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**ON THE STRUCTURE OF BERGMAN AND POLY–BERGMAN SPACES***

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Let $\Pi$ be the upper half-plane in $\mathbb{C}$, consider the Bergman space $\mathcal{A}^2(\Pi)$, the subspace of all analytic functions from $L_2(\Pi)$. The complete decomposition of $L_2(\Pi)$ onto Bergman and Bergman type spaces of poly-analytic and poly-anti-analytic functions is obtained. The orthogonal Bergman type projections onto each of these subspaces are described. Connections with the Hardy spaces and the Szegö projections are established.

# 1 Introduction

Let $H^2_+(\mathbb{R})$ be the Hardy space on the upper half-plane $\Pi$ in $\mathbb{C}$, and $P^+_{\mathbb{R}}$ be the (orthogonal) Szegö projection of $L_2(\mathbb{R})$ onto $H^2_+(\mathbb{R})$. Recall one old and well known result: *the Fourier transform $F$ gives an isometric isomorphism of the space $L_2(R)$, under which*

1. *the Hardy space $H^2_+(\mathbb{R})$ is mapped onto $L_2(\mathbb{R}_+)$*

$$F : H^2_+(\mathbb{R}) \longrightarrow L_2(\mathbb{R}_+),$$

2. *the Szegö projection $P^+_{\mathbb{R}} : L_2(\mathbb{R}) \longrightarrow H^2_+(\mathbb{R})$ is unitary equivalent to the following one*

$$F\, P^+_{\mathbb{R}}\, F^{-1} = \chi_+ I.$$

Furthermore, if $H^2_-(\mathbb{R})$ is the Hardy space on the lower half-plane and $P^-_{\mathbb{R}}$ is the (orthogonal) Szegö projection of $L_2(\mathbb{R})$ onto $H^2_-(\mathbb{R})$, then

$$L_2(\mathbb{R}) = H^2_+(\mathbb{R}) \oplus H^2_-(\mathbb{R}),$$

and the *same* Fourier transform gives

1. $F(H^2_-(\mathbb{R})) = L_2(\mathbb{R}_-)$,

2. $F\, P^-_{\mathbb{R}}\, F^{-1} = \chi_- I.$

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Here we use the obvious notations: $\mathbb{R}_+$ ($\mathbb{R}_-$) is the positive (negative) half-line, $\chi_+$ ($\chi_-$) is the characteristic function of $\mathbb{R}_+$ ($\mathbb{R}_-$).

Consider now the space $L_2(\Pi)$ and its Bergman subspace $\mathcal{A}^2(\Pi)$, consisting of all analytic in $\Pi$ functions. A number of natural questions arises:

- how much room does the Bergman space $\mathcal{A}^2(\Pi)$ occupy incide $L_2(\Pi)$,
- does there exist a complete decomposition of $L_2(\Pi)$ onto analytic and analytic like spaces,
- does there exist an unitary operator on $L_2(\Pi)$, which *simultaneously* reduces all the pieces of decomposition to some simple understandable form,
- how do the projections onto these pieces look like,
- are there any connections between these pieces and the Hardy spaces?

In the article we give answers to all these questions. To describe the results introduce necessary notations. For each $n \in \mathbb{N}$ denote by $\mathcal{A}_n^2(\Pi)$ ($\widetilde{\mathcal{A}}_n^2(\Pi)$) the subspace of $L_2(\Pi)$, consisting of all $n$-analytic ($n$-anti-analytic) functions, i.e., the functions satisfying the equation $(\partial/\partial\overline{w})^n\varphi = 0$ ($(\partial/\partial w)^n\varphi = 0$). And let $\mathcal{A}_{(n)}^2(\Pi) = \mathcal{A}_n^2(\Pi) \ominus \mathcal{A}_{n-1}^2(\Pi)$ ($\widetilde{\mathcal{A}}_n^2(\Pi) = \widetilde{\mathcal{A}}_n^2(\Pi) \ominus \widetilde{\mathcal{A}}_{n-1}^2(\Pi)$) be the space of true-$n$-analytic (true-$n$-anti-analytic) functions. We will use the following ortonormal base in $L_2(\mathbb{R}_+)$: $\{\ell_n(y)\}_{n=0}^\infty$, where $\ell_n(y) = e^{y/2}L_n(y)$, $L_n(y)$ is the Laguerre polinomial of degree $n$. Denote by $L_n$ the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by $\ell_n(y)$, and by $P_n$ the orthogonal projection of $L_2(\mathbb{R}_+)$ onto $L_n$.

The following main results obtained:

1. The space $L_2(\Pi)$ admits the following decomposition
$$L_2(\Pi) = \bigoplus_{k=1}^{\infty} \mathcal{A}_{(k)}^2(\Pi) \oplus \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}_{(k)}^2(\Pi).$$

2. There exists an unitary operator $U : L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$, such that for all $n \in \mathbb{N}$
$$\begin{aligned} U &: \mathcal{A}_{(n)}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+) \otimes L_{n-1}, \\ U &: \widetilde{\mathcal{A}}_{(n)}^2(\Pi) \longrightarrow L_2(\mathbb{R}_-) \otimes L_{n-1}. \end{aligned}$$

3. The orthogonal projections $B_\Pi^{(n)}$ and $\widetilde{B}_\Pi^{(n)}$ of $L_2(\Pi)$ onto $\mathcal{A}_{(n)}^2(\Pi)$ and $\widetilde{\mathcal{A}}_{(n)}^2(\Pi)$ have the following form
$$\begin{aligned} (B_\Pi^{(n)}\varphi)(w) &= \int_\Pi K_\Pi^{(n)}(w, \overline{w}; \omega, \overline{\omega})\, \varphi(\omega)\, d\mu(\omega), \\ (\widetilde{B}_\Pi^{(n)}\varphi)(w) &= \int_\Pi K_\Pi^{(n)}(\omega, \overline{\omega}; w, \overline{w})\, \varphi(\omega)\, d\mu(\omega), \end{aligned}$$

where

$$K_{\Pi}^{(n)}(w,\overline{w};\omega,\overline{\omega}) = -\frac{1}{\pi}\cdot\frac{1}{(w-\overline{\omega})^2}\cdot\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\kappa_{j,k}^{n-1}\left(\frac{w-\overline{w}}{w-\overline{\omega}}\right)^j\left(\frac{\omega-\overline{\omega}}{w-\overline{\omega}}\right)^k,$$

and $\kappa_{j,k}^{n-1}$ are given by (4.5).

4. There exists an unitary operator $W: L_2(\Pi) = L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+)$, such that for all $n\in\mathbb{N}$

$$\begin{aligned} W &: \mathcal{A}^2_{(n)}(\Pi) \longrightarrow H^2_+(\mathbb{R})\otimes L_{n-1}, \\ W &: \widetilde{\mathcal{A}}^2_{(n)}(\Pi) \longrightarrow H^2_-(\mathbb{R})\otimes L_{n-1}. \end{aligned}$$

# 2 Bergman space and Bergman projection

Let $\Pi$ be the upper half-plane in $\mathbb{C}$, introduce the space $L_2(\Pi)$, with usual Lebesgue volume measure $d\mu(w) = dudv$, $w = u+iv$, and its subspace $\mathcal{A}^2(\Pi)$, the Bergman space, consisting of all functions analytic in $\Pi$. It is well known that the orthogonal projection $B_\Pi$ of $L_2(\Pi)$ onto $\mathcal{A}^2(\Pi)$, the so-called Bergman projection, is given by

$$(B_\pi\varphi)(w) = \int_\Pi K_\Pi(w,\overline{\omega})\varphi(\omega)\,d\mu(\omega),$$

where the Bergman kern-function $K_\Pi(w,\overline{\omega})$ of the upper half-plane is given by

$$K_\Pi(w,\overline{\omega}) = \frac{1}{\pi}\cdot\frac{1}{(w-\overline{\omega})^2}.$$

Note, that the Bergman space $\mathcal{A}^2(\Pi)$ can be described alternatively as the (closed) subspace of $L_2(\Pi)$, which consists of all functions satisfying the equation

$$2\frac{\partial}{\partial\overline{w}}\,\varphi = (\frac{\partial}{\partial u}+i\frac{\partial}{\partial v})\,\varphi = 0,$$

where $w = u+iv$.

Introduce the unitary operator

$$U_1 = F\otimes I:\ L_2(\Pi) = L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+),$$

where the Fourier transform $F: L_2(\mathbb{R})\to L_2(\mathbb{R})$ is given by

$$(Ff)(u) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-iu\xi}f(\xi)\,d\xi.$$

Then the image of the Bergman space $\mathcal{A}^2_1 = U_1(\mathcal{A}^2(\Pi))$ can be described as the (closed) subspace of $L_2(\Pi)$, which consists of all functions satisfying the equation

$$(F\otimes I)2\frac{\partial}{\partial\overline{w}}(F^{-1}\otimes I)\,\varphi = i(u+\frac{\partial}{\partial v})\,\varphi = 0. \tag{2.1}$$

The equation (2.1) is easy to solve, and its general solution has the form

$$\varphi(u,v) = \psi(u) \cdot e^{-uv}.$$

The condition $\varphi \in L_2(\Pi)$ suggests (and we will check this at the moment) that the space $\mathcal{A}_1^2$ consists of all the functions

$$\varphi_0(u,v) = \chi_+(u)\,\sqrt{2u}\,f(u) \cdot e^{-uv}, \tag{2.2}$$

where $f(u) \in L_2(\mathbb{R})$, and $\chi_+(u)$ is the characteristic function of the positive half-line, here, in addition

$$\|\varphi_0(u,v)\|_{\mathcal{A}_1^2} = \|f(u)\|_{L_2(\mathbb{R}_+)}.$$

The orthogonal projection $B_1 : L_2(\Pi) \to \mathcal{A}_1^2$ has obviously the form

$$B_1 = (F \otimes I) B_\Pi (F^{-1} \otimes I).$$

Calculate

$$\begin{aligned}
(F^{-1} \otimes I) B_1 \varphi &= B_\Pi (F^{-1} \otimes I)\varphi = \langle (F^{-1} \otimes I)\varphi, \overline{K_\Pi(w,\overline{\omega})} \rangle \\
&= \langle \varphi, (F \otimes I)\overline{K_\Pi(w,\overline{\omega})} \rangle = \langle \varphi, (F \otimes I) K_\Pi(\omega,\overline{w}) \rangle.
\end{aligned}$$

By [1] the Fourier transform (with respect to $\xi = \operatorname{Re}\omega$) of the function

$$K_\Pi(\omega,\overline{w}) = -\frac{1}{\pi} \cdot \frac{1}{(\omega - \overline{w})^2} = \frac{1}{\pi} \cdot \frac{1}{(i\overline{w} + \eta - i\xi)^2}$$

is equal to

$$\sqrt{\frac{2}{\pi}}\,\chi_+(\xi)\,\xi \cdot e^{-(\eta + i\overline{w})\xi}.$$

Thus

$$\begin{aligned}
(F^{-1} \otimes I) B_1 \varphi &= \langle \varphi, (F \otimes I) K_\Pi(\omega,\overline{w}) \rangle \\
&= \sqrt{\frac{2}{\pi}} \int_\Pi \varphi(\xi,\eta)\,\chi_+(\xi)\,\xi \cdot e^{-(\eta - iw)\xi}\, d\xi\, d\eta \\
&= \sqrt{\frac{2}{\pi}} \int_\Pi \varphi(\xi,\eta)\,\chi_+(\xi)\,\xi \cdot e^{-(\eta + v)\xi} \cdot e^{iu\xi}\, d\xi\, d\eta \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left( \int_{\mathbb{R}_+} 2\xi\,\chi_+(\xi)\,\varphi(\xi,\eta) \cdot e^{-(\eta+v)\xi}\, d\eta \right) e^{iu\xi}\, d\xi \\
&= (F^{-1} \otimes I) \left( 2u\,\chi_+(u) \cdot e^{-uv} \int_{\mathbb{R}_+} \varphi(u,\eta) \cdot e^{-\eta u}\, d\eta \right).
\end{aligned}$$

Or

$$(B_1\varphi)(u,v) = 2u\,\chi_+(u) \cdot e^{-uv} \int_{\mathbb{R}_+} \varphi(u,\eta) \cdot e^{-\eta u}\, d\eta.$$

Thus for each function $\varphi \in L_2(\Pi)$ its image $(B_1\varphi)(u,v)$ has the form (2.2) with

$$f(u) = \sqrt{2u} \int_{\mathbb{R}_+} \varphi(u,\eta) \cdot e^{-\eta u} \, d\eta,$$

and furthermore, for each function $\varphi_0$ of the form (2.2) we have

$$(B_1\varphi_0)(u,v) = \varphi_0(u,v).$$

Really

$$\begin{aligned}
(B_1\varphi_0)(u,v) &= 2u\,\chi_+(u) \cdot e^{-uv} \int_{\mathbb{R}_+} \varphi_0(u,\eta) \cdot e^{-\eta u} \, d\eta \\
&= 2u\,\chi_+(u) \cdot e^{-uv} \sqrt{2u}\, f(u) \int_{\mathbb{R}_+} e^{-2\eta u} \, d\eta \\
&= \chi_+(u) \sqrt{2u}\, f(u) \cdot e^{-uv} \int_{\mathbb{R}_+} 2u\, e^{-2\eta u} \, d\eta \\
&= \chi_+(u) \sqrt{2u}\, f(u) \cdot e^{-uv} = \varphi_0(u,v).
\end{aligned}$$

Thus $\mathcal{A}_1^2$ coincides with the space of all functions of the form (2.2).

Introduce the unitary operator

$$U_2 : L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

by the rule

$$U_2 : \varphi(u,v) \longmapsto \frac{1}{\sqrt{2|x|}} \varphi(x, \frac{y}{2|x|}).$$

Then the inverse operator $U_2^{-1} = U_2^* : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ is given by

$$U_2^{-1} : \varphi(x,y) \longmapsto \sqrt{2|u|}\, \varphi(u, 2|u| \cdot v).$$

Denote $\mathcal{A}_2^2 = U_2(\mathcal{A}_1^2)$, the operator $B_2 = U_2\, B_1\, U_2^{-1}$ is obviously the orthogonal projection of $L_2(\Pi)$ onto $\mathcal{A}_2^2$. Calculate

$$\begin{aligned}
(B_2\varphi)(x,y) &= U_2(2u\,\chi_+(u)\, e^{-uv} \int_{\mathbb{R}_+} e^{-u\eta} \sqrt{2u}\, \varphi(u, 2u\eta)\, d\eta) \\
&= U_2(\sqrt{2u}\,\chi_+(u)\, e^{-uv} \int_{\mathbb{R}_+} \varphi(u,\nu)\, e^{-\nu/2}\, d\nu) \\
&= \chi_+(x)\, e^{-y/2} \int_{\mathbb{R}_+} \varphi(x,\nu)\, e^{-\nu/2}\, d\nu.
\end{aligned}$$

Introduce $\ell_0(y) = e^{-y/2}$, we have $\ell_0(y) \in L_2(\mathbb{R}_+)$ and $\|\ell_0(y)\| = 1$. Denote by $L_0$ the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_0(y)$, then the one-dimensional projection $P_0$ of $L_2(\mathbb{R}_+)$ onto $L_0$ has the form

$$(P_0\psi)(y) = \langle \psi, \ell_0 \rangle \cdot \ell_0 = e^{-y/2} \int_{\mathbb{R}_+} \psi(\nu)\, e^{-\nu/2}\, d\nu. \tag{2.3}$$

Thus

$$B_2 = \chi_+(x) I \otimes P_0.$$

The above work leads up to the following theorem.

**Theorem 2.1** *The unitary operator $U = U_2 U_1$ gives an isometrical isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ under which*

1. *the Bergman space $\mathcal{A}^2(\Pi)$ is mapped onto $L_2(\mathbb{R}_+) \otimes L_0$*

$$U : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+) \otimes L_0,$$

*where $L_0$ is the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by $\ell_0 = e^{-y/2}$,*

2. *the Bergman projection $B_\Pi$ is unitary equivalent to the following one*

$$U\, B_\Pi\, U^{-1} = \chi_+ I \otimes P_0,$$

*where $P_0$ is the one-dimensional projection (2.3) of $L_2(\mathbb{R}_+)$ onto $L_0$.*

**Remark 2.2** The above result describes the structure of the Bergman space inside $L_2(\Pi)$, and is a "Begrman version" of the following long ago well known "Hardy version" result: *the Fourier transform $F$ gives an isometrical isomorphism of the space $L_2(R)$, under which*

1. *the Hardy space $H^2_+(\mathbb{R})$ is mapped onto $L_2(\mathbb{R}_+)$*

$$F : H^2_+(\mathbb{R}) \longrightarrow L_2(\mathbb{R}_+),$$

2. *the Szegö projection $P^+_{\mathbb{R}} : L_2(\mathbb{R}) \longrightarrow H^2_+(\mathbb{R})$ is unitary equivalent to the following one*

$$F\, P^+_{\mathbb{R}}\, F^{-1} = \chi_+ I.$$

Introduce the isometrical imbedding

$$R_0 : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

by the rule

$$(R_0 f)(x,y) = \chi_+(x)\, f(x)\, \ell_0(y).$$

The image of $R_0$ is obviously coincides with the space $\mathcal{A}^2_2$. The adjoint operator $R_0^* : L_2(\Pi) \to L_2(\mathbb{R}_+)$ is given by

$$(R_0^* \varphi)(x) = \chi_+(x) \int_{\mathbb{R}_+} \varphi(x,\eta)\, \ell_0(\eta)\, d\eta,$$

and

$$\begin{array}{rcl} R_0^* R_0 = I & : & L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+) \\ R_0 R_0^* = B_2 & : & L_2(\Pi) \longrightarrow \mathcal{A}^2_2 = L_2(\mathbb{R}_+) \otimes L_0. \end{array}$$

Now the operator $R = R_0^* U$ maps the space $L_2(\Pi)$ onto $L_2(\mathbb{R}_+)$, and the restriction

$$R|_{\mathcal{A}^2(\Pi)} : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+)$$

is an isometrical isomorphism. The adjoint operator

$$R^* = U^* R_0 : L_2(\mathbb{R}_+) \longrightarrow \mathcal{A}^2(\Pi) \subset L_2(\Pi)$$

is an isometrical isomorphism of $L_2(\mathbb{R}_+)$ onto the subspace $\mathcal{A}^2(\Pi)$ of the space $L_2(\Pi)$.

**Remark 2.3** We have

$$\begin{aligned} R\,R^* = I \quad &: \quad L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+) \\ R^* R = B_\Pi \quad &: \quad L_2(\Pi) \longrightarrow \mathcal{A}^2(\Pi). \end{aligned}$$

**Theorem 2.4** *The isometrical isomorphism*

$$R^* = U^* R_0 : L_2(\mathbb{R}_+) \longrightarrow \mathcal{A}^2(\Pi)$$

*is given by*

$$(R^* f)(z) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} \sqrt{\xi}\, f(\xi)\, e^{iz\cdot\xi}\, d\xi.$$

Proof. Calculate

$$\begin{aligned} (R^* f)(z) &= (U_1^* U_2^* R_0 f)(z) \\ &= (F^{-1} \otimes I)(\chi_+(\xi)\, f(\xi)\, \sqrt{2\xi}\, e^{-xy}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_+(\xi)\, f(\xi)\, \sqrt{2\xi}\, e^{-xy}\, e^{ix\xi}\, d\xi \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} \sqrt{\xi}\, f(\xi)\, e^{i(x+iy)\cdot\xi}\, d\xi. \end{aligned}$$

□

**Corollary 2.5** *The inverse isomorphism*

$$R : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+)$$

*is given by*

$$(R\varphi)(x) = \sqrt{x}\, \frac{1}{\sqrt{\pi}} \int_\Pi \varphi(w)\, e^{-i\overline{w}\cdot x}\, d\mu(w).$$

## 3 Connections between Bergman and Hardy spaces

The first immediate result follows directly from Theorem 2.1 and the statement inside Remark 2.2.

**Theorem 3.1** *The unitary operator $W = (F^{-1} \otimes I)U_2(F \otimes I)$ gives an isometrical isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ under which*

1. *the Bergman $\mathcal{A}^2(\Pi)$ and the Hardy $H^2_+(\mathbb{R})$ spaces are connected by the formula*

$$W(\mathcal{A}^2(\Pi)) = H^2_+(\mathbb{R}) \otimes L_0,$$

2. *the Bergman $B_\Pi$ and the Szegö $P^+_{\mathbb{R}}$ projections are connected by the formula*

$$W\, B_\Pi\, W^{-1} = P^+_{\mathbb{R}} \otimes P_0,$$

*where $P_0$ is the one-dimensional projection (2.3) of $L_2(\mathbb{R}_+)$ onto one-dimensional space $L_0$ generated by $\ell_0(y) = e^{-y/2} \in L_2(\mathbb{R}_+)$.*

For a domain $D$ (which will be $\Pi$ later on) besides the Bergman space $\mathcal{A}^2(D)$ introduce the space $\widetilde{\mathcal{A}}^2(D)$ as the (closed) subspace of $L_2(D)$ consisting of all functions, anti-analytic in $D$. It is well known (see, for example, [4]) that if $K_D(z, \overline{\zeta})$ is the Bergman kern-function of the domain $D$, then the orthogonal projection $\widetilde{B}_D$ of $L_2(D)$ onto $\widetilde{\mathcal{A}}^2(D)$ is given by

$$(\widetilde{B}_D\varphi)(z) = \int_D K_D(\zeta, \overline{z})\, \varphi(\zeta)\, d\mu(\zeta).$$

**Theorem 3.2** *The unitary operator $U = U_2U_1$ gives an isometrical isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ under which*

1. *the space $\widetilde{\mathcal{A}}^2(\Pi)$ is mapped onto $L_2(\mathbb{R}_-) \otimes L_0$*

$$U : \widetilde{\mathcal{A}}^2(\Pi) \longrightarrow L_2(\mathbb{R}_-) \otimes L_0,$$

   *where $L_0$ is the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by $\ell_0 = e^{-y/2}$,*

2. *the projection $\widetilde{B}_\Pi$ is unitary equivalent to the following one*

$$U\, \widetilde{B}_\Pi\, U^{-1} = \chi_- I \otimes P_0,$$

   *where $\chi_-$ is the characteristic function of the negative half-line, and $P_0$ is the one-dimensional projection (2.3) of $L_2(\mathbb{R}_+)$ onto $L_0$.*

PROOF. Follows all the steps of the proof of Theorem 2.1. Note only, that all the $L_2(\Pi)$ – solutions of the equation

$$(u - \frac{\partial}{\partial v})\varphi = 0$$

are of the form

$$\varphi(u,v) = \chi_-(u)\,\sqrt{2|u|}\,f(u)\,e^{u\,v},$$

where $f(u) \in L_2(\mathbb{R})$, and that the Fourier image (with respect to $\xi = \operatorname{Re}\omega$) of the function

$$\overline{K_\Pi(\omega.\overline{w})} = -\frac{1}{\pi}\,\frac{1}{(\overline{\omega} - w)^2}$$

is equal (see, for example, [1]) to the following function

$$-\sqrt{\frac{2}{\pi}}\,\xi\,\chi_-(\xi)\,e^{(\eta - iw)\cdot\xi}.$$

The projection $\widetilde{B}_1 = U_1\,\widetilde{B}_\Pi\,U_1^{-1}$ onto the space $\widetilde{\mathcal{A}}_1^2 = U_1(\widetilde{\mathcal{A}}^2(\Pi))$ has the form

$$(\widetilde{B}_1\varphi)(u.v) = 2|u|\,\chi_-(u)\,e^{uv}\int_{\mathbb{R}_+}\varphi(u,\eta)\,e^{u\eta}\,d\eta.$$

□

Introduce now the Hardy space $H_-^2(\mathbb{R})$ as the subspace of $L_2(\mathbb{R})$ consisting of the functions having analytic extension to the lower half-plane. It is well known that the operator $P_{\mathbb{R}}^- = F^{-1}\,\chi_-(\xi)\,F$ is the orthogonal (Szegö) projection of $L_2(\mathbb{R})$ onto $H_-^2(\mathbb{R})$.

**Corollary 3.3** *The unitary operator $W = (F^{-1} \otimes I)U_2(F \otimes I)$ gives an isometrical isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ under which*

1. *the spaces $\widetilde{\mathcal{A}}^2(\Pi)$ and $H_-^2(\mathbb{R})$ are connected by the formula*

$$W(\widetilde{\mathcal{A}}^2(\Pi)) = H_-^2(\mathbb{R}) \otimes L_0,$$

2. *the projections $\widetilde{B}_\Pi$ and $P_{\mathbb{R}}^-$ are connected by the formula*

$$W\,\widetilde{B}_\Pi\,W^{-1} = P_{\mathbb{R}}^- \otimes P_0,$$

*where $P_0$ is the one-dimensional projection (2.3) of $L_2(\mathbb{R}_+)$ onto one-dimensional space $L_0$ generated by $\ell_0(y) = e^{-y/2} \in L_2(\mathbb{R}_+)$.*

**Remark 3.4** The direct sum of both Hardy spaces coincides with the whole space

$$L_2(\mathbb{R}) = H_+^2(\mathbb{R}) \oplus H_-^2(\mathbb{R}),$$

while the direct sum $\mathcal{A}^2(\Pi) \oplus \widetilde{\mathcal{A}}^2(\Pi)$ has the (sufficiently large) orthogonal complement

$$\mathcal{A}^\perp(\Pi) = L_2(\Pi) \ominus (\mathcal{A}^2(\Pi) \oplus \widetilde{\mathcal{A}}^2(\Pi)),$$

whose image under the mapping $W$ is equal to

$$W(\mathcal{A}^\perp(\Pi)) = L_2(\mathbb{R}) \otimes L_0^\perp.$$

## 4 Poly-Bergman spaces, decomposition of $L_2(\Pi)$

Analogously to the Bergman spaces $\mathcal{A}^2(\Pi)$ and $\widetilde{\mathcal{A}}^2(\Pi)$, of respectively analytic and anti-analytic functions from $L_2(\Pi)$, introduce the spaces of poly-analytic and poly-anti-analytic functions (see, [3, 4] for bounded domains), the poly-Bergman spaces.

Define the space $\mathcal{A}_n^2(\Pi)$ of $n$-analytic functions as the (closed) subspace of $L_2(\Pi)$ of all functions $\varphi = \varphi(w, \overline{w}) = \varphi(u, v)$, which satisfy the equation

$$\left(\frac{\partial}{\partial \overline{w}}\right)^n \varphi = \frac{1}{2^n}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)^n \varphi = 0.$$

Similarly, define the space $\widetilde{\mathcal{A}}_n^2(\Pi)$ of $n$-anti-analytic functions as the (closed) subspace of $L_2(\Pi)$ of all functions $\varphi = \varphi(w, \overline{w}) = \varphi(u, v)$, which satisfy the equation

$$\left(\frac{\partial}{\partial w}\right)^n \varphi = \frac{1}{2^n}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)^n \varphi = 0.$$

Of course, we have $\mathcal{A}_1^2(\Pi) = \mathcal{A}^2(\Pi)$ and $\widetilde{\mathcal{A}}_1^2(\Pi) = \widetilde{\mathcal{A}}^2(\Pi)$, for $n = 1$, as well as $\mathcal{A}_n^2(\Pi) \subset \mathcal{A}_{n+1}^2(\Pi)$ and $\widetilde{\mathcal{A}}_n^2(\Pi) \subset \widetilde{\mathcal{A}}_{n+1}^2(\Pi)$, for each $n \in \mathbb{N}$.

Recall (see, for example, [2]), that the Laguerre polinomial $L_n(y)$ of degree $n$, $n = 0, 1, 2, ...$, and type 0 is defined by

$$\begin{aligned} L_n(y) &= L_n^0(y) = \frac{e^y}{n!}\frac{d^n}{dy^n}(e^{-y}\, y^n) \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{(-y)^k}{k!}, \qquad y \in \mathbb{R}_+, \end{aligned} \tag{4.1}$$

and that the system of functions

$$\ell_n(y) = e^{-y/2} L_n(y), \qquad n = 0, 1, 2, ... \tag{4.2}$$

form an ortonormal base in the space $L_2(\mathbb{R}_+)$.

Denote by $L_n$, $n = 0, 1, 2, ...$, the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by the function $\ell_n(y)$. Note, that for $n = 0$ this definition gives exactly previously defined space $L_0$. And let

$$L_n^{\oplus} = \bigoplus_{k=0}^{n} L_k$$

be the direct sum of the first $(n+1)$ spaces.

**Theorem 4.1** *The unitary operator $U : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ maps the space $\mathcal{A}_n^2(\Pi)$ of $n$-analytic functions onto the space $L_2(\mathbb{R}_+) \otimes L_{n-1}^{\oplus}$.*

Proof. The space $U(\mathcal{A}_n^2(\Pi))$ obviously coincides with the set of all functions from $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$, which satisfy the equation

$$\begin{aligned} U\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)^n U^{-1}\varphi &= i^n\, U_2\left(u + \frac{\partial}{\partial v}\right)^n U_2^{-1}\varphi \\ &= i^n\, |x|^n \left(\operatorname{sign}(x) + 2\frac{\partial}{\partial y}\right)^n \varphi = 0. \end{aligned}$$

It is easy to see, that the intersection of the general solution of this equation with the space $L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ coincides with the set of all functions

$$\sum_{k=0}^{n-1} \chi_+(x)\, \psi_k(x)\, y^k e^{-y/2},$$

where $\psi_k(x) \in L_2(\mathbb{R})$, for all $k = \overline{0, n-1}$, or rearranging polynomials on $y$, with the set of all functions

$$\sum_{k=0}^{n-1} \chi_+(x)\, f_k(x)\, L_k(y) e^{-y/2} = \sum_{k=0}^{n-1} \chi_+(x)\, f_k(x)\, \ell_k(y),$$

where $f_k(x) \in L_2(\mathbb{R})$, for all $k = \overline{0, n-1}$. □

Introduce the space $\mathcal{A}^2_{(n)}$ of true-$n$-analytic functions by

$$\mathcal{A}^2_{(n)} = \mathcal{A}^2_n \ominus \mathcal{A}^2_{n-1},$$

for $n > 1$, and by $\mathcal{A}^2_{(1)} = \mathcal{A}^2_1$, for $n = 1$, then, of course,

$$\mathcal{A}^2_n = \bigoplus_{k=1}^{n} \mathcal{A}^2_{(k)}.$$

**Corollary 4.2** *The unitary operator $U : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ maps the space $\mathcal{A}^2_{(n)}(\Pi)$ of true-$n$-analytic functions onto the space $L_2(\mathbb{R}_+) \otimes L_{n-1}$.*

Analogously in the anti-analytic situations we have the following assertions.

**Theorem 4.3** *The unitary operator $U : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ maps the space $\widetilde{\mathcal{A}}^2_n(\Pi)$ of $n$-anti-analytic functions onto the space $L_2(\mathbb{R}_-) \otimes L^{\oplus}_{n-1}$.*

Proof. Follows all the steps of the proof of Theorem 4.1. The space $U(\widetilde{\mathcal{A}}^2_n(\Pi))$ coincides with the set of all functions from $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$, which satisfy the equation

$$\begin{aligned} U\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)^n U^{-1}\varphi &= i^n\, U_2 \left(u - \frac{\partial}{\partial v}\right)^n U_2^{-1}\varphi \\ &= i^n\, |x|^n \left(\operatorname{sign}(x) - 2\frac{\partial}{\partial y}\right)^n \varphi = 0. \end{aligned}$$

Now the intersection of the general solution of this equation with the space $L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ coincides with the set of all functions

$$\sum_{k=0}^{n-1} \chi_-(x)\, \psi_k(x)\, y^k e^{-y/2},$$

where $\psi_k(x) \in L_2(\mathbb{R})$, for all $k = \overline{0, n-1}$, or rearranging polynomials on $y$, with the set of all functions

$$\sum_{k=0}^{n-1} \chi_-(x)\, f_k(x)\, L_k(y) e^{-y/2} = \sum_{k=0}^{n-1} \chi_-(x)\, f_k(x)\, \ell_k(y),$$

where $f_k(x) \in L_2(\mathbb{R})$, for all $k = \overline{0, n-1}$. $\square$

Symmetrically, introduce the space $\widetilde{\mathcal{A}}^2_{(n)}$ of true-$n$-anti-analytic functions by

$$\widetilde{\mathcal{A}}^2_{(n)} = \widetilde{\mathcal{A}}^2_n \ominus \widetilde{\mathcal{A}}^2_{n-1},$$

for $n > 1$, and by $\widetilde{\mathcal{A}}^2_{(1)} = \widetilde{\mathcal{A}}^2_1$, for $n = 1$, analogously,

$$\widetilde{\mathcal{A}}^2_n = \bigoplus_{k=1}^{n} \widetilde{\mathcal{A}}^2_{(k)}.$$

**Corollary 4.4** *The unitary operator* $U : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ *maps the space* $\widetilde{\mathcal{A}}^2_{(n)}(\Pi)$ *of true-n-anti-analytic functions onto the space* $L_2(\mathbb{R}_-) \otimes L_{n-1}$.

The above results lead up to the following theorem.

**Theorem 4.5** *We have the following isometrical isomorphisms and decompositions of the spaces:*

1. *Isomorphic images of poly-analytic spaces*

$$\begin{aligned} W &: \mathcal{A}^2_{(n)}(\Pi) \longrightarrow H^2_+(\mathbb{R}) \otimes L_{n-1}, \\ W &: \mathcal{A}^2_n(\Pi) \longrightarrow H^2_+(\mathbb{R}) \otimes \bigoplus_{k=0}^{n-1} L_k, \\ W &: \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)}(\Pi) \longrightarrow H^2_+(\mathbb{R}) \otimes L_2(\mathbb{R}_+). \end{aligned}$$

2. *Isomorphic images of poly-anti-analytic spaces*

$$\begin{aligned} W &: \widetilde{\mathcal{A}}^2_{(n)}(\Pi) \longrightarrow H^2_-(\mathbb{R}) \otimes L_{n-1}, \\ W &: \widetilde{\mathcal{A}}^2_n(\Pi) \longrightarrow H^2_-(\mathbb{R}) \otimes \bigoplus_{k=0}^{n-1} L_k, \\ W &: \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}^2_{(k)}(\Pi) \longrightarrow H^2_-(\mathbb{R}) \otimes L_2(\mathbb{R}_+). \end{aligned}$$

*3. Decomposition of the space $L_2(\Pi)$*

$$L_2(\Pi) = \bigoplus_{k=1}^{\infty}(\mathcal{A}^2_{(k)}(\Pi) \oplus \widetilde{\mathcal{A}}^2_{(k)}(\Pi)) \tag{4.3}$$
$$= \bigoplus_{k=1}^{\infty}\mathcal{A}^2_{(k)}(\Pi) \oplus \bigoplus_{k=1}^{\infty}\widetilde{\mathcal{A}}^2_{(k)}(\Pi).$$

*4. Decomposition of $\mathcal{A}^{\perp}(\Pi)$*

$$\mathcal{A}^{\perp}(\Pi) = \bigoplus_{k=2}^{\infty}(\mathcal{A}^2_{(k)}(\Pi) \oplus \widetilde{\mathcal{A}}^2_{(k)}(\Pi)).$$

*Here $L_n$ is the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by the function $\ell_n(y) = e^{-y/2}L_n(y)$, where the Laguerre polinomial $L_n(y)$ of degree $n$ is given by (4.1).*

The procedure reverse to one of the first section allows us to give exact formulas for the orthogonal projections onto each summand in the decomposition (4.3). Of course it is sufficient to describe the projections onto the spaces of true-$n$-analytic and true-$n$-anti-analytic functions. Start with the projection onto the space $\mathcal{A}^2_{(n)}(\Pi)$ of true-$n$-analytic functions.

**Theorem 4.6** *The orthogonal projection $B_{\Pi}^{(n)}$ of $L_2(\Pi)$ onto the space $\mathcal{A}^2_{(n)}(\Pi)$ of true-$n$-analytic functions is given by*

$$(B_{\Pi}^{(n)}\varphi)(w) = \int_{\Pi} K_{\Pi}^{(n)}(w,\overline{\omega})\,\varphi(\omega)\,d\mu(\omega),$$

*where*

$$K_{\Pi}^{(n)}(w,\overline{\omega}) = K_{\Pi}(w,\overline{\omega}) \cdot \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \kappa_{j,k}^{n-1}\left(\frac{2iv}{w-\overline{\omega}}\right)^j \left(\frac{2i\eta}{w-\overline{\omega}}\right)^k, \tag{4.4}$$

*and*

$$K_{\Pi}(w,\overline{\omega}) = -\frac{1}{\pi}\cdot\frac{1}{(w-\overline{\omega})^2}$$

*is the Bergman kern-function of the domain $\Pi$, $w = u+iv$, $\omega = \xi + i\eta$,*

$$\kappa_{j,k}^{n-1} = (-1)^{j+k}\left(\frac{n!}{j!\,k!}\right)^2 \frac{(j+k+1)!}{(n-j)!\,(n-k)!}. \tag{4.5}$$

PROOF. The orthogonal projection $U\,B_{\Pi}^{(n)}\,U^{-1} = B_2^{(n)} : L_2(\Pi) \to L_2(\mathbb{R}_+) \otimes L_{n-1}$ is obviously given by

$$B_2^{(n)} = \chi_+(x) \otimes P_{n-1},$$

where

$$(P_{n-1}\psi)(y) = \langle\psi, \ell_{n-1}\rangle\,\ell_{n-1} = \ell_{n-1}(y)\int_{\mathbb{R}_+}\psi(\nu)\,\ell_{n-1}(\nu)\,d\nu \tag{4.6}$$

is the orthogonal projection of $L_2(\mathbb{R}_+)$ onto the one-dimensional space $L_{n-1}$, generated by the function $\ell_{n-1}(y)$ given by (4.2). Or,

$$(B_2^{(n)}\varphi)(x,y) = \chi_+(x)\ell_{n-1}(y)\int_{\mathbb{R}_+}\varphi(x,\nu)\,\ell_{n-1}(\nu)\,d\nu.$$

Calculate the projection $B_1^{(n)} = U_2^{-1}\,B_2^{(n)}\,U_2$

$$\begin{aligned}(B_1^{(n)}\varphi)(u,v) &= \sqrt{2|u|}\,\chi_+(u)\,\ell_{n-1}(2|u|v)\int_{\mathbb{R}_+}\frac{1}{\sqrt{2|u|}}\varphi(u,\frac{\nu}{2|u|})\,\ell_{n-1}(\nu)\,d\nu\\ &= \chi_+(u)\,2u\,e^{-uv}\,L_{n-1}(2uv)\int_{\mathbb{R}_+}\varphi(u,\eta)\,e^{-u\eta}\,L_{n-1}(2u\eta)\,d\eta.\end{aligned}$$

Calculate now

$$\begin{aligned}(F^{-1}\otimes I)B_1^{(n)}\varphi &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}_+}\chi_+(\xi)2\xi e^{-v\xi}L_{n-1}(2\xi v)L_{n-1}(2\xi\eta)\varphi(\xi,\eta)e^{-\eta\xi}d\eta\right)e^{iu\xi}d\xi\\ &= \sqrt{\frac{2}{\pi}}\int_{\Pi}\varphi(\xi,\eta)\chi_+(\xi)\xi L_{n-1}(2\xi v)L_{n-1}(2\xi\eta)e^{-(\eta-iw)\xi}d\xi d\eta.\end{aligned}$$

From $B_2^{(n)} = U_2(F\otimes I)B_\Pi^{(n)}(F^{-1}\otimes I)U_2^{-1}$ it follows that

$$B_1^{(n)} = U_2^{-1}\,B_2^{(n)}\,U_2 = (F\otimes I)B_\Pi^{(n)}(F^{-1}\otimes I),$$

and that

$$\begin{aligned}(F^{-1}\otimes I)B_1^{(n)}\varphi &= B_\Pi^{(n)}(F^{-1}\otimes I)\varphi\\ &= \langle (F^{-1}\otimes I)\varphi, \overline{K_\Pi^{(n)}(w,\overline{\omega})}\rangle\\ &= \langle \varphi, (F\otimes I)\overline{K_\Pi^{(n)}(w,\overline{\omega})}\rangle.\end{aligned}$$

Thus

$$(F\otimes I)\overline{K_\Pi^{(n)}(w,\overline{\omega})} = \sqrt{\frac{2}{\pi}}\,\chi_+(\xi)\,\xi\,L_{n-1}(2\xi v)\,L_{n-1}(2\xi\eta)\,e^{-(\eta+i\,\overline{w})\xi}.$$

By (4.1)

$$L_{n-1}(y) = \sum_{k=0}^{n-1}\frac{n!}{k!\,(n-k)!}\,\frac{(-y)^k}{k!} = \sum_{k=0}^{n-1}\lambda_k^{n-1}\,y^k,$$

where

$$\lambda_k^{n-1} = \frac{(-1)^k\,n!}{k!\,(n-k)!\,k!}. \tag{4.7}$$

And thus

$$(F\otimes I)\overline{K_\Pi^{(n)}(w,\overline{\omega})} = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\lambda_j^{n-1}\,\lambda_k^{n-1}\,2^{j+k}\,v^j\,\eta^k\,(\sqrt{\frac{2}{\pi}}\,\chi_+(\xi)\,\xi^{j+k+1}\,e^{-(\eta+i\,\overline{w})\xi}).$$

By [1] the the function

$$\sqrt{\frac{2}{\pi}}\,\chi_+(\xi)\,\xi^m\,e^{-(\eta+i\,\overline{w})\xi}$$

is the Fourier transform of the following function

$$\frac{1}{\pi}\,\frac{m!}{(i\,\overline{w}+\eta-i\xi)^{m+1}}.$$

Thus

$$\overline{K_{\Pi}^{(n)}(w,\overline{\omega})} = \frac{1}{\pi}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\lambda_j^{n-1}\,\lambda_k^{n-1}\,2^{j+k}\,v^j\,\eta^k\,\frac{(j+k+1)!}{(i\,\overline{w}+\eta-i\xi)^{j+k+2}},$$

or

$$K_{\Pi}^{(n)}(w,\overline{\omega}) = -\frac{1}{\pi}\,\frac{1}{(w-\overline{\omega})^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\lambda_j^{n-1}\,\lambda_k^{n-1}\,(j+k+1)!\,2^{j+k}\frac{(iv)^j(i\eta)^k}{(w-\overline{\omega})^{j+k}}.$$

Introducing

$$\begin{aligned}\kappa_{j,k}^{n-1} &= \lambda_j^{n-1}\,\lambda_k^{n-1}\,(j+k+1)!\\ &= (-1)^{j+k}\left(\frac{n!}{j!\,k!}\right)^2\frac{(j+k+1)!}{(n-j)!\,(n-k)!},\end{aligned}$$

we have finally

$$K_{\Pi}^{(n)}(w,\overline{\omega}) = -\frac{1}{\pi}\,\frac{1}{(w-\overline{\omega})^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\kappa_{j,k}^{n-1}\left(\frac{2iv}{w-\overline{\omega}}\right)^j\left(\frac{2i\eta}{w-\overline{\omega}}\right)^k.$$

□

Describe now the orthogonal projection $\widetilde{B}_{\Pi}^{(n)}$ of $L_2(\Pi)$ onto the space $\widetilde{\mathcal{A}}_{(n)}^2(\Pi)$ of true-$n$-anti-analytic functions.

**Theorem 4.7** *The orthogonal projection $\widetilde{B}_{\Pi}^{(n)}$ of $L_2(\Pi)$ onto the space $\widetilde{\mathcal{A}}_{(n)}^2(\Pi)$ of true-$n$-anti-analytic functions is given by*

$$(\widetilde{B}_{\Pi}^{(n)}\varphi)(w) = \int_{\Pi}\widetilde{K}_{\Pi}^{(n)}(\overline{w},\omega)\,\varphi(\omega)\,d\mu(\omega),$$

*where*

$$\widetilde{K}_{\Pi}^{(n)}(\overline{w},\omega) = -\frac{1}{\pi}\cdot\frac{1}{(\overline{w}-\omega)^2}\cdot\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\kappa_{j,k}^{n-1}\left(\frac{-2iv}{\overline{w}-\omega}\right)^j\left(\frac{-2i\eta}{\overline{w}-\omega}\right)^k,$$

*$w=u+iv$, $\omega=\xi+i\eta$, and $\kappa_{j,k}^{n-1}$ are given by (4.5).*

Proof. Follows all the steps of the proof of Theorem 4.6, that is why we will mention only the key formulas. We have the following connections between the projections

$$\begin{aligned}\widetilde{B}_{\Pi}^{(n)} &= (F^{-1}\otimes I)\widetilde{B}_{1}^{(n)}(F\otimes I)\\ \widetilde{B}_{1}^{(n)} &= U_2^1\,\widetilde{B}_{2}^{(n)}\,U_2.\end{aligned}$$

The projections $\widetilde{B}_2^{(n)}$ and $\widetilde{B}_1^{(n)}$ are given by

$$(\widetilde{B}_2^{(n)}\varphi)(x,y) = \chi_-(x)\,\ell_{n-1}(y)\int_{\mathbb{R}_+}\varphi(x,\nu)\,\ell_{n-1}(\nu)\,d\nu,$$

where $\ell_{n-1}$ is defined in (4.2), and by

$$\begin{aligned}(\widetilde{B}_1^{(n)}\varphi)(u,v) &= \sqrt{2|u|}\chi_-(u)\ell_{n-1}(2|u|v)\int_{\mathbb{R}_+}\frac{1}{\sqrt{2|u|}}\varphi(u,\frac{\nu}{2|u|})\ell_{n-1}(\nu)d\nu \\ &= -2u\chi_-(u)e^{uv}L_{n-1}(-2uv)\int_{\mathbb{R}_+}\varphi(u,\eta)e^{u\eta}L_{n-1}(-2u\eta)d\eta,\end{aligned}$$

where $L_{n-1}(y)$ is the Laguerre polynomial (4.1).

Calculate

$$\begin{aligned}(F^{-1}\otimes I)\widetilde{B}_1^{(n)}\varphi &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}_+}(-2\xi)\chi_-(\xi)e^{v\xi}L_{n-1}(-2v\xi)L_{n-1}(-2\eta\xi)\varphi(\xi,\eta)e^{\eta\xi}d\eta\right)e^{iu\xi}d\xi \\ &= \sqrt{\frac{2}{\pi}}\int_{\Pi}\varphi(\xi,\eta)(-\xi)\chi_-(\xi)L_{n-1}(-2v\xi)L_{n-1}(-2\eta\xi)e^{(\eta+i\,\overline{w})\xi}d\xi d\eta.\end{aligned}$$

Because of

$$\begin{aligned}(F^{-1}\otimes I)\widetilde{B}_1^{(n)}\varphi &= \widetilde{B}_{\Pi}^{(n)}(F^{-1}\otimes I)\varphi \\ &= \langle (F^{-1}\otimes I)\varphi, \overline{\widetilde{K}_{\Pi}^{(n)}(\overline{w},\omega)}\rangle \\ &= \langle \varphi, (F\otimes I)\overline{\widetilde{K}_{\Pi}^{(n)}(\overline{w},\omega)}\rangle\end{aligned}$$

we have

$$\begin{aligned}(F\otimes I)\overline{\widetilde{K}_{\Pi}^{(n)}(\overline{w},\omega)} &= \sqrt{\frac{2}{\pi}}\,\chi_-(\xi)\,(-\xi)\,L_{n-1}(-2v\xi)L_{n-1}(-2\eta\xi)\,e^{(\eta-i\,w)\xi} \\ &= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\lambda_j^{n-1}\lambda_k^{n-1}2^{j+k}v^j\eta^k(\sqrt{\frac{2}{\pi}}\,\chi_-(\xi)\,(-\xi)^{j+k+1}\,e^{(\eta-i\,w)\xi}),\end{aligned}$$

where $\lambda_k^{n-1}$ is introduced in (4.7).

By [1] the the function

$$\sqrt{\frac{2}{\pi}}\,\chi_-(\xi)\,(-\xi)^m\,e^{(\eta-i\,w)\xi}$$

is the Fourier transform of the following function

$$\frac{1}{\pi}\,\frac{m!}{(i\xi+\eta-i\,w)^{m+1}}.$$

Thus

$$\overline{\widetilde{K}_{\Pi}^{(n)}(\overline{w},\omega)} = \frac{1}{\pi}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\lambda_j^{n-1}\,\lambda_k^{n-1}\,2^{j+k}\,v^j\,\eta^k\,\frac{(j+k+1)!}{(i\xi+\eta-i\,w)^{j+k+2}},$$

or

$$\begin{aligned}\widetilde{K}_{\Pi}^{(n)}(\overline{w},\omega) &= -\frac{1}{\pi}\,\frac{1}{(\overline{w}-\omega)^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\lambda_j^{n-1}\,\lambda_k^{n-1}\,(j+k+1)!\,2^{j+k}\frac{(-iv)^j(-i\eta)^k}{(\overline{w}-\omega)^{j+k}}\\ &= -\frac{1}{\pi}\,\frac{1}{(\overline{w}-\omega)^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\kappa_{j,k}^{n-1}\left(\frac{-2iv}{\overline{w}-\omega}\right)^j\left(\frac{-2i\eta}{\overline{w}-\omega}\right)^k,\end{aligned}$$

where $\kappa_{j,k}^{n-1}$ is given by (4.5). □

To uniform the statements of Theorem 4.6 and Theorem 4.7 rewrite the kernel $K_{\Pi}^{(n)}$ of the projection $B_{\Pi}^{(n)}$, given by (4.4), in the following form

$$K_{\Pi}^{(n)}(w,\overline{w};\omega,\overline{\omega}) = -\frac{1}{\pi}\cdot\frac{1}{(w-\overline{\omega})^2}\cdot\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\kappa_{j,k}^{n-1}\left(\frac{w-\overline{w}}{w-\overline{\omega}}\right)^j\left(\frac{\omega-\overline{\omega}}{w-\overline{\omega}}\right)^k, \tag{4.8}$$

where $\kappa_{j,k}^{n-1}$ is given by (4.5). The function $K_{\Pi}^{(n)}(w,\overline{w};\omega,\overline{\omega})$ is obviously Hermitian symmetric, i.e.,

$$\overline{K_{\Pi}^{(n)}(\omega,\overline{\omega};w,\overline{w})} = K_{\Pi}^{(n)}(w,\overline{w};\omega,\overline{\omega}),$$

and for $n=1$ coincides with the Bergman kern-function: $K_{\Pi}^{(1)}(w,\overline{w};\omega,\overline{\omega}) = K_{\Pi}(w,\overline{\omega})$.

Now we have

**Theorem 4.8** *The orthogonal projections $B_{\Pi}^{(n)}$ of $L_2(\Pi)$ onto the space $\mathcal{A}^2_{(n)}(\Pi)$ of true-n-analytic functions, and $\widetilde{B}_{\Pi}^{(n)}$ of $L_2(\Pi)$ onto the space $\widetilde{\mathcal{A}}^2_{(n)}(\Pi)$ of true-n-anti-analytic functions are given by the following formulas*

$$(B_{\Pi}^{(n)}\varphi)(w) = \int_{\Pi} K_{\Pi}^{(n)}(w,\overline{w};\omega,\overline{\omega})\,\varphi(\omega)\,d\mu(\omega),$$

*and*

$$\begin{aligned}(\widetilde{B}_{\Pi}^{(n)}\varphi)(w) &= \int_{\Pi}\overline{K_{\Pi}^{(n)}(w,\overline{w};\omega,\overline{\omega})}\,\varphi(\omega)\,d\mu(\omega)\\ &= \int_{\Pi}K_{\Pi}^{(n)}(\omega,\overline{\omega};w,\overline{w})\,\varphi(\omega)\,d\mu(\omega),\end{aligned}$$

*where the function $K_{\Pi}^{(n)}(w,\overline{w};\omega,\overline{\omega})$ is defined by (4.8).*

**Corollary 4.9** *The unitary operator $W : L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+)$ gives the following connections between the true-poly-Bergman projections $B_{\Pi}^{(n)}$, $\widetilde{B}_{\Pi}^{(n)}$ and the Szegö projections $P_{\Pi}^{+}$, $P_{\Pi}^{-}$*

$$\begin{aligned}W\,B_{\Pi}^{(n)}\,W^{-1} &= P_{\Pi}^{+}\otimes P_{n-1},\\ W\,\widetilde{B}_{\Pi}^{(n)}\,W^{-1} &= P_{\Pi}^{-}\otimes P_{n-1},\end{aligned}$$

*where $P_{n-1}$ in the orthogonal projection (4.6) of $L_2(\mathbb{R}_+)$ onto the one-dimensional space $L_{n-1}$, generated by the function $\ell_{n-1}(y)$, defined in (4.2).*

Proof. Follows directly from Corollaries 4.2 and 4.4. □

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