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On Quaternionic Bergman and Poly–Bergman Spaces*

N. L. VASILEVSKI

Departamento de Matemáticas, CINVESTAV del I.P.N., México-City, México
nvasilev@math.cinvestav.mx

Let Π be the upper half-space in $\mathbb{R}^4$, consider the Bergman space ${}^{\psi}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$, the subspace of all left-ψ-hyper-holomorphic $\mathbb{H}(\mathbb{C})$ valued functions from $L_2(\Pi, \mathbb{H}(\mathbb{C}))$. Complete decomposition of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ onto Bergman and Bergman type spaces of poly-ψ-hyper-holomorphic and poly-$\overline{\psi}$-hyper-holomorphic functions is obtained. Simple isomorphic descriptions of all these spaces are given. Connections with the Hardy spaces and the Szegö projections are established.

Keywords: Bergman space, quaternionic hyperholomorphic functions.

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1 Introduction

Start with well known one-dimensional complex analysis result. Consider the Hardy space $H^2_+(\mathbb{R})$ on the upper half-plane Π in $\mathbb{C}$, and the (orthogonal) Szegö projection $P^+_{\mathbb{R}}$ of $L_2(\mathbb{R})$ onto $H^2_+(\mathbb{R})$.

Theorem 1.1 *The Fourier transform F gives an isometric isomorphism of the space $L_2(R)$, under which*

1. *the Hardy space $H^2_+(\mathbb{R})$ is mapped onto $L_2(\mathbb{R}_+)$*

$$F : H^2_+(\mathbb{R}) \longrightarrow L_2(\mathbb{R}_+),$$

2. *the Szegö projection $P^+_{\mathbb{R}} : L_2(\mathbb{R}) \longrightarrow H^2_+(\mathbb{R})$ is unitary equivalent to the following one*

$$F\, P^+_{\mathbb{R}}\, F^{-1} = \chi_+ I.$$

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Furthermore, if $H_{-}^{2}(\mathbb{R})$ is the Hardy space on the lower half-plane and $P_{\mathbb{R}}^{-}$ is the (orthogonal) Szegö projection of $L_2(\mathbb{R})$ onto $H_{-}^{2}(\mathbb{R})$, then

$$L_2(\mathbb{R}) = H_{+}^{2}(\mathbb{R}) \oplus H_{-}^{2}(\mathbb{R}),$$

and the *same* Fourier transform gives

1. $F(H_{-}^{2}(\mathbb{R})) = L_2(\mathbb{R}_{-})$,
2. $F\, P_{\mathbb{R}}^{-}\, F^{-1} = \chi_{-} I$.

Here we use the obvious notations: $\mathbb{R}_{+}$ ($\mathbb{R}_{-}$) is the positive (negative) half-line, χ_{+} (χ_{-}) is the characteristic function of $\mathbb{R}_{+}$ ($\mathbb{R}_{-}$).

Theorem 2.1 below gives the exact analog of this result for quaternionic hyperholomorphic functions.

Consider now the quaternionic Bergman space of hyperholomorphic functions in the upper half-space Π in $\mathbb{R}^4$. A number of natural questions appears:

- how much room does the quaternionic Bergman space occupy incide corresponding $L_2(\Pi)$,
- does there exist a complete decomposition of $L_2(\Pi)$ onto hyperholomorphic and hyperholomorphic like spaces,
- does there exist an unitary operator on $L_2(\Pi)$, which *simultaneously* reduces all the pieces of decomposition to some simple transparent form,
- are there any connections between these pieces and the Hardy spaces?

In the article we answer to all these questions. To formulate main results introduce necessary notations. Consider the space $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ of complex quaternion valued functions. Let $\psi = \{\psi^0, \psi^1, \psi^2, \psi^3\}$ be the set of orthonormal (in the usual Euclidean sense) real quaternions, with $\psi^0 = 1$. For each $n \in \mathbb{N}$ denote by ${}^{\psi}\mathcal{A}_n^2(\Pi, \mathbb{H}(\mathbb{C}))$ (${}^{\overline{\psi}}\mathcal{A}_n^2(\Pi, \mathbb{H}(\mathbb{C}))$) the subspace of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$, consisting of all n-ψ-hyperholomorphic (n-$\overline{\psi}$-hyperholomorphic) functions, i.e., the functions satisfying the equation

$$\left(\sum_{k=0}^{3} \psi^k \frac{\partial}{\partial x_k}\right)^n f = 0, \qquad \left(\left(\sum_{k=0}^{3} \overline{\psi}^k \frac{\partial}{\partial x_k}\right)^n f = 0\right).$$

And let ${}^{\psi}\mathcal{A}_{(n)}^2(\Pi, \mathbb{H}(\mathbb{C})) = {}^{\psi}\mathcal{A}_n^2(\Pi, \mathbb{H}(\mathbb{C})) \ominus {}^{\psi}\mathcal{A}_{n-1}^2(\Pi, \mathbb{H}(\mathbb{C}))$ (${}^{\overline{\psi}}\mathcal{A}_{(n)}^2(\Pi, \mathbb{H}(\mathbb{C})) = {}^{\overline{\psi}}\mathcal{A}_n^2(\Pi, \mathbb{H}(\mathbb{C})) \ominus {}^{\overline{\psi}}\mathcal{A}_{n-1}^2(\Pi, \mathbb{H}(\mathbb{C}))$) be the space of true-n-ψ-hyperholomorphic (true-n-$\overline{\psi}$-hyperholomorphic) functions.

The multiplication operators ${}^{\psi}Q_+ = {}^{\psi}q_+(\xi)I$ and ${}^{\psi}Q_- = {}^{\psi}q_-(\xi)I$, where

$$ {}^{\psi}q_{\pm}(\xi) = \frac{1}{2}(1 \pm i\sum_{k=1}^{3} \psi^k \frac{\xi_k}{|\xi|}), \qquad \xi \in \mathbb{R}^3, $$

are mutually orthogonal projections on $L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, and ${}^{\psi}Q_+ + {}^{\psi}Q_- = I$.

We will use the following orthonormal base in $L_2(\mathbb{R}_+)$: $\{\ell_n(y)\}_{n=0}^{\infty}$, where $\ell_n(y) = e^{y/2}L_n(y)$, $L_n(y)$ is the Laguerre polinomial of degree n. Denote by L_n the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by $\ell_n(y)$, and by P_n the orthogonal projection of $L_2(\mathbb{R}_+)$ onto L_n.

The following main results obtained:

1. The space $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ admits the following decomposition

$$ L_2(\Pi, \mathbb{H}(\mathbb{C})) = \bigoplus_{n=1}^{\infty} {}^{\psi}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C})) \oplus \bigoplus_{n=1}^{\infty} {}^{\overline{\psi}}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C})). $$

2. There exists an unitary operator

$$ U : L_2(\Pi, \mathbb{H}(\mathbb{C})) = L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \longrightarrow L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), $$

such that for all $n \in \mathbb{N}$

$$ \begin{aligned} U &: {}^{\psi}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow L_{n-1} \otimes \operatorname{Im} {}^{\psi}Q_+, \\ U &: {}^{\overline{\psi}}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow L_{n-1} \otimes \operatorname{Im} {}^{\psi}Q_-, . \end{aligned} $$

3. There exists an unitary operator

$$ W : L_2(\Pi, \mathbb{H}(\mathbb{C})) = L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \longrightarrow L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), $$

such that for all $n \in \mathbb{N}$

$$ \begin{aligned} W : {}^{\psi}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C})) &\longrightarrow L_{n-1} \otimes {}^{\psi}H^2_+(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \\ W : {}^{\overline{\psi}}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C})) &\longrightarrow L_{n-1} \otimes {}^{\psi}H^2_-(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \\ W\, {}^{\psi}B^{(n)}_{\Pi}\, W^{-1} &= P_{n-1} \otimes {}^{\psi}P^+_{\mathbb{R}^3}, \\ W\, {}^{\overline{\psi}}B^{(n)}_{\Pi}\, W^{-1} &= P_{n-1} \otimes {}^{\psi}P^-_{\mathbb{R}^3}, \end{aligned} $$

where ${}^{\psi}B^{(n)}_{\Pi}$ and ${}^{\overline{\psi}}B^{(n)}_{\Pi}$ are Bergman orthogonal projections of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ onto the spaces ${}^{\psi}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C}))$ and ${}^{\overline{\psi}}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C}))$, respectively, and ${}^{\psi}P^+_{\mathbb{R}^3}$ and ${}^{\psi}P^-_{\mathbb{R}^3}$ are Szegö orthogonal projections of $L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ onto the Hardy spaces ${}^{\psi}H^2_+(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ and ${}^{\psi}H^2_-(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, respectively.

2 Preliminaries

Recall here necessary facts from the quaternionic analysis (for proofs and details see, for example, [2, 3, 4]).

Denote by $\mathbb{H}$ the skew field of real quaternions, i.e.,

$$\mathbb{H} = \{q = i_0 q_0 + i_1 q_1 + i_2 q_2 + i_3 q_3 : q_k \in \mathbb{R}\},$$

where $i_0 = 1 \in \mathbb{R}$, and the multiplication rule of imaginary units is given by

$$i_1 \cdot i_2 = -i_2 \cdot i_1 = i_3, \quad i_2 \cdot i_3 = -i_3 \cdot i_2 = i_1, \quad i_3 \cdot i_1 = -i_1 \cdot i_3 = i_2.$$

The main involution in $\mathbb{H}$, the quaternionic conjugation, is defined by

$$\overline{q} = i_0 q_0 - i_1 q_1 - i_2 q_2 - i_3 q_3,$$

note, that $\overline{q_1 \cdot q_2} = \overline{q_2} \cdot \overline{q_1}$, for each pair $q_1, q_2 \in \mathbb{H}$.

Let Ω be a domain in $\mathbb{R}^4$. Consider $\mathbb{H}$-valued functions defined in Ω:

$$f = \sum_{k=0}^{3} i_k f_k : \Omega \longrightarrow \mathbb{H},$$

where f_k, are real valued in Ω functions.

The real space $L_2(\Omega, \mathbb{H})$, consisting of all square-integrable (with respect to the usual Lebesgue measure) $\mathbb{H}$-valued functions in Ω, forms, at the same time, a right module over $\mathbb{H}$. In the sequel we will consider, as well, the set $\mathbb{H}(\mathbb{C}) = \mathbb{H} \otimes \mathbb{C}$ of complex quaternions, $\mathbb{H}(\mathbb{C})$-valued functions, and the complex Hilbert space $L_2(\Omega, \mathbb{H}(\mathbb{C}))$, being, at the same time, a right module over $\mathbb{H}(\mathbb{C})$.

A system $\psi = \{\psi^0, \psi^1, \psi^2, \psi^3\}$ of orthonormal (in the usual Euclidean sense) real quaternions is called a structural set. Without loss of generality we may assume that $\psi^0 = 1$, and thus ψ^k, $k = \overline{1, 3}$, are pure imaginary quaternions.

On the set $C^1(\Omega, \mathbb{H})$ (or $C^1(\Omega, \mathbb{H}(\mathbb{C}))$) define the quaternionic Cauchy–Riemann operator

$${}^{\psi}D = \sum_{k=0}^{3} \psi^k \frac{\partial}{\partial x_k}.$$

Functions $f : \Omega \to \mathbb{H}$ (or $f : \Omega \to \mathbb{H}(\mathbb{C})$), which satisfy the equation ${}^{\psi}Df = 0$, are called left-ψ-hyper-holomorphic, or (in this paper) just ψ-hyper-holomorphic.

Let $d\hat{x}_k$ be the differential form $dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$ with the multiple dx_k omitted, and let

$$\sigma_{\psi,x} = \sum_{k=0}^{3} (-1)^k \psi^k d\hat{x}_k.$$

Denote by $\mathcal{K}_\psi$ the fundamental solution of the operator ${}^{\psi}D$:

$$\mathcal{K}_\psi(x) = \frac{1}{2\pi^2 |x|^4} \sum_{k=0}^{3} \overline{\psi^k} \cdot x_k,$$

which plays a role of the Cauchy kernel in quaternionic analysis, i.e., we have the following **Cauchy integral formula:** *Let Ω be a bounded domain in $\mathbb{R}^4$ with a smooth boundary Γ, and f be ψ-hyper-holomorphic in Ω and continuous on $\overline{\Omega}$. Then*

$$\int_\Gamma \mathcal{K}_\psi(\tau - x) \cdot \sigma_{\psi,\tau} \cdot f(\tau) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^4 \setminus \overline{\Omega} \end{cases}.$$

The singular integral operator

$$({}^{\psi}S_\Gamma f)(t) = 2 \int_\Gamma \mathcal{K}_\psi(\tau - t) \cdot \sigma_{\psi,\tau} \cdot f(\tau)$$

is bounded on $L_2(\Gamma, \mathbb{H}(\mathbb{C}))$, selfadjoint and

$${}^{\psi}S_\Gamma^2 = I.$$

For the case $\Gamma = \mathbb{R}^3 = \{(0, x_1, x_2, x_3)\} \subset \mathbb{R}^4$ we have

$$\begin{aligned} ({}^{\psi}S_{\mathbb{R}^3} f)(t) &= \frac{1}{\pi^2} \int_{\mathbb{R}^3} \frac{1}{|\tau - t|^4} \sum_{k=1}^{3} \overline{\psi}^k (\tau_k - t_k)\, f(\tau)\, d\tau \\ &= \sum_{k=1}^{3} \overline{\psi}^k \frac{1}{\pi^2} \int_{\mathbb{R}^3} \frac{\tau_k - t_k}{|\tau - t|^4}\, f(\tau)\, d\tau \\ &= \sum_{k=1}^{3} \overline{\psi}^k R_k = -\sum_{k=1}^{3} \psi^k R_k, \end{aligned} \tag{2.1}$$

where

$$(R_k f)(t) = \frac{1}{\pi^2} \int_{\mathbb{R}^3} \frac{\tau_k - t_k}{|\tau - t|^4}\, f(\tau)\, d\tau, \qquad k = 1, 2, 3, \tag{2.2}$$

are well-known Riesz operators.

Introduce the direct and the inverse Fourier transforms of functions from $L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ as follows

$$
\begin{aligned}
(Ff)(t) &= (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-it\cdot\xi} \cdot f(\xi)\, d\xi, \\
(F^{-1}f)(t) &= (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{it\cdot\xi} \cdot f(\xi)\, d\xi
\end{aligned}
\tag{2.3}
$$

Then for the Riesz operators (2.2) we have

$$
R_k = F^{-1} i \frac{\xi_k}{|\xi|} F. \tag{2.4}
$$

Denote by ${}^{\psi}H^2_+(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ $(\subset L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))\,)$ the Hardy space consisting of all functions, which admit ψ-hyper-holomorphic extension on the upper half-space $\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 > 0\}$, and by ${}^{\psi}H^2_-(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ $(\subset L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))\,)$ the Hardy space consisting of all functions, which admit ψ-hyper-holomorphic extension on the lower half-space $\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 < 0\}$. The operators

$$
{}^{\psi}P^+_{\mathbb{R}^3} = \frac{1}{2}(I - {}^{\psi}S_{\mathbb{R}^3}) \quad \text{and} \quad {}^{\psi}P^-_{\mathbb{R}^3} = \frac{1}{2}(I + {}^{\psi}S_{\mathbb{R}^3}) \tag{2.5}
$$

are the Szegö orthogonal projections of $L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ onto the Hardy spaces ${}^{\psi}H^2_+(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ and ${}^{\psi}H^2_-(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, respectively.

Introduce the functions

$$
{}^{\psi}q_+(\xi) = \frac{1}{2}\Big(1 + i \sum_{k=1}^{3} \psi^k \frac{\xi_k}{|\xi|}\Big), \qquad {}^{\psi}q_-(\xi) = \frac{1}{2}\Big(1 - i \sum_{k=1}^{3} \psi^k \frac{\xi_k}{|\xi|}\Big), \qquad \xi \in \mathbb{R}^3, \tag{2.6}
$$

and corresponding multiplication operators

$$
{}^{\psi}Q_+ = {}^{\psi}q_+(\xi) I, \qquad {}^{\psi}Q_- = {}^{\psi}q_-(\xi) I, \tag{2.7}
$$

acting on $L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$. Easy calculations show that ${}^{\psi}Q_+$ and ${}^{\psi}Q_-$ are mutually orthogonal orthogonal projections, and

$$
{}^{\psi}Q_+ + {}^{\psi}Q_- = I. \tag{2.8}
$$

Denote by ${}^{\psi}L_2^+ = {}^{\psi}L_2^+(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ and ${}^{\psi}L_2^- = {}^{\psi}L_2^-(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ the images of the projections ${}^{\psi}Q_+$ and ${}^{\psi}Q_-$, respectively.

Now we have the following analog of Theorem 1.1.

Theorem 2.1 *The Fourier integral transform F gives the isometric isomorphism*

$$F : L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \longrightarrow L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})),$$

under which

1. *the Hardy spaces ${}^{\psi}H^2_{\pm}(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ are mapped onto ${}^{\psi}L_2^{\pm}$*

$$\begin{aligned} F &: {}^{\psi}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \longrightarrow {}^{\psi}L_2^{+}, \\ F &: {}^{\psi}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \longrightarrow {}^{\psi}L_2^{-}, \end{aligned}$$

2. *the Szegö projections ${}^{\psi}P^{\pm}_{\mathbb{R}^3}$ are unitary equivalent to ${}^{\psi}Q_{\pm}$*

$$\begin{aligned} F\,{}^{\psi}P^{+}_{\mathbb{R}^3}F^{-1} &= {}^{\psi}Q_{+}, \\ F\,{}^{\psi}P^{-}_{\mathbb{R}^3}F^{-1} &= {}^{\psi}Q_{-}. \end{aligned}$$

And

$$L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) = {}^{\psi}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \oplus {}^{\psi}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})). \tag{2.9}$$

Proof. Apply Fourier transform to the projections (2.5), use the representation of the operator (2.1) via the operators (2.2), and finally use the formulas (2.4), (2.7) and (2.8). □

Corollary 2.2 *Besides (2.9) we have the following direct sum splittings*

$$\begin{aligned} L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) &= {}^{\psi}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \oplus {}^{\overline{\psi}}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \\ L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) &= {}^{\overline{\psi}}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \oplus {}^{\overline{\psi}}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \\ L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) &= {}^{\overline{\psi}}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \oplus {}^{\psi}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \end{aligned}$$

and the following connections between the Szegö projections and between the Hardy spaces

$$\begin{aligned} {}^{\psi}P^{+}_{\mathbb{R}^3} = {}^{\overline{\psi}}P^{-}_{\mathbb{R}^3} &= F^{-1}\,{}^{\psi}q_{+}(\xi)F = F^{-1}\,{}^{\overline{\psi}}q_{-}(\xi)F, \\ {}^{\psi}P^{-}_{\mathbb{R}^3} = {}^{\overline{\psi}}P^{+}_{\mathbb{R}^3} &= F^{-1}\,{}^{\psi}q_{-}(\xi)F = F^{-1}\,{}^{\overline{\psi}}q_{+}(\xi)F, \\ {}^{\psi}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) &= {}^{\overline{\psi}}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \\ {}^{\psi}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) &= {}^{\overline{\psi}}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})). \end{aligned}$$

Proof. Follows from ${}^{\overline{\psi}}S_{\mathbb{R}^3} = -{}^{\psi}S_{\mathbb{R}^3}$ and ${}^{\psi}q_{-}(\xi) = {}^{\overline{\psi}}q_{+}(\xi)$. □

3 Quaternionic Bergman space

Let $\Pi = \{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 > 0\}$ be the upper half-space in $\mathbb{R}^4$. The points of Π we will denote by $x = (x_0, x')$, where $x_0 \in \mathbb{R}_+$, and $x' = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Given structural set $\psi = \{\psi^0, \psi^1, \psi^2, \psi^3\}$, introduce the subspace ${}^{\psi}\!\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$ of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$, the Bergman space, consisting of all ψ-hyper-holomorphic functions in Π, i.e., the functions $f : \Pi \to \mathbb{H}(\mathbb{C})$, which satisfy the quaternionic Cauchy-Riemann equation

$$ {}^{\psi}\!Df = \sum_{k=0}^{3} \psi^k \frac{\partial f}{\partial x_k} = 0. $$

Denote by ${}^{\psi}\!B_{\Pi}$ the Bergman orthogonal projection of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ onto ${}^{\psi}\!\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$.

Introduce the unitary operator $U_1 = I \otimes F$, acting on $L_2(\Pi, \mathbb{H}(\mathbb{C})) = L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, where F is the Fourier transform (2.3). Then the image of the Bergman space ${}^{\psi}\!\mathcal{A}_1^2 = U_1({}^{\psi}\!\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C})))$ can be described as the (closed) subspace of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$, which consists of all functions satisfying the equation

$$ (I \otimes F)\, {}^{\psi}\!D\, (I \otimes F^{-1})\, f = \Big(\frac{\partial}{\partial x_0} + i \sum_{k=1}^{3} \psi^k x_k\Big) f = 0. \tag{3.1} $$

The equation (3.1) is easy to solve, and its general solution has the form

$$ \varphi(x_0, x') = e^{-|x'| \cdot x_0} \Big(1 + i \sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|}\Big) h(x'). $$

The condition $\varphi \in L_2(\Pi, \mathbb{H}(\mathbb{C}))$ implies, that the space ${}^{\psi}\!\mathcal{A}_1^2$ consists of all the functions

$$ f(x_0, x') = e^{-|x'| \cdot x_0} \frac{1}{2} \Big(1 + i \sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|}\Big) \sqrt{2|x'|}\, g(x'), $$

where $g(x') \in L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, and, in addition,

$$ \|f(x_0, x')\|_{{}^{\psi}\!\mathcal{A}_1^2} = \|({}^{\psi}Q_+ g)(x')\|_{L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))}, $$

where the projection ${}^{\psi}Q_+$ is given by (2.7).

Let U_2 be the unitary operator, acting on $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ by the rule

$$ U_2 : f(x_0, x') \longmapsto \frac{1}{\sqrt{2|x'|}} f\Big(\frac{x_0}{2|x'|}, x'\Big). $$

Then the inverse operator $U_2^{-1} = U_2^*$ is given by

$$U_2^{-1} : f(x_0, x') \longmapsto \sqrt{2|x'|}\, f(2|x'| \cdot x_0, x').$$

The space ${}^{\psi}\!\mathcal{A}_2^2 = U_2({}^{\psi}\!\mathcal{A}_1^2)$ consists obviously of all the functions

$$\begin{aligned} f(x_0, x') &= U_2(e^{-|x'| \cdot x_0}\, \frac{1}{2}\, (1 + i \sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|})\, \sqrt{2|x'|}\, g(x')) \\ &= e^{-x_0/2}\, \frac{1}{2}\, (1 + i \sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|})\, g(x'). \end{aligned}$$

Introduce $\ell_0(x_0) = e^{-x_0/2}$, we have $\ell_0(x_0) \in L_2(\mathbb{R}_+)$ and $\|\ell_0(x_0)\| = 1$. Denote by L_0 the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_0(x_0)$, then the one-dimensional projection P_0 of $L_2(\mathbb{R}_+)$ onto L_0 has the form

$$(P_0 \psi)(x_0) = \langle \psi, \ell_0 \rangle \cdot \ell_0 = e^{-x_0/2} \int_{\mathbb{R}_+} \psi(\nu)\, e^{-\nu/2}\, d\nu. \tag{3.2}$$

Now the orthogonal projection ${}^{\psi}\!B_2$ of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ onto ${}^{\psi}\!\mathcal{A}_2^2$ has obviously the form

$${}^{\psi}\!B_2 = P_0 \otimes {}^{\psi}\!Q_+,$$

where the projection ${}^{\psi}\!Q_+$ is given by (2.7).

The above work leads up to the following theorem.

Theorem 3.1 *The unitary operator $U = U_2 U_1$ gives an isometric isomorphism of the space $L_2(\Pi, \mathbb{H}(\mathbb{C})) = L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ under which*

1. *the Bergman space ${}^{\psi}\!\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$ is mapped onto $L_0 \otimes \operatorname{Im} {}^{\psi}\!Q_+$*

$$U : {}^{\psi}\!\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow L_0 \otimes \operatorname{Im} {}^{\psi}\!Q_+,$$

where L_0 is the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by $\ell_0 = e^{-x_0/2}$, and

$${}^{\psi}\!Q_+ = \frac{1}{2}\, (1 + i \sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|}) I,$$

2. the Bergman projection ${}^{\psi}B_{\Pi}$ is unitary equivalent to the following one

$$U\,{}^{\psi}B_{\Pi}\,U^{-1} = P_0 \otimes {}^{\psi}Q_+,$$

where P_0 is the one-dimensional projection (3.2) of $L_2(\mathbb{R}_+)$ onto L_0.

For the image ${}^{\psi}L_2^+ = {}^{\psi}L_2^+(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ of the projection ${}^{\psi}Q_+$ introduce the isometric imbedding

$$R_0 : {}^{\psi}L_2^+ \longrightarrow L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$$

by the rule

$$(R_0 g)(x_0, x') = \ell_0(x_0)\,({}^{\psi}Q_+ g)(x').$$

The image of R_0 is obviously coincides with the space ${}^{\psi}\!\mathcal{A}_2^2$. The adjoint operator $R_0^* : L_2(\Pi, \mathbb{H}(\mathbb{C})) \to {}^{\psi}L_2^+$ is given by

$$(R_0^* f)(x') = \frac{1}{2}\,(1 + i\sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|}) \int_{\mathbb{R}_+} f(x_0, x')\,\ell_0(x_0)\,dx_0,$$

and

$$\begin{aligned} R_0^* R_0 = I \quad &: \quad {}^{\psi}L_2^+ \longrightarrow {}^{\psi}L_2^+ \\ R_0 R_0^* = {}^{\psi}B_2 \quad &: \quad L_2(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow {}^{\psi}\!\mathcal{A}_2^2 = L_0 \otimes {}^{\psi}L_2^+. \end{aligned}$$

Now the operator $R = R_0^* U$ maps the space $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ onto ${}^{\psi}L_2^+$, and the restriction

$$R|_{{}^{\psi}\!\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))} : {}^{\psi}\!\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow {}^{\psi}L_2^+$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : {}^{\psi}L_2^+ \longrightarrow {}^{\psi}\!\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C})) \subset L_2(\Pi, \mathbb{H}(\mathbb{C}))$$

is an isometric isomorphism of ${}^{\psi}L_2^+$ onto the subspace ${}^{\psi}\!\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$ of the space $L_2(\Pi, \mathbb{H}(\mathbb{C}))$.

Remark 3.2 We have

$$\begin{aligned} R\,R^* = I \quad &: \quad {}^{\psi}L_2^+ \longrightarrow {}^{\psi}L_2^+ \\ R^* R = {}^{\psi}B_{\Pi} \quad &: \quad L_2(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow {}^{\psi}\!\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C})). \end{aligned}$$

Theorem 3.3 *The isometric isomorphism*

$$R^* = U^* R_0 : {}^{\psi}L_2^+ \longrightarrow {}^{\psi}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$$

is given by

$$(R^*g)(x_0, x') = \frac{1}{4\pi^{3/2}} \int_{\mathbb{R}^3} e^{ix'\cdot\xi - |\xi|\cdot x_0} \, (1 + i \sum_{k=1}^{3} \psi^k \frac{\xi_k}{|\xi|}) \, |\xi|^{1/2} \, g(\xi) \, d\xi.$$

Proof. Calculate

$$\begin{aligned}
(R^*g)(x_0, x') &= (U_1^* U_2^* R_0 g)(x_0, x') \\
&= (I \otimes F^{-1})(e^{-|x'|\cdot x_0} \, \frac{1}{2} \, (1 + i \sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|}) \, \sqrt{2|x'|} \, g(x')) \\
&= (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix'\cdot\xi} \, e^{-|\xi|\cdot x_0} \, \frac{1}{2} \, (1 + i \sum_{k=1}^{3} \psi^k \frac{\xi_k}{|\xi|}) \, \sqrt{2|\xi|} \, g(\xi) \, d\xi \\
&= \frac{1}{4\pi^{3/2}} \int_{\mathbb{R}^3} e^{ix'\cdot\xi - |\xi|\cdot x_0} \, (1 + i \sum_{k=1}^{3} \psi^k \frac{\xi_k}{|\xi|}) \, |\xi|^{1/2} \, g(\xi) \, d\xi.
\end{aligned}$$

□

Corollary 3.4 *The inverse isomorphism*

$$R : {}^{\psi}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow {}^{\psi}L_2^+$$

is given by

$$(Rf)(x') = \frac{|x'|^{1/2}}{4\pi^{3/2}} \, (1 + i \sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|}) \int_{\Pi} e^{-ix'\cdot y' - |x'|\cdot y_0} \, f(y) \, dy.$$

4 Connections between Bergman and Hardy spaces

The next result follows directly from Theorems 2.1 and 3.1.

Theorem 4.1 *The unitary operator* $W = (I \otimes F^{-1}) U_2 (I \otimes F)$ *gives an isometric isomorphism of the space* $L_2(\Pi, \mathbb{H}(\mathbb{C})) = L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ *under which*

1. *the Bergman* ${}^{\psi}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$ *and the Hardy* ${}^{\psi}H^2_+(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ *spaces are connected by the formula*
$$W({}^{\psi}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))) = L_0 \otimes {}^{\psi}H^2_+(\mathbb{R}^3, \mathbb{H}(\mathbb{C})),$$
2. *the Bergman* ${}^{\psi}B_{\Pi}$ *and the Szegö* ${}^{\psi}P^+_{\mathbb{R}^3}$ *projections are connected by the formula*
$$W\, {}^{\psi}B_{\Pi}\, W^{-1} = P_0 \otimes {}^{\psi}P^+_{\mathbb{R}^3},$$

where P_0 *is the one-dimensional projection (3.2) of* $L_2(\mathbb{R}_+)$ *onto one-dimensional space* L_0 *generated by* $\ell_0(x_0) = e^{-x_0/2} \in L_2(\mathbb{R}_+)$.

Besides the Bergman space ${}^{\psi}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$ of ψ-hyper-holomorphic functions introduce the space ${}^{\overline{\psi}}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$ as the (closed) subspace of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ consisting of all $\overline{\psi}$-hyper-holomorphic (or anti-ψ-hyper-holomorphic) functions in Π. And denote by ${}^{\overline{\psi}}B_{\Pi}$ corresponding Bergman projection.

Theorem 4.2 *The unitary operator* $U = U_2U_1$ *gives an isometric isomorphism of the space* $L_2(\Pi, \mathbb{H}(\mathbb{C})) = L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ *under which*

1. *the space* ${}^{\overline{\psi}}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$ *is mapped onto* $L_0 \otimes \operatorname{Im} {}^{\psi}Q_-$
$$U : {}^{\overline{\psi}}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow L_0 \otimes \operatorname{Im} {}^{\psi}Q_-,$$
where L_0 *is the one-dimensional subspace of* $L_2(\mathbb{R}_+)$, *generated by* $\ell_0 = e^{-x_0/2}$, *and the orthogonal projection* ${}^{\psi}Q_-$, *acting on* $L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, *is given by (2.7),*
2. *the projection* ${}^{\overline{\psi}}B_{\Pi}$ *is unitary equivalent to the following one*
$$U\, {}^{\overline{\psi}}B_{\Pi}\, U^{-1} = P_0 \otimes {}^{\psi}Q_-,$$
where P_0 *is the one-dimensional projection (3.2) of* $L_2(\mathbb{R}_+)$ *onto* L_0.

Proof. Follows all the steps of the proof of Theorem 3.1. Note only, that all the $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ – solutions of the equation

$$\begin{aligned}(I \otimes F)\, {}^{\overline{\psi}}D\, (I \otimes F^{-1})\, f &= (\frac{\partial}{\partial x_0} + i \sum_{k=1}^{3} \overline{\psi}^k x_k)\, f \\ &= (\frac{\partial}{\partial x_0} - i \sum_{k=1}^{3} \psi^k x_k)\, f = 0.\end{aligned}$$

are of the form

$$f(x_0, x') = e^{-|x'|\cdot x_0}\, \frac{1}{2}\,(1 - i\sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|})\, \sqrt{2|x'|}\, g(x'),$$

where $g(x') \in L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, and, in addition,

$$\|f(x_0, x')\|_{\overline{\psi}\mathcal{A}_1^2} = \|({}^{\psi}Q_{-}g)(x')\|_{L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))},$$

where the projection ${}^{\psi}Q_{-} = {}^{\overline{\psi}}Q_{+}$ is given by (2.7). $\square$

Corollary 4.3 *The unitary operator $W = (I \otimes F^{-1})U_2(I \otimes F)$ gives an isometric isomorphism of the space $L_2(\Pi, \mathbb{H}(\mathbb{C})) = L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ under which*

1. *the Bergman ${}^{\overline{\psi}}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$ and the Hardy spaces are connected by*

$$\begin{aligned} W({}^{\overline{\psi}}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))) &= L_0 \otimes {}^{\psi}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \\ &= L_0 \otimes {}^{\overline{\psi}}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \end{aligned}$$

2. *the Bergman ${}^{\overline{\psi}}B_{\Pi}$ and the Szegö projections are connected by*

$$W\,{}^{\overline{\psi}}B_{\Pi}\,W^{-1} = P_0 \otimes {}^{\psi}P^{-}_{\mathbb{R}^3} = P_0 \otimes {}^{\overline{\psi}}P^{+}_{\mathbb{R}^3},$$

where P_0 is the one-dimensional projection (3.2) of $L_2(\mathbb{R}_+)$ onto one-dimensional space L_0 generated by $\ell_0(x_0) = e^{-x_0/2} \in L_2(\mathbb{R}_+)$.

Remark 4.4 The direct sum of both Hardy spaces coincides with the whole space

$$L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) = {}^{\psi}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \oplus {}^{\psi}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})),$$

while the direct sum ${}^{\psi}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C})) \oplus {}^{\overline{\psi}}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$ has the (sufficiently large) orthogonal complement

$${}^{\overline{\psi}}\mathcal{A}^{\perp}(\Pi, \mathbb{H}(\mathbb{C})) = L_2(\Pi, \mathbb{H}(\mathbb{C})) \ominus ({}^{\overline{\psi}}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C})) \oplus {}^{\overline{\psi}}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))),$$

whose image under the mapping W is equal to

$$W({}^{\overline{\psi}}\mathcal{A}^{\perp}(\Pi, \mathbb{H}(\mathbb{C}))) = L_0^{\perp} \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})).$$

5 Poly-Bergman spaces, decomposition of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$

Analogously to the Bergman spaces ${}^{\psi}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$ and ${}^{\overline{\psi}}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$, of respectively ψ-hyper-holomorphic and $\overline{\psi}$-hyper-holomorphic functions from $L_2(\Pi, \mathbb{H}(\mathbb{C}))$, introduce the spaces of poly-ψ-hyper-holomorphic and poly-$\overline{\psi}$-hyper-holomorphic functions, the poly-Bergman spaces.

Define the space ${}^{\psi}\mathcal{A}^2_n(\Pi, \mathbb{H}(\mathbb{C}))$ of n-ψ-hyper-holomorphic functions as the (closed) subspace of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ of all functions $f = f(x) = f(x_0, x')$, which satisfy the equation

$$\left({}^{\psi}D\right)^n f = \left(\sum_{k=0}^{3} \psi^k \frac{\partial}{\partial x_k}\right)^n f = 0.$$

Then the space ${}^{\overline{\psi}}\mathcal{A}^2_n(\Pi, \mathbb{H}(\mathbb{C}))$ of n-$\overline{\psi}$-hyper-holomorphic functions is the (closed) subspace of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ of all functions $f = f(x) = f(x_0, x')$, which satisfy the equation

$$\left({}^{\overline{\psi}}D\right)^n f = \left(\sum_{k=0}^{3} \overline{\psi}^k \frac{\partial}{\partial x_k}\right)^n f = 0.$$

Of course, we have ${}^{\psi}\mathcal{A}^2_1(\Pi, \mathbb{H}(\mathbb{C})) = {}^{\psi}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$ and ${}^{\overline{\psi}}\mathcal{A}^2_1(\Pi, \mathbb{H}(\mathbb{C})) = {}^{\overline{\psi}}\mathcal{A}^2(\Pi, \mathbb{H}(\mathbb{C}))$, for $n = 1$, as well as ${}^{\psi}\mathcal{A}^2_n(\Pi, \mathbb{H}(\mathbb{C})) \subset {}^{\psi}\mathcal{A}^2_{n+1}(\Pi, \mathbb{H}(\mathbb{C}))$ and ${}^{\overline{\psi}}\mathcal{A}^2_n(\Pi, \mathbb{H}(\mathbb{C})) \subset {}^{\overline{\psi}}\mathcal{A}^2_{n+1}(\Pi, \mathbb{H}(\mathbb{C}))$, for each $n \in \mathbb{N}$.

Recall (see, for example, [1]), that the Laguerre polinomial $L_n(y)$ of degree n, $n = 0, 1, 2, ...$, and type 0 is defined by

$$\begin{aligned} L_n(y) &= L_n^0(y) = \frac{e^y}{n!} \frac{d^n}{dy^n}(e^{-y}\, y^n) \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{(-y)^k}{k!}, \qquad y \in \mathbb{R}_+, \end{aligned} \tag{5.1}$$

and that the system of functions

$$\ell_n(y) = e^{-y/2} L_n(y), \qquad n = 0, 1, 2, ... \tag{5.2}$$

form an orthonormal base in the space $L_2(\mathbb{R}_+)$.

Denote by L_n, $n = 0, 1, 2, ...$, the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by the function $\ell_n(y)$. Note, that for $n = 0$ this definition gives exactly previously defined space L_0. The operator

$$(P_n\psi)(y) = \langle \psi, \ell_n \rangle\, \ell_n = \ell_n(y) \int_{\mathbb{R}_+} \psi(\nu)\, \ell_n(\nu)\, d\nu \tag{5.1}$$

is obviously the orthogonal projection of $L_2(\mathbb{R}_+)$ onto the one-dimensional space L_n. And let

$$L_n^{\oplus} = \bigoplus_{k=0}^{n} L_k$$

be the direct sum of first $(n+1)$ spaces.

Theorem 5.1 *The unitary operator U, acting on the space $L_2(\Pi, \mathbb{H}(\mathbb{C})) = L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, maps the space ${}^{\psi}\!\mathcal{A}_n^2(\Pi, \mathbb{H}(\mathbb{C}))$ of n-ψ-hyper-holomorphic functions onto the space $L_{n-1}^{\oplus} \otimes {}^{\psi}\!L_2^+$.*

Proof. The space $U({}^{\psi}\!\mathcal{A}_n^2(\Pi, \mathbb{H}(\mathbb{C})))$ obviously coincides with the set of all functions from $L_2(\Pi, \mathbb{H}(\mathbb{C})) = L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, which satisfy the equation

$$\begin{aligned} U\left(\sum_{k=0}^{3} \psi^k \frac{\partial}{\partial x_k}\right)^n U^{-1} f &= U_2\left(\frac{\partial}{\partial x_0} + i\sum_{k=1}^{3} \psi^k x_k\right)^n U_2^{-1} f \\ &= |x'|^n \left(2\frac{\partial}{\partial x_0} + i\sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|}\right)^n f = 0. \end{aligned}$$

It is easy to see, that the intersection of the general solution of this equation with the space $L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ coincides with the set of all functions

$$\sum_{m=0}^{n-1} x_0^m e^{-x_0/2}\, \frac{1}{2}(1 + i\sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|})\, h_m(x'),$$

where $h_m(x') \in L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, for all $m = \overline{0, n-1}$, or rearranging polynomials on x_0, with the set of all functions

$$\sum_{m=0}^{n-1} L_m(x_0) e^{-x_0/2}\, \frac{1}{2}(1 + i\sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|})\, g_m(x') = \sum_{m=0}^{n-1} \ell_m(x_0)\, \frac{1}{2}(1 + i\sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|})\, g_m(x'),$$

where $g_m(x') \in L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, for all $m = \overline{0, n-1}$. $\square$

Introduce the space ${}^{\psi}\!\mathcal{A}_{(n)}^2(\Pi, \mathbb{H}(\mathbb{C}))$ of true-n-ψ-hyper-holomorphic functions by

$${}^{\psi}\!\mathcal{A}_{(n)}^2(\Pi, \mathbb{H}(\mathbb{C})) = {}^{\psi}\!\mathcal{A}_n^2(\Pi, \mathbb{H}(\mathbb{C})) \ominus {}^{\psi}\!\mathcal{A}_{n-1}^2(\Pi, \mathbb{H}(\mathbb{C})),$$

for $n > 1$, and by ${}^{\psi}\!\mathcal{A}_{(1)}^2(\Pi, \mathbb{H}(\mathbb{C})) = {}^{\psi}\!\mathcal{A}_1^2(\Pi, \mathbb{H}(\mathbb{C}))$, for $n = 1$, then, of course,

$${}^{\psi}\!\mathcal{A}_n^2(\Pi, \mathbb{H}(\mathbb{C})) = \bigoplus_{k=1}^{n} {}^{\psi}\!\mathcal{A}_{(k)}^2(\Pi, \mathbb{H}(\mathbb{C})).$$

Corollary 5.2 *The unitary operator U maps the space ${}^{\psi}\!\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C}))$ of true-n-ψ-hyper-holomorphic functions onto the space $L_{n-1} \otimes {}^{\psi}L_2^+$.*

Analogously for n-$\overline{\psi}$-hyper-holomorphic functions we have the following assertions.

Theorem 5.3 *The unitary operator U maps the space ${}^{\overline{\psi}}\!\mathcal{A}^2_n(\Pi, \mathbb{H}(\mathbb{C}))$ of n-$\overline{\psi}$-hyper-holomorphic functions onto the space $L^{\oplus}_{n-1} \otimes {}^{\psi}L_2^-$, where ${}^{\psi}L_2^-$ is the image of the projection ${}^{\psi}Q_-$, given by (2.7).*

Proof. Follows all the steps of the proof of Theorem 5.1. The space $U({}^{\overline{\psi}}\!\mathcal{A}^2_n(\Pi, \mathbb{H}(\mathbb{C})))$ coincides with the set of all functions from $L_2(\Pi, \mathbb{H}(\mathbb{C})) = L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, which satisfy the equation

$$\begin{aligned} U\left(\sum_{k=0}^{3} \overline{\psi}^k \frac{\partial}{\partial x_k}\right)^n U^{-1} f &= U_2\left(\frac{\partial}{\partial x_0} - i\sum_{k=1}^{3} \psi^k x_k\right)^n U_2^{-1} f \\ &= |x'|^n \left(2\frac{\partial}{\partial x_0} - i\sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|}\right)^n f = 0. \end{aligned}$$

Now the intersection of the general solution of this equation with the space $L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ coincides with the set of all functions

$$\sum_{m=0}^{n-1} x_0^m e^{-x_0/2}\, \frac{1}{2}(1 - i\sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|})\, h_m(x'),$$

where $h_m(x') \in L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, for all $m = \overline{0, n-1}$, or rearranging polynomials on x_0, with the set of all functions

$$\sum_{m=0}^{n-1} L_m(x_0) e^{-x_0/2}\, \frac{1}{2}(1 - i\sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|})\, g_m(x') = \sum_{m=0}^{n-1} \ell_m(x_0)\, \frac{1}{2}(1 - i\sum_{k=1}^{3} \psi^k \frac{x_k}{|x'|})\, g_m(x'),$$

where $g_m(x') \in L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$, for all $m = \overline{0, n-1}$. $\square$

Corollary 5.4 *The unitary operator U maps the space ${}^{\overline{\psi}}\!\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C}))$ of true-n-$\overline{\psi}$-hyper-holomorphic functions onto the space $L_{n-1} \otimes {}^{\psi}L_2^-$.*

Denote by ${}^{\psi}B^{(n)}_{\Pi}$ $({}^{\overline{\psi}}B^{(n)}_{\Pi})$ the Bergman orthogonal projection of $L_2(\Pi, \mathbb{H}(\mathbb{C}))$ onto the space ${}^{\psi}\!\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C}))$ of true-n-ψ-hyperholomorphic functions (onto the space ${}^{\overline{\psi}}\!\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C}))$ of true-n-$\overline{\psi}$-hyperholomorphic functions).

The above results lead up to the following theorem.

Theorem 5.5 *We have the following isometric isomorphisms, decompositions of the spaces, and the description of the projections:*

1. *Isomorphic images of poly-ψ-hyper-holomorphic spaces*

$$\begin{aligned} W &: {}^{\psi}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow L_{n-1} \otimes {}^{\psi}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \\ W &: {}^{\psi}\mathcal{A}^2_{n}(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow \bigoplus_{k=0}^{n-1} L_k \otimes {}^{\psi}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \\ W &: \bigoplus_{k=1}^{\infty} {}^{\psi}\mathcal{A}^2_{(k)}(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow L_2(\mathbb{R}_+) \otimes {}^{\psi}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})). \end{aligned}$$

2. *Isomorphic images of poly-$\overline{\psi}$-hyper-holomorphic spaces*

$$\begin{aligned} W &: {}^{\overline{\psi}}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow L_{n-1} \otimes {}^{\psi}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \\ W &: {}^{\overline{\psi}}\mathcal{A}^2_{n}(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow \bigoplus_{k=0}^{n-1} L_k \otimes {}^{\psi}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \\ W &: \bigoplus_{k=1}^{\infty} {}^{\overline{\psi}}\mathcal{A}^2_{(k)}(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow L_2(\mathbb{R}_+) \otimes {}^{\psi}H^2_{-}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})). \end{aligned}$$

   or

$$\begin{aligned} W &: {}^{\overline{\psi}}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow L_{n-1} \otimes {}^{\overline{\psi}}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \\ W &: {}^{\overline{\psi}}\mathcal{A}^2_{n}(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow \bigoplus_{k=0}^{n-1} L_k \otimes {}^{\overline{\psi}}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})), \\ W &: \bigoplus_{k=1}^{\infty} {}^{\overline{\psi}}\mathcal{A}^2_{(k)}(\Pi, \mathbb{H}(\mathbb{C})) \longrightarrow L_2(\mathbb{R}_+) \otimes {}^{\overline{\psi}}H^2_{+}(\mathbb{R}^3, \mathbb{H}(\mathbb{C})). \end{aligned}$$

3. *Decomposition of the space $L_2(\Pi, \mathbb{H}(\mathbb{C}))$*

$$\begin{aligned} L_2(\Pi, \mathbb{H}(\mathbb{C})) &= \bigoplus_{k=1}^{\infty} ({}^{\psi}\mathcal{A}^2_{(k)}(\Pi, \mathbb{H}(\mathbb{C})) \oplus {}^{\overline{\psi}}\mathcal{A}^2_{(k)}(\Pi, \mathbb{H}(\mathbb{C})) \\ &= \bigoplus_{k=1}^{\infty} {}^{\psi}\mathcal{A}^2_{(k)}(\Pi, \mathbb{H}(\mathbb{C})) \oplus \bigoplus_{k=1}^{\infty} {}^{\overline{\psi}}\mathcal{A}^2_{(k)}(\Pi, \mathbb{H}(\mathbb{C})). \end{aligned}$$

4. *Decomposition of* ${}^{\psi}\mathcal{A}^{\perp}(\Pi, \mathbb{H}(\mathbb{C}))$

$$
{}^{\psi}\mathcal{A}^{\perp}(\Pi, \mathbb{H}(\mathbb{C})) = \bigoplus_{k=2}^{\infty}({}^{\psi}\mathcal{A}^2_{(k)}(\Pi, \mathbb{H}(\mathbb{C})) \oplus {}^{\overline{\psi}}\mathcal{A}^2_{(k)}(\Pi, \mathbb{H}(\mathbb{C}))).
$$

5. *Unitary equivalence of the projections*

$$
\begin{aligned}
W\, {}^{\psi}B^{(n)}_{\Pi}\, W^{-1} &= P_{n-1} \otimes {}^{\psi}P^{+}_{\mathbb{R}^3} = P_{n-1} \otimes {}^{\overline{\psi}}P^{-}_{\mathbb{R}^3}, \\
W\, {}^{\overline{\psi}}B^{(n)}_{\Pi}\, W^{-1} &= P_{n-1} \otimes {}^{\psi}P^{-}_{\mathbb{R}^3} = P_{n-1} \otimes {}^{\overline{\psi}}P^{+}_{\mathbb{R}^3}.
\end{aligned}
$$

Here L_n *is the one-dimensional subspace of* $L_2(\mathbb{R}_+)$*, generated by the function* $\ell_n(x_0) = e^{-x_0/2}L_n(x_0)$*, where the Laguerre polinomial* $L_n(x_0)$ *of degree* n *is given by (5.1), and* P_{n-1} *is the orthogonal projection (5.1) of* $L_2(\mathbb{R}_+)$ *onto* L_{n-1}*.*

6 Real quaternion valued functions and corresponding spaces

It is clear that the only place in what have done above, where we forced to consider complex quaternions, was the use of the Fourier transform with the consequent consideration of the functions ${}^{\psi}q_{\pm}(\xi)$, projections ${}^{\psi}Q_{\pm}$, and their images - the subspaces ${}^{\psi}L_2^{\pm}$ of the space $L_2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$. Thus, if we start with real quaternion valued functions, after the applications of the Fourier transform we have to consider certain real subspaces in complex spaces ${}^{\psi}L_2^{\pm}$, which are invariant with respect to the operators ${}^{\psi}Q_{\pm}$. The consequent application of the inverse Fourier transform (in the operator W) returns all the functions to real quaternion valued ones.

Thus, preserving the notations for the Bergman and the Szegö projections, we have the following exact analog of Theorem 5.5 for the real quaternion valued hyperholomorphic functions.

Theorem 6.1 *There are the following isometric isomorphisms, decompositions of the spaces, and the description of the projections:*

1. *Isomorphic images of poly-ψ-hyper-holomorphic spaces*

$$
W \ : \ {}^{\psi}\mathcal{A}^2_{(n)}(\Pi, \mathbb{H}) \longrightarrow L_{n-1} \otimes {}^{\psi}H^2_{+}(\mathbb{R}^3, \mathbb{H}),
$$

$$
\begin{aligned}
W &: {}^{\psi}\!\mathcal{A}^2_n(\Pi,\mathbb{H}) \longrightarrow \bigoplus_{k=0}^{n-1} L_k \otimes {}^{\psi}H^2_+(\mathbb{R}^3,\mathbb{H}),\\
W &: \bigoplus_{k=1}^{\infty} {}^{\psi}\!\mathcal{A}^2_{(k)}(\Pi,\mathbb{H}) \longrightarrow L_2(\mathbb{R}_+) \otimes {}^{\psi}H^2_+(\mathbb{R}^3,\mathbb{H}).
\end{aligned}
$$

2. *Isomorphic images of poly-$\overline{\psi}$-hyper-holomorphic spaces*

$$
\begin{aligned}
W &: {}^{\overline{\psi}}\!\mathcal{A}^2_{(n)}(\Pi,\mathbb{H}) \longrightarrow L_{n-1} \otimes {}^{\psi}H^2_-(\mathbb{R}^3,\mathbb{H}),\\
W &: {}^{\overline{\psi}}\!\mathcal{A}^2_n(\Pi,\mathbb{H}) \longrightarrow \bigoplus_{k=0}^{n-1} L_k \otimes {}^{\psi}H^2_-(\mathbb{R}^3,\mathbb{H}),\\
W &: \bigoplus_{k=1}^{\infty} {}^{\overline{\psi}}\!\mathcal{A}^2_{(k)}(\Pi,\mathbb{H}) \longrightarrow L_2(\mathbb{R}_+) \otimes {}^{\psi}H^2_-(\mathbb{R}^3,\mathbb{H}).
\end{aligned}
$$

or

$$
\begin{aligned}
W &: {}^{\overline{\psi}}\!\mathcal{A}^2_{(n)}(\Pi,\mathbb{H}) \longrightarrow L_{n-1} \otimes {}^{\overline{\psi}}H^2_+(\mathbb{R}^3,\mathbb{H}),\\
W &: {}^{\overline{\psi}}\!\mathcal{A}^2_n(\Pi,\mathbb{H}) \longrightarrow \bigoplus_{k=0}^{n-1} L_k \otimes {}^{\overline{\psi}}H^2_+(\mathbb{R}^3,\mathbb{H}),\\
W &: \bigoplus_{k=1}^{\infty} {}^{\overline{\psi}}\!\mathcal{A}^2_{(k)}(\Pi,\mathbb{H}) \longrightarrow L_2(\mathbb{R}_+) \otimes {}^{\overline{\psi}}H^2_+(\mathbb{R}^3,\mathbb{H}).
\end{aligned}
$$

3. *Decomposition of the space $L_2(\Pi,\mathbb{H})$*

$$
\begin{aligned}
L_2(\Pi,\mathbb{H}) &= \bigoplus_{k=1}^{\infty} ({}^{\psi}\!\mathcal{A}^2_{(k)}(\Pi,\mathbb{H}) \oplus {}^{\overline{\psi}}\!\mathcal{A}^2_{(k)}(\Pi,\mathbb{H})\\
&= \bigoplus_{k=1}^{\infty} {}^{\psi}\!\mathcal{A}^2_{(k)}(\Pi,\mathbb{H}) \oplus \bigoplus_{k=1}^{\infty} {}^{\overline{\psi}}\!\mathcal{A}^2_{(k)}(\Pi,\mathbb{H}).
\end{aligned}
$$

4. *Unitary equivalence of the projections*

$$
\begin{aligned}
W\, {}^{\psi}B^{(n)}_{\Pi}\, W^{-1} &= P_{n-1} \otimes {}^{\psi}P^{+}_{\mathbb{R}^3} = P_{n-1} \otimes {}^{\overline{\psi}}P^{-}_{\mathbb{R}^3},\\
W\, {}^{\overline{\psi}}B^{(n)}_{\Pi}\, W^{-1} &= P_{n-1} \otimes {}^{\psi}P^{-}_{\mathbb{R}^3} = P_{n-1} \otimes {}^{\overline{\psi}}P^{+}_{\mathbb{R}^3}.
\end{aligned}
$$

Here L_n is the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by the function $\ell_n(x_0) = e^{-x_0/2}L_n(x_0)$, where the Laguerre polinomial $L_n(x_0)$ of degree n is given by (5.1), and P_{n-1} is the orthogonal projection (5.1) of $L_2(\mathbb{R}_+)$ onto L_{n-1}.

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