown for arbitrary n-tuple $T = \{T_1, ..., T_n\}$. If each T_k is a finite union of intervals $[a_j, b_j)$, $-\infty \le a_j < b_j \le +\infty$, the description of the algebra can be done on base of [9]. In this case the algebras $\widetilde{\mathcal{R}}_T$ (independently of the form of T) are irreducible and contain the ideal of compact operators. Their quotient algebras, in general, are different for different kind of T, and have only one- and two-dimensional irreducible representations. The case, when $n = 3$ and the two first T_k are just intervals (having or no having common boundary point) has been considered in details in [14].

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AMS Classification: 46L35, 47D25