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ALGEBRAS OF SINGULAR INTEGRAL OPERATORS GENERATED BY THREE ORTHOGONAL PROJECTIONS*

E. RAMIREZ DE ARELLANO and N. L. VASILEVSKI†

The structure of the C^*-algebra with identity generated by two orthogonal projections is well understood. All irreducible representations of this algebra are either two–dimensional or one–dimensional. The situation becomes unpredictable in the case of the C^*-algebra generated by three orthogonal projections. Even in the more specific case when two of the projections commute, the algebra under consideration may have infinite dimensional irreducible representations.

In this paper we produce three concrete realizations of the algebra generated by three orthogonal projections in that specific case. It turns out that these algebras have quite different structure.

1 Introduction

The structure of the C^*-algebra with identity generated by two orthogonal projections, say P_1 and P_2, is well understood (see, for example, [5, 6, 9]). If they are in "generic position" [5] (that means that we exclude the trivial commutative part of the projections), then the only information one needs to describe this algebra is the spectrum $\mathrm{sp}\,(P_1 - P_2)^2$, which *a priori* is a compact subset of $[0,1]$.

The algebra $\mathcal{R}(\mathbb{C}; P_1, P_2)$ generated by the generic position orthogonal projections P_1 and P_2 is isomorphic and isometric to the algebra of all 2×2 matrix–valued functions continuous on $\mathrm{sp}\,(P_1 - P_2)^2$ and diagonal at the points of $\{0,1\}\cap \mathrm{sp}\,(P_1 - P_2)^2$. This isomorphism is generated by the following mapping:

$$\begin{aligned} P_1 &\longmapsto \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix}, \\ P_2 &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

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†On leave from the Odessa State University, Odessa, Ukraine.

where $x \in \mathrm{sp}\,(P_1 - P_2)^2$.

The situation becomes unpredictable in the case of the C^*-algebra generated by three orthogonal projections. Even in the particular case when two of the projections commute or, equivalently, when two of the projections are mutually orthogonal, the algebra under consideration may have infinite dimensional irreducible representations.

The aim of this paper is to produce some concrete realizations of the algebra generated by three orthogonal projections in the above mentioned case. The algebra generated by two concrete (non commuting) orthogonal projections appeared for the first time in the study by I. Gohberg and N. Krupnik of one–dimensional singular integral operators with piecewise continuous coefficients (see, for example, [3]). This specific research has greatly aided understanding the general case. It seems natural to consider the same objects in the three–projection situation. We are studying here three concrete algebras generated by orthogonal projections based on a one–dimensional singular integral operator and piecewise constant functions. It turns out that these algebras have quite different structure.

2 Preliminaries

2.1 Let Γ be the unit circle, and denote by S_Γ the one dimensional singular integral operator

$$(S_\Gamma \varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau) d\tau}{\tau - t} d\tau$$

acting on $L_2(\Gamma)$.

It is well known that $S_\Gamma^2 = I$ and $S_\Gamma^* = S_\Gamma$. The operator $P_\Gamma = \frac{1}{2}(I + S_\Gamma)$ is the orthogonal projection of $L_2(\Gamma)$ onto the Hardy space $H^2(\Gamma)$.

We will use the following two simple facts. Let

$$x = \omega(t) = \frac{at + b}{ct + d}$$

be a Moebius transformation of the unit disk onto the upper half–plane.

Introduce the unitary operator $W : L_2(\mathbb{R}) \to L_2(\Gamma)$ by the rule

$$(W\varphi)(t) = \frac{(ad - bc)^{\frac{1}{2}}}{ct + d} \varphi(\omega(t)),$$

with

$$W^{-1} : \varphi \in L_2(\Gamma) \mapsto \frac{(ad - bc)^{\frac{1}{2}}}{-cx + d} \varphi(\omega^{-1}(x)).$$

Then $W^{-1} S_\Gamma W = S_\mathbb{R}$, where

$$(S_\mathbb{R} \varphi)(x) = \frac{1}{\pi i} \int_\mathbb{R} \frac{\varphi(y) dy}{y - x}.$$

Let t_1, t_2, t_3 be three points on Γ, ordered $t_1 \prec t_2 \prec t_3$ with respect to the positive orientation on Γ.

The Moebius transformation

$$x = \omega(t) = \frac{(t-t_2)(t_3-t_2)}{(t-t_3)(t_1-t_2)}$$

maps the unit circle onto the real line, and under this mapping

$$\omega(t_1) = -1, \quad \omega(t_2) = 0, \quad \omega(t_3) = \infty.$$

2.2 Let t', t'' be two points on Γ, and let ℓ be the arc of Γ starting at t' and ending at t''. Denote by $\chi_\ell(t)$ the characteristic function of the arc ℓ.

Introduce the algebra $\mathcal{R}_\ell = \mathcal{R}(\mathbb{C}; P_\Gamma, Q_\ell)$ with identity I generated by the two orthogonal projections

$$P_\Gamma = \frac{1}{2}(I + S_\Gamma) \text{ and } Q_\ell = \chi_\ell(t) I$$

acting on $L_2(\Gamma)$. Denote by $\mathfrak{S}$ the C^*–algebra of all 2×2 matrix–valued functions continuous on $[0,1]$, which are diagonal at the points 0 and 1.

2.3 Theorem *The algebra $\mathcal{R}_\ell = \mathcal{R}(\mathbb{C}; P_\Gamma, Q_\ell)$ is isomorphic and isometric to the algebra $\mathfrak{S}$. The isomorphism*

$$\sigma : \quad \mathcal{R}_\ell \to \mathfrak{S}$$

is generated by the following mapping of the generators of $\mathcal{R}_\ell$:

$$\begin{aligned} \sigma \ &: \ P_\Gamma \longmapsto \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix}, \\ \sigma \ &: \ Q_\ell \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

where $x \in [0,1]$.

2.4 Remark. Theorem 2.3 is well known as well as its different proofs. Nevertheless we will sketch a proof for several reasons, in particular to use details of it later on. Also we would like to recall the "direct" proof, independent of the general "two projection theory", by constructing the isometric isomorphism $T : L_2([0,1]) \to L_2(\Gamma)$ with

$$T^{-1}\mathcal{R}_\ell T = \mathfrak{S}.$$

This is almost the only known case, among many successful applications of the two–projection theory, when one can easily construct such an intertwining operator.

2.5 Sketch of the Proof (H. O. Cordes [1], A. Dynin [4], p.149). Let $t_0 \notin \ell$ be a point of Γ. Then, for the corresponding Moebius transformation (with $t_1 = t_0$, $t_2 = t'$, $t_3 = t''$) and the corresponding operator W, we have

$$W^{-1}\mathcal{R}_\ell W = \mathcal{R}_+ = \mathcal{R}(\mathbb{C}; P_\mathbb{R}, Q_+),$$

where $\mathcal{R}_+$ is the algebra generated by the two orthogonal projections

$$P_\mathbb{R} = \frac{1}{2}(I + S_\mathbb{R}) \text{ and } Q_+ = \chi_{\mathbb{R}_+}(x) I$$

acting on $L_2(\mathbb{R})$, or equevalently generated by the operators $S_\mathbb{R}$ and Q_+.

Introduce the isomorphism $V: L_2^2(\mathbb{R}_+) \to L_2(\mathbb{R})$ by the rule

$$V: (\varphi_1, \varphi_2) \mapsto \varphi(x) = \begin{cases} \varphi_1(x), & x \in \mathbb{R}_+, \\ \varphi_2(-x), & x \in \mathbb{R} \backslash \mathbb{R}_+, \end{cases}$$

then

$$V^{-1}: \varphi(x) \in L_2(\mathbb{R}) \mapsto (\varphi|_{\mathbb{R}_+}(x), \varphi|_{\mathbb{R}\backslash\mathbb{R}_+}(-x)) \in L_2^2(\mathbb{R}_+).$$

For the generators of the algebra $\mathcal{R}_+$ we have obviously

$$V^{-1} S_{\mathbb{R}} V = \begin{pmatrix} S_{\mathbb{R}_+} & -N \\ N & -S_{\mathbb{R}_+} \end{pmatrix},$$

$$V^{-1} Q_+ V = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$(S_{\mathbb{R}_+}\varphi)(x) = \frac{1}{\pi i} \int_{\mathbb{R}_+} \frac{\varphi(y)dy}{y - x}, \quad (N\varphi)(x) = \frac{1}{\pi i} \int_{\mathbb{R}_+} \frac{\varphi(y)dy}{y + x}.$$

The composition of the Fourier transform $F: L_2(\mathbb{R}) \to L_2(\mathbb{R})$,

$$(F\varphi)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\lambda x} \varphi(x) dx$$

with the unitary operator $U_1 : L_2(\mathbb{R}_+) \to L_2(\mathbb{R})$,

$$(U_1\varphi)(x) = e^{\frac{x}{2}} \varphi(e^x),$$

is the Mellin transform

$$M = FU_1 : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}),$$

where

$$(M\varphi)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} x^{-i\lambda - \frac{1}{2}} \varphi(x) dx;$$

and

$$M^{-1}\psi = M^*\psi = U_1^{-1} F^{-1} \psi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^{i\lambda - \frac{1}{2}} \psi(\lambda) d\lambda.$$

Then

$$\begin{aligned} M S_{\mathbb{R}_+} M^{-1} &= -\tanh \pi\lambda I, \\ M N M^{-1} &= i(\coth \pi\lambda)^{-1} I, \end{aligned}$$

where $\lambda \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Introduce now the isomorphism $U_2 : L_2([0,1]) \to L_2(\mathbb{R})$ by the rule

$$(U_2\varphi)(\lambda) = \frac{\pi}{e^{-\pi\lambda} + e^{\pi\lambda}} \varphi(\frac{e^{\pi\lambda}}{e^{-\pi\lambda} + e^{\pi\lambda}}).$$

Then

$$\begin{aligned} U_2^{-1}(-\tanh \pi\lambda I)U_2 &= (1-2x)I, \\ U_2^{-1}(i(\coth \pi\lambda)^{-1}I)U_2 &= 2i\sqrt{x(1-x)}I, \end{aligned}$$

where $x \in [0,1]$, and thus

$$\begin{aligned} (U_2^{-1}M \otimes e_2)V^{-1}S_{\mathbb{R}}V(M^{-1}U_2 \otimes e_2) &= \begin{pmatrix} 1-2x & -2i\sqrt{x(1-x)} \\ 2i\sqrt{x(1-x)} & 2x-1 \end{pmatrix} I, \\ (U_2^{-1}M \otimes e_2)V^{-1}Q_+V(M^{-1}U_2 \otimes e_2) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} I, \end{aligned}$$

where $x \in [0,1]$, e_2 is the 2×2 unit matrix.

Let finally

$$m = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Then the operator

$$T = WV(M^{-1}U_2 \otimes e_2)mI : L_2^2([0,1]) \to L_2(\Gamma)$$

is an isometric isomorphism,

$$\begin{aligned} T^{-1}P_\Gamma T &= \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix} I, \\ T^{-1}Q_\ell T &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} I, \end{aligned}$$

where $x \in [0,1]$, and as a corollary,

$$T^{-1}\mathcal{R}_\ell T = \mathfrak{S}.$$

$\square$

3 Invariant subspaces for the algebra of singular integral operators generated by two projections

3.1 In this section we describe all invariant subspaces of the algebra $\mathcal{R}_\ell = \mathcal{R}(\mathbb{C}; P_\Gamma, Q_\ell)$ of the previous Section.

In this connection note that the less mysterious classical C^*–algebra $\mathcal{R} = \mathcal{R}(C(\Gamma); S_\Gamma)$ generated by all the operators

$$A = a(t)I + b(t)S_\Gamma$$

acting on $L_2(\Gamma)$, with $a, b \in C(\Gamma)$, or generated just by two generators tI and S_Γ, does not have invariant subspaces (the identical representation of this algebra is irreducible), while the generators tI and S_Γ have many of them.

The different nature of the operators $Q_\ell = \chi_\ell(t)I$ and S_Γ suggests that they should not have invariant subspaces. Nevertheless they do; moreover, the C^*–algebra generated by them, as well as each algebra generated by two projections, has an uncountable family of invariant subspaces.

3.2 Fix $\ell \subset \Gamma$ and consider the algebra $\mathcal{R}_\ell = \mathcal{R}(\mathbb{C}; S_\Gamma, Q_\ell)$, the operator W, and the corresponding algebra $\mathcal{R}_+ = \mathcal{R}(\mathbb{C}; S_\mathbb{R}, Q_+)$.

Introduce the algebra

$$\mathfrak{S}_\mathbb{R} = (M \otimes e_2)V^{-1}W^{-1}\mathcal{R}_\ell WV(M^{-1} \otimes e_2).$$

As follows from 2.5, the C^*–algebra $\mathfrak{S}_\mathbb{R}$ consists of all multiplication operators by 2×2 matrix–valued functions continuous on $\overline{\mathbb{R}}$ and diagonal at the points $\pm\infty$, acting on $L_2^2(\mathbb{R})$.

3.3 Lemma *The commutant $\mathfrak{S}'_\mathbb{R}$ of the algebra $\mathfrak{S}$ is given by*

$$\mathfrak{S}'_\mathbb{R} = L_\infty(\mathbb{R}) \otimes e_2.$$

PROOF. Obvious.

3.4 Let $X \subset \mathbb{R}$ be a set with positive measure. Denote by L_X the subspace of $L_2(\mathbb{R})$ of all functions of the form

$$\varphi(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \int_X x^{i\lambda - \frac{1}{2}} \psi_1(\lambda)d\lambda, & x \geq 0 \\ \frac{1}{\sqrt{2\pi}} \int_X (-x)^{i\lambda - \frac{1}{2}} \psi_2(\lambda)d\lambda, & x < 0 \end{cases}$$

where $\psi_1, \psi_2 \in L_2(X)$.

3.5 Theorem *All invariant subspaces of the algebras $\mathcal{R}_+ = \mathcal{R}(\mathbb{C}; S_\mathbb{R}, Q_+)$ and $\mathcal{R}_\ell = \mathcal{R}(\mathbb{C}; S_\Gamma, Q_\ell)$ are of the form L_X and $W(L_X)$ respectively, where X is an arbitrary set in $\mathbb{R}$ having positive measure.*

PROOF. A subspace of $L_2^2(\mathbb{R})$ is invariant for the algebra $\mathfrak{S}_\mathbb{R}$ if and only if the orthogonal projection onto this subspace commutes with all the operators of $\mathfrak{S}_\mathbb{R}$, or belongs to $\mathfrak{S}'_\mathbb{R}$. But all orthogonal projections in $\mathfrak{S}_\mathbb{R}$ have the form $\chi_X(\lambda)I \otimes e_2$, for a measurable positive measure set X in $\mathbb{R}$. Thus all invariant subspaces for the algebra $\mathfrak{S}_\mathbb{R}$ have the form

$$(\chi_X I \otimes e_2)(L_2^2(\mathbb{R})) = L_2(X) \oplus L_2(X).$$

To describe the invariant subspaces for the algebras $\mathcal{R}_+$ and $\mathcal{R}_\ell$ one needs to use the isometries $V(M^{-1} \otimes e_2)$ and $WV(M^{-1} \otimes e_2)$ respectively. □

4 Algebras of singular integral operators generated by three orthogonal projections

4.1 Let $c_3 \prec c_2 \prec c_1$ be three points on the unit circle Γ. Denote by ℓ_1 the arc on Γ with starting point c_2 and end point c_1, and by ℓ_2 an arc on Γ with starting point c_3 having no common interior points with ℓ_1. Thus for the end point c' of ℓ_2 we have either $c' = c_2$, or $c_3 \prec c' \prec c_2$.

The operators $Q_1 = \chi_{\ell_1}(t)I$ and $Q_2 = \chi_{\ell_2}(t)I$, where $\chi_{\ell_k}(t)$ is the characteristic function of ℓ_k, $k = 1, 2$, are obviously orthogonal projections on $L_2(\Gamma)$ with

$$Q_1 \cdot Q_2 = Q_2 \cdot Q_1 = 0.$$

Denote by $\mathcal{R} = \mathcal{R}(\mathbb{C}; P, Q_1, Q_2)$ the C^*–algebra with identity generated by the orthogonal projections

$$P = \frac{1}{2}(I + S_\Gamma), \quad Q_1 \text{ and } \ Q_2$$

acting on $L_2(\Gamma)$.

4.2 Theorem *The identical representation*

$$id: \quad \mathcal{R} \longrightarrow \mathcal{L}(L_2(\Gamma))$$

of the algebra $\mathcal{R}$ is irreducible.

PROOF. For $t_1 = c_3$, $t_2 = c_2$ and $t_3 = c_1$ introduce the corresponding Moebius transformation $x = \omega(t)$ (see 1.1) and consider the isometric isomorphism of the algebra $\mathcal{R}$,

$$\nu: \quad \mathcal{R} \longrightarrow \widetilde{\mathcal{R}} = \widetilde{T}^{-1}\mathcal{R}\widetilde{T},$$

with $\widetilde{T} = WVU_1^{-1}$, and the operators W, V, U_1^{-1} as in 2.5.

The operators of the algebra $\widetilde{\mathcal{R}}$ act on $L_2^2(\mathbb{R})$, $\widetilde{Q}_2 = \nu(Q_2) = \chi_{[a,b]}I \otimes e_2$, where $[a, b] = \ln(\omega(\ell_1))$. By 2.5 and Lemma 3.3, the commutant $\widetilde{\mathfrak{S}}'$ of the algebra $\widetilde{\mathfrak{S}} = \nu(\mathcal{R}_{\ell_1}) = \nu(\mathcal{R}(\mathbb{C}; P, Q_1))$ is equal to

$$\widetilde{\mathfrak{S}}' = F^{-1}L_\infty(\mathbb{R})F \otimes e_2.$$

Now it is easy to see that none of the operators (except of course the scalars $kI = F^{-1}kF$, $k \in \mathbb{C}$) from the algebra $\widetilde{\mathfrak{S}}'$ commutes with the projection $\widetilde{Q}_2$. Thus the commutant of the algebra $\widetilde{\mathcal{R}}$, generated by elements of $\widetilde{\mathfrak{S}}$ and $\widetilde{Q}_2$, is scalar, and thus, with the help of the isometry $\widetilde{T}$, the commutator of the algebra $\mathcal{R}$ is scalar, so this algebra is irreducible. □

4.3 Corollary *The algebra $\mathcal{R} = \mathcal{R}(\mathbb{C}; P, Q_1, Q_2)$ contains the ideal $\mathcal{K}$ of all compact operators on $L_2(\Gamma)$.*

PROOF. It is sufficient to find at least one compact operator in $\mathcal{R}$. The operators $Q_1S_\Gamma Q_2$ for $c' \neq c_2$, and $Q_1S_\Gamma Q_2S_\Gamma(I - Q_1 - Q_2)$ for $c' = c_2$, are obviously compact. □

4.4 Corollary *Each irreducible representation of the algebra $\mathcal{R} = \mathcal{R}(\mathbb{C}; P, Q_1, Q_2)$ is either the identical representation or an irreducible representation of the symbol (Calkin) algebra* $\operatorname{Sym}\mathcal{R} = \mathcal{R}/\mathcal{K}$.

4.5 We will give descriptions of the symbol algebras for two algebras $\mathcal{R} = \mathcal{R}(\mathbb{C}; P, Q_1, Q_2)$: the first one is the algebra $\mathcal{R}_1$ for the case when $c' = c_2$, the second one is the algebra $\mathcal{R}_2$ when $c' \neq c_2$.

These descriptions can be easily deduced from the well–known description (see for example [3]) of the symbol algebra for one–dimensional singular integral operators with piecewise continuous coefficients (having in our case three or four points of discontinuity respectively).

One needs only to check (using for example, the standard local principle of Douglas–Varela [2, 8]) that the algebras $\operatorname{Sym}\mathcal{R}_1$ and $\operatorname{Sym}\mathcal{R}_2$ consist of *all* elements of the type described.

To formulate the results we introduce two C^*–algebras. Denote by $\mathfrak{S}_1$ the algebra of all 2×2 matrix–valued functions $\sigma = (\sigma_{ij})_{i,j=1}^2$ continuous on $\Delta_1 = \bigsqcup_{k=1}^{3} \Delta^{(k)}$, where $\Delta^{(k)} = [0, 1]$, $k = \overline{1, 3} = 1, 2, 3$, such that

1) σ is diagonal at the points of the disjoint union

$$\partial\Delta_1 = \bigsqcup_{k=1}^{3} \partial\Delta^{(k)};$$

2) the diagonal entries at the points of $\partial\Delta_1$ satisfy the following conditions:

$$\begin{array}{ll} \sigma_{11}^{(1)}(0) = \sigma_{11}^{(3)}(1), & \sigma_{22}^{(1)}(0) = \sigma_{22}^{(2)}(1), \\ \sigma_{11}^{(2)}(0) = \sigma_{11}^{(1)}(1), & \sigma_{22}^{(2)}(0) = \sigma_{22}^{(3)}(1), \\ \sigma_{11}^{(3)}(0) = \sigma_{11}^{(2)}(1), & \sigma_{22}^{(3)}(0) = \sigma_{22}^{(1)}(1), \end{array}$$

where $\sigma^{(k)} = \sigma|_{\Delta^{(k)}}$, $k = \overline{1,3}$.

Denote by $\mathfrak{S}_2$ the algebra of all 2×2 matrix–valued functions $\sigma = (\sigma_{ij})_{i,j=1}^2$ continuous on $\Delta_2 = \Delta^{(1)} \bigsqcup \Delta^{(2)}$, where $\Delta^{(k)} = [0,1]$, $k = 1,2$, such that:

1) σ is diagonal at the points of

$$\partial\Delta_2 = \partial\Delta^{(1)} \bigsqcup \partial\Delta^{(2)};$$

2) the diagonal entries at the points of $\partial\Delta_2$ satisfy the following conditions:

$$\sigma_{11}^{(1)}(1) = \sigma_{11}^{(2)}(1), \quad \sigma_{22}^{(1)}(0) = \sigma_{22}^{(2)}(0).$$

4.6 Theorem *The symbol algebra* $\operatorname{Sym}\mathcal{R}_1 = \mathcal{R}_1/\mathcal{K}$ *of the algebra* $\mathcal{R}_1$ $(c' = c_2)$ *is isomorphic and isometric to the algebra* $\mathfrak{S}_1$*. The homomorphism*

$$\operatorname{sym} : \mathcal{R}_1 \longrightarrow \mathfrak{S}_1$$

is generated by the following mapping of the projections:

$$\operatorname{sym} : P \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad x \in \Delta_1,$$

$$\operatorname{sym} : Q_1 \longmapsto \begin{cases} \begin{pmatrix} x & -\sqrt{x(1-x)} \\ -\sqrt{x(1-x)} & 1-x \end{pmatrix}, & x \in \Delta^{(1)}, \\ \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix}, & x \in \Delta^{(2)}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & x \in \Delta^{(3)}, \end{cases}$$

$$\operatorname{sym} : Q_2 \longmapsto \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & x \in \Delta^{(1)}, \\ \begin{pmatrix} x & -\sqrt{x(1-x)} \\ -\sqrt{x(1-x)} & 1-x \end{pmatrix}, & x \in \Delta^{(2)}, \\ \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix}, & x \in \Delta^{(3)}. \end{cases}$$

4.7 Theorem *The symbol algebra* $\mathrm{Sym}\,\mathcal{R}_2 = \mathcal{R}_2/\mathcal{K}$ *of the algebra* $\mathcal{R}_2$ $(c' \neq c_2)$ *is isomorphic and isometric to the algebra* $\mathfrak{S}_2$. *The homomorphism*

$$\mathrm{sym} : \mathcal{R}_2 \longrightarrow \mathfrak{S}_2$$

is generated by the following mapping of the projections

$$\mathrm{sym} : P \quad \longmapsto \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad x \in \Delta_2$$

$$\mathrm{sym} : Q_1 \quad \longmapsto \quad \begin{cases} \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix}, & x \in \Delta^{(1)}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & x \in \Delta^{(2)}, \end{cases}$$

$$\mathrm{sym} : Q_2 \quad \longmapsto \quad \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & x \in \Delta^{(1)}, \\ \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix}, & x \in \Delta^{(2)}. \end{cases}$$

4.8 Remark. Both algebras $\mathcal{R}_1$ and $\mathcal{R}_2$ are generated by the orthogonal projections P, Q_1 and Q_2 with the same relations between them. But Theorems 4.6 and 4.7 show that these algebras have different structures. Namely, both algebras are irreducible, contain the ideal $\mathcal{K}$ of all compact operators, but their quotient algebras $\mathcal{R}_k/\mathcal{K}$ are quite different, as well as the system of their finite–dimensional irreducible representations.

5 Algebras of bisingular operators generated by three orthogonal projections

5.1 For each angle $\alpha \in [0, 2\pi)$ introduce a rotation of $\mathbb{R}^2 = \mathbb{C}$ by the rule

$$\omega_\alpha(x) = e^{-i\alpha}x, \text{ where } x = x_1 + ix_2,$$

and the unitary operator

$$(W_\alpha\varphi)(x) = \varphi(\omega_\alpha(x))$$

acting on $L_2(\mathbb{R}^2)$. Let F and F^{-1} be the direct and inverse Fourier transformations on $\mathbb{R}^2$ respectively. We have obviously

$$FW_\alpha = W_\alpha F \text{ and } F^{-1}W_\alpha = W_\alpha F^{-1}.$$

For $\alpha \in [0, 2\pi)$, let $A_1[\alpha]$, $A_2[\alpha]$ be the images of the standard coordinate axes A_1, A_2 under ω_{-a}, and let $(x_1[\alpha], x_2[\alpha])$ denote the coordinates of the point $x = (x_1, x_2)$ with respect to the rotated axes $A_1[\alpha]$, $A_2[\alpha]$.

Let $S_\mathbb{R}$ be, as before, the one dimensional singular integral operator over the reals. For any axis $L \subset \mathbb{R}^2$, S_L is the operator corresponding to this inclusion. Let χ_+ be the characteristic function of the positive reals, thus $\chi_+(x_1[\alpha])$ is the characteristic function on the positive half of $A_1[\alpha]$.

Given angles α_1 and α_2, introduce the three orthogonal projections

$$\begin{aligned} P &= \frac{1}{2}(I+S_{A_1})\otimes I_{\mathbb{A}_{\not\equiv}}, \\ Q_1 &= \chi_+(\cos\alpha_1\cdot x_1+\sin\alpha_1\cdot x_2)I=\chi_+(x_1[\alpha_1])I, \\ Q_2 &= \chi_+(\cos\alpha_2\cdot x_1+\sin\alpha_2\cdot x_2)I=\chi_+(x_1[\alpha_2])I. \end{aligned}$$

We have obviously $Q_2\cdot Q_2=Q_2\cdot Q_1$.

In this section we study the C^*–algebra with identity $\mathcal{R}=\mathcal{R}(\alpha_1,\alpha_2)=\mathcal{R}(\mathbb{C};P,Q_1,Q_2)$ generated by these three projections.

The dual algebra $\mathcal{R}'=F\mathcal{R}F^{-1}$ to the algebra $\mathcal{R}$ has obviously the same structure and is generated by the following projections:

$$\begin{aligned} P'=FPF^{-1} &= \chi_+(x_1)I \\ Q_1'=FQ_1F^{-1} &= \frac{1}{2}(I_{A_1[\alpha_1]}+S_{A_1[\alpha_1]})\otimes I_{A_2[\alpha_1]}, \\ Q_2'=FQ_2F^{-1} &= \frac{1}{2}(I_{A_1[\alpha_2]}+S_{A_1[\alpha_2]})\otimes I_{A_2[\alpha_2]}. \end{aligned}$$

The "rotated" algebra

$$\mathcal{R}'(\beta)=W_\beta\mathcal{R}'W_\beta^{-1},$$

which is generated by the following projections

$$\begin{aligned} P'_{(\beta)} &= W_\beta P'W_\beta^{-1}=\chi_+(x_1[\beta])I, \\ Q_1'(\beta) &= W_\beta Q_1'W_\beta^{-1}=\frac{1}{2}(I_{A_1[\alpha_1+\beta]}+S_{A_1[\alpha_1+\beta]})\otimes I_{A_2[\alpha_1+\beta]}, \\ Q_2'(\beta) &= W_\beta Q_2'W_\beta^{-1}=\frac{1}{2}(I_{A_1[\alpha_2+\beta]}+S_{A_1[\alpha_2+\beta]})\otimes I_{A_2[\alpha_2+\beta]} \end{aligned}$$

is obviously isomorphic to each of the algebras $\mathcal{R}$ and $\mathcal{R}'$.

Note that some operators from the special case of the algebra $\mathcal{R}'(\beta)$ (with $\beta=\frac{3\pi}{4}$, $\alpha_1+\beta=0$, $\alpha_2+\beta=\frac{\pi}{2}$), i.e., from the algebra generated by

$$S_{\mathbb{R}}\otimes I,\quad I\otimes S_{\mathbb{R}}\quad\text{and}\quad\chi_+(x_2-x_1)I$$

have been considered in [7].

5.2 Given $\alpha_1\neq\alpha_2$, we consider the algebra $\mathcal{R}=\mathcal{R}(\alpha_1,\alpha_2)=\mathcal{R}(\mathbb{C};P,Q_1,Q_2)$ generated by the projections P, Q_1, Q_2 introduced in Subsection 5.1.

Changing the numbering and changing Q_k for $I-Q_k$, $k=1,2$, if necessary, we can assume that

$$-\frac{\pi}{2}<\alpha_1<\alpha_2\leq\frac{\pi}{2}.$$

We exclude the following elementary case: if $\alpha_2=\frac{\pi}{2}$ then Q_2 commutes with both P and Q_1, and $\mathcal{R}$ is just the direct sum of the two algebras generated by *two* projections. Thus we are studying the algebra $\mathcal{R}=\mathcal{R}(\alpha_1,\alpha_2)$, with $-\frac{\pi}{2}<\alpha_1<\alpha_2<\frac{\pi}{2}$.

5.3 Theorem *The algebra $\mathcal{R}=\mathcal{R}(\alpha_1,\alpha_2)$, with $-\frac{\pi}{2}<\alpha_1<\alpha_2<\frac{\pi}{2}$, splits into a direct integral of the algebras $\mathcal{R}(x_2)$, $x_2\in\mathbb{R}\backslash\{0\}$, of operators acting on $L_2(\mathbb{R})=L_2(A_1)$.*

For each $x_2 \in \mathbb{R}\backslash\{0\}$, the algebra $\mathcal{R}(x_2)$ is isomorphic to the algebra $\mathcal{R}_1$ of Subsection 4.5.

PROOF. Introduce the unitary operator $V : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ by the rule

$$(V\varphi)(x_1, x_2) = |x_2|^{1/2}\varphi(|x_2| \cdot x_1, x_2).$$

Then obviously

$$(V^{-1}\varphi)(x_1, x_2) = |x_2|^{-1/2}\varphi(|x_2|^{-1} \cdot x_1, x_2).$$

The algebra $\widetilde{\mathcal{R}} = V\mathcal{R}(\alpha_1, \alpha_2)V^{-1}$ is naturally isomorphic to the algebra $\mathcal{R}(\alpha_1, \alpha_2)$. Direct calculation shows that the generators of the algebra $\widetilde{\mathcal{R}}$ acting on the space $L_2(\mathbb{R}^2) = L_2(A_1)\otimes L_2(A_2)$ have the following form:

$$\begin{aligned}
\widetilde{P} &= VPV^{-1} = P = \widetilde{P}(x_2) \otimes I, \\
\widetilde{Q}_1 &= VQ_1V^{-1} = \chi_+(x_1 + \tan\alpha_1 \cdot \frac{x_2}{|x_2|})I = \widetilde{Q}_1(x_2) \otimes I, \\
\widetilde{Q}_2 &= VQ_2V^{-1} = \chi_+(x_1 + \tan\alpha_2 \cdot \frac{x_2}{|x_2|})I = \widetilde{Q}_2(x_2) \otimes I,
\end{aligned}$$

where the operator–valued functions

$$\widetilde{P}(\cdot), \quad \widetilde{Q}_1(\cdot), \quad \widetilde{Q}_2 : \quad \mathbb{R}\backslash\{0\} \to \mathcal{L}(L_2(\mathbb{R}))$$

are defined by the rule

$$\widetilde{P}(x_2) = \frac{1}{2}(I + S_{\mathbb{R}}), \quad x_2 \in \mathbb{R}\backslash\{0\}$$

$$\widetilde{Q}_k(x_2) = \begin{cases} \chi_+(x_1 + \tan\alpha_k)I, & x_2 > 0 \\ \chi_+(x_1 - \tan\alpha_k)I, & x_2 < 0 \end{cases},$$

$k = 1, 2$.

Thus the algebra $\widetilde{\mathcal{R}}$ splits onto a direct integral of the algebras $\widetilde{\mathcal{R}}(x_2)$, $x_2 \in \mathbb{R}\backslash\{0\}$, generated by the projections

$$\widetilde{P}(x_2), \quad \widetilde{Q}_1(x_2) \ \text{ and } \ \widetilde{Q}_2(x_2)$$

acting on $L_2(\mathbb{R}) = L_2(A_1)$. Introduce the unitary operators τ_h, $h \in \mathbb{R}$, and δ_r, $r \in \mathbb{R}_+$, acting on $L_2(\mathbb{R})$ by the rule

$$\begin{aligned}
(\tau_n)\varphi(x) &= \varphi(x - h), \\
(\delta_r\varphi)(x) &= r^{1/2}\varphi(rx).
\end{aligned}$$

Then the algebra $\widehat{\mathcal{R}}(x_2) = \delta_r\tau_h\widetilde{\mathcal{R}}(x_2)\tau_h^{-1}\delta_r^{-1}$, where $h = \tan\alpha_1$ for $x_2 > 0$ and $h = -\tan\alpha_2$ for $x_2 < 0$, $r = \tan\alpha_2 - \tan\alpha_1$, is naturally isomorphic to the algebra $\widetilde{\mathcal{R}}(x_2)$, and is generated by the following projections:

$$\widehat{P}(x_2) = \frac{1}{2}(I + S_{\mathbb{R}}), \quad x_2 \in \mathbb{R}\backslash\{0\},$$

$$\widehat{Q}_1(x_2) = \begin{cases} \chi_+(x_1), & x_2 > 0 \\ \chi_+(x_1 + 1), & x_2 < 0 \end{cases},$$

$$\widehat{Q}_2(x_2) = \begin{cases} \chi_+(x_1+1), & x_2 > 0 \\ \chi_+(x_1), & x_2 < 0 \end{cases}.$$

Applying the operator W from Subsection 2.1 we have now for each $x_2 \in \mathbb{R}\backslash\{0\}$ the isomorphism

$$\widehat{\mathcal{R}}(x_2) \longrightarrow W\widehat{\mathcal{R}}(x_2)W^{-1} = \mathcal{R}_1$$

between the algebra $\widehat{\mathcal{R}}(x_2)$ and the algebra $\mathcal{R}_1$ (with $c' = c_2$) of Subsection 4.5. $\square$

5.4 Corollary *The algebra $\mathcal{R} = \mathcal{R}(\alpha_1, \alpha_2)$ is isomorphic to the algebra $\mathcal{R}_1 \oplus \mathcal{R}_1$, with $\mathcal{R}_1$ as in Subsection 4.5. The isomorphism*

$$\mu : \quad \mathcal{R}(\alpha_1, \alpha_2) \longrightarrow \mathcal{R}_1 \oplus \mathcal{R}_1$$

is defined by the following mapping of the generators of the algebra $\mathcal{R}$,

$$\begin{array}{lll} \mu & : & P \longmapsto (1/2(I + S_\Gamma),\, 1/2(I + S_\Gamma)), \\ \mu & : & Q_1 \longmapsto (\chi_{\ell_1} I,\, \chi_{\ell_1 \cup \ell_2} I), \\ \mu & : & Q_2 \longmapsto (\chi_{\ell_1 \cup \ell_2} I,\, \chi_{\ell_1} I), \end{array}$$

or in a more transparent notation:

$$\begin{array}{lll} \mu & : & P \longmapsto (p,\, p), \\ \mu & : & Q_1 \longmapsto (q_1,\, q_1 + q_2), \\ \mu & : & Q_2 \longmapsto (q_1 + q_2,\, q_1), \end{array}$$

where we denote the generators of the algebra $\mathcal{R}_1$ by p, q_1 and q_2.

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Departamento de Matemáticas
CINVESTAV del I.P.N.
Apartado Postal 14-740
07000 México, D.F.
MEXICO

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