TWO TYPES OF ANALYSIS ASSOCIATED TO THE NOTION OF HURWITZ PAIRS *

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1. Introduction

The Hurwitz problem stated by him in 1898 [H1] and its further development [H2] were the motivation to introduce and to study the so–called Hurwitz pairs (see [Sh], [SY] for an extensive literature).

In our works [RSV1], [RSV2] the precise relation between Hurwitz pairs and Clifford algebras has been established. Two canonical algorithms have been described for constructing an irreducible representation of a certain Clifford algebra for a given Hurwitz pair and, conversely, for constructing all possible Hurwitz pairs for given Clifford algebra and its irreducible representation.

All this has been inspired by our wish to develop the so–called Hurwitz analysis initiated in [KR1], [KR2].

The present article gives our vision of the situation, that is, which analytic theories can be related to a given Hurwitz pair. Our previous studies ([RSV1],[RSV2], see also Section 1) show that a Hurwitz pair generates two types of multiplication which are essentially different, in general. In accordance with this fact, in Section 2 we introduce two types of generalized Cauchy–Riemann operators and show that all main formulas for the functions from their kernels can be obtained in the traditional way.

We stop our considerations at the place where it is perfectly clear how to develop both corresponding theories. We are going to explain how they are connected with the Clifford analysis elsewhere.

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2. Algebraic fundamentals of Hurwitz pairs

2.1 Let S be a $(p+1)$–dimensional real vector space with basis $\{\varepsilon_\alpha\}, \alpha \in \{0\} \cup N_p, N_p := \{1, 2, \ldots, p\}$, and let the R–bilinear form

$$(\cdot, \cdot)_S : S \times S \to R$$

be defined by the following metric matrix

$$\eta := [\eta_{\alpha\beta}] := [(\varepsilon_\alpha, \varepsilon_\beta)_S] = \text{ diag } (\underbrace{1, \ldots, 1}_{r+1}; \underbrace{-1, \ldots, -1}_{s}),$$

where $p = r + s$.

Introduce also the n–dimensional real vector space V with the basis $\{e_j\}, j \in N_n$, provided with an R–bilinear form

$$(\cdot, \cdot)_V : V \times V \to R,$$

which is defined by the following nonsingular metric matrix

$$\kappa := [\kappa_{kj}] := [(e_k, e_j)_V].$$

We assume also that the form $(\cdot, \cdot)_V$ in V is either symmetric: $\kappa = \kappa^t$, or antisymmetric: $\kappa = -\kappa^t$, where "t" means transposition.

Let

$$\circ : S \times V \to V$$

be an R–bilinear mapping. We call it (see, for instance [LR]) a Hurwitz multiplication (of elements from V by elements of S on the left–hand side) if the following axioms are fulfilled:

H.1 for all $\{f, g\} \subset V$ and all $a \in S$

$$(a, a)_S (f, g)_V = (a \circ f, a \circ g)_V;$$

H.2 there exists the unit element $\varepsilon \in S$ with respect to the mapping "$\circ$", i.e. for all $f \in V$

$$\varepsilon \circ f = f;$$

H.3 the mapping "$\circ$" does not leave invariant any proper subspace of V.

The set $(S, V, \circ)$ is called a Hurwitz pair.

2.2 Introduce the R–linear isomorphisms

$$\nu_S : S \to R^{p+1}$$

and

$$\nu_V : V \to R^n$$

by the rules

$$\nu_S : a = \sum_{\alpha=0}^{p} a_\alpha \varepsilon_\alpha \longmapsto \tilde{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{pmatrix} \in R^{p+1}$$

and

$$\nu_V : f = \sum_{j=1}^{n} f_j e_j \longmapsto \widetilde{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in R^n.$$

For each basis elements $\varepsilon_\alpha \in S$ and $e_j \in V$ we have $\varepsilon_\alpha \circ \varepsilon_j \in V$. Thus for some real constants $c^k_{\alpha j}$:

$$\varepsilon_\alpha \circ e_j = \sum_{k=1}^{n} c^k_{\alpha j} e_k,$$

and for every $f = \sum\limits_{j=1}^{n} f_j e_j \in V$:

$$\varepsilon_\alpha \circ f = \sum_{k=1}^{n} (\sum_{j=1}^{n} c^k_{\alpha j} f_j) e_k.$$

Applying the isomorphism ν_V to the both sides of this equality we obtain

$$\nu_V(\varepsilon_\alpha \circ f) = C_\alpha \cdot \nu_V(f),$$

where $C_\alpha := [c^k_{\alpha j}]^n_{j,k=1}$.

Thus, each element $\varepsilon_\alpha \in S$ determines uniquely the matrix C_α (and vice versa), and the following diagram

$$\begin{array}{ccc} V & \xrightarrow{m(\varepsilon_\alpha)} & V \\ \downarrow{\scriptstyle \nu_V} & & \downarrow{\scriptstyle \nu_V} \\ R^n & \xrightarrow{m(C_\alpha)} & R^n \end{array}$$

is commutative. Here $m(\varepsilon_\alpha) : f \mapsto \varepsilon_\alpha \circ f$ and $m(C_\alpha) : \widetilde{f} \mapsto C_\alpha \cdot \widetilde{f}$.

2.3 Without loss of generality we may assume that the unit element ($\equiv$ identity $\varepsilon \in S$) coincides with ε_0.

Under this assumption we have (see [RSV1]): for each $\{\alpha, \beta\} \subset N_p$

$$C_\alpha \cdot C_\beta + C_\beta \cdot C_\alpha = -2\eta_{\alpha\beta} I_n.$$

2.4 Consider a fixed Hurwitz pair $(S, V, \circ)$ with the unit element $\varepsilon = \varepsilon_0$. Each element $a \in S$ generates in a natural way the operator of "Hurwitz multiplication by a" acting on V by the rule

$$f \in V \longmapsto a \circ f \in V.$$

Denote this operator by $m(a)$. It is clear that for $a = \sum\limits_{\alpha=0}^{p} a_\alpha \varepsilon_\alpha$

$$m(a) = \sum_{\alpha=0}^{p} a_\alpha m(\varepsilon_\alpha),$$

where $m(\varepsilon_0) = I$, the identity operator on V.

The mapping

$$\mu : a \longmapsto \mu(a) \in \text{ Hom } (V, V)$$

gives a linear isomorphism between the space S and some linear subspace of Hom (V, V).

Denote by $Al(S, V, \circ)$ the algebra generated by all operators $m(a)$ acting on V. We have obviously

$$\mu(S) \subset Alg(S, V, \circ) \subset Hom(V, V) \cong R(n),$$

and in general both inclusions are proper.

All above said can be found in [RSV1], [RSV2], but to construct the corresponding function theory we need to complement the algebraic part of those works with some new results.

2.5 The algebra $Alg(S, V, \circ)$ allows us to introduce a multiplication on elements of S in such a way that the algebra $\widetilde{S}$ generated by this multiplication becomes isomorphic to $Al(S, V, \circ)$. The corresponding isomorphism will be an extension of the linear mapping μ from S onto the algebra $\widetilde{S}$.

Let us describe this procedure.

Denote by $\check{S}$ the *free* algebra generated by the elements of S and let

$$\check{\mu} : \check{S} \longrightarrow Alg(S, V, \circ)$$

be the real algebra homomorphism which is generated by the following mapping of the generators $a \in S$ of the algebra $\check{S}$:

$$\check{\mu} : a \in S \longmapsto \check{\mu}(a) := \mu(a) = m(a) \in \text{ Hom } (V, V),$$

i.e. the mapping $\check{\mu}$ is an extension (up to a real algebra homomorphism) of the mapping

$$\mu : S \longrightarrow Alg(S, V, \circ).$$

Now introduce the algebra $\widetilde{S} := \check{S}/ker\check{\mu}$ and the mapping

$$\widetilde{\mu} : \widetilde{S} \longrightarrow Alg(S, V, \circ)$$

which is defined by the rule

$$\widetilde{\mu} : [\check{a}] \longmapsto \check{\mu}(\check{a}),$$

where $[\check{a}] := \check{a} + ker\check{\mu}$ for $\check{a} \in \check{S}$.

We will denote by "$*$" the multiplication symbol in $\widetilde{S}$.

The described procedure provides a natural imbedding of S into $\widetilde{S}$. Identifying, as usual, S and its image under this imbedding, we can say now that S is a subset (and a linear subspace) of the algebra $\widetilde{S}$. It is clear that

$$\widetilde{\mu}|S = \mu.$$

The mapping $\widetilde{\mu}$ gives now a real algebra isomorphism of $\widetilde{S}$ onto $Alg(S, V, \circ) \subset$ Hom (V, V), and thus also a representation of the algebra $\widetilde{S}$ on the space V.

Moreover, we have a well–defined extension (from $S \times V$ onto $\widetilde{S} \times V$) of the Hurwitz multiplication, also denoted by "$\circ$", as follows:

$$\text{for each} \qquad s \in \widetilde{S} \quad \text{and} \quad f \in V$$

$$s \circ f := \widetilde{\mu}(s)(f) \in V.$$

2.6 Remark. For any elements $s_1, s_2 \in \widetilde{S}$ and $f \in V$ the following "associativity law" is true

$$(s_1 * s_2) \circ f = s_1 \circ (s_2 \circ f),$$

and thus we can write $s_1 * s_2 \circ f := s_1 \circ s_2 \circ f := (s_1 * s_2) \circ f = s_1 \circ (s_2 \circ f)$.

3. Conceptions of Hurwitz analyses

3.1 Given a Hurwitz pair $(S, V, \circ)$ (with no restrictions on η), denote by l any integer with the condition $2 \leq l \leq p$, and let Ω be a domain in R^{l+1}. For any set of vectors $(\psi^0, \psi^1, \ldots, \psi^l) =: \psi \in S^{l+1}$ we can write the formal expression

$$\sum_{\alpha=0}^{l} \psi^\alpha \circ \frac{\partial}{\partial x_\alpha} \tag{1}$$

where $\frac{\partial}{\partial x_\alpha}$ denotes the operation of the usual partial derivation of a given ($\widetilde{S}$–valued or V–valued) function defined in Ω. Depending on the type of function the expression (1) allows to introduce two kinds of operators.

3.2 For an arbitrary function $f \in \mathcal{C}^1(\Omega, V)$ define the operator $\mathcal{D}_\psi$ by the rule

$$\mathcal{D}_\psi[f] := \sum_{\alpha=0}^{l} \psi^\alpha \circ \frac{\partial}{\partial x_\alpha}[f] := \sum_{\alpha=0}^{l} \psi^\alpha \circ \frac{\partial f}{\partial x_\alpha}.$$

The operator $\mathcal{D}_\psi$ becomes later the Cauchy–Riemann operator in the analysis of V–valued functions. It is necessary to emphasize strongly that just here we see the consequence of the Hurwitz multiplication asymmetry: in contrast with the usual hyperholomorphic settings we can introduce the left operator only, not the left and the right ones (compare with what will be done below).

3.3 Consider an arbitrary $g \in \mathcal{C}^1(\Omega, \widetilde{S})$. Introduce the left ${}^\psi\mathfrak{D}$ and the right $\mathfrak{D}^\psi$ analogs of the operator $\mathcal{D}_\psi$ by the rules:

$${}^\psi\mathfrak{D}[g] := \sum_{\alpha=0}^{l} \psi^\alpha * \frac{\partial}{\partial x_\alpha}[g] =: \sum_{\alpha=0}^{l} \psi^\alpha * \frac{\partial g}{\partial x_\alpha}$$

and

$$\mathfrak{D}^\psi[g] := \sum_{\alpha=0}^{l} \frac{\partial}{\partial x_\alpha} * m(\psi^\alpha)[g] := \sum_{\alpha=0}^{l} \frac{\partial g}{\partial x_\alpha} * \psi^\alpha,$$

where m is the map defined in 1.6.

3.4 Using the notion of the "natural conjugation" on S we introduce the "conjugate" operators $\mathcal{D}_\psi^{\#}$, ${}^{\psi}\mathfrak{D}^{\#}$ and $\mathfrak{D}^{\psi,\#}$:

$$\begin{aligned}
\mathcal{D}_\psi^{\#}[f] &:= \sum_{\alpha=0}^{l} \psi^{\alpha\#} \circ \frac{\partial}{\partial x_\alpha}[f], \\
{}^{\psi}\mathfrak{D}^{\#}[g] &:= \sum_{\alpha=0}^{l} \psi^{\alpha\#} * \frac{\partial}{\partial x_\alpha}[g], \\
\mathfrak{D}^{\psi\#}[g] &:= \sum_{\alpha=0}^{l} \frac{\partial}{\partial x_\alpha}[g] * \psi^{\alpha\#},
\end{aligned}$$

where "#" is a linear mapping on S defined on the basis elements ε_k by the rule

$$\varepsilon_k^{\#} = -\varepsilon_k, \ k \in N_p$$

3.5 Remark. To develop the corresponding function theory it is necessary to be able to multiply the above defined operators.

The following peculiarities arise from the asymmetry of the Hurwitz multiplication. We have sets of S–valued functions, of V–valued functions, of $\widetilde{S}$–valued operators (that is, differential operators with coefficients from $\widetilde{S}$), etc. We can "multiply" V–valued functions on the *left*–hand side by $\widetilde{S}$–valued operators obtaining in V–valued functions: ${}^{\psi}\mathfrak{D} \circ f := {}^{\psi}\mathfrak{D}[f]$.

We can "multiply" various $\widetilde{S}$–valued operators on both sides resulting in $\widetilde{S}$–valued operators: ${}^{\psi}\mathfrak{D} * {}^{\psi}\mathfrak{D}^{\#} := {}^{\psi}\mathfrak{D} \cdot {}^{\psi}\mathfrak{D}^{\#}$ with the "$\cdot$" denoting the usual operator product. We should take into account that, according to the definition of ${}^{\psi}\mathfrak{D}$ and ${}^{\psi}\mathfrak{D}^{\#}$, the result of the multiplication is an operator acting on $\widetilde{S}$–valued (not on V–valued) functions.

Finally we can "multiply", in the sense of the Hurwitz multiplication, operators of the type of $\mathcal{D}_\psi$:

$$\mathcal{D}_\psi \circ \mathcal{D}_\psi : f \mapsto \mathcal{D}_\psi \circ (\mathcal{D}_\psi \circ f) = \mathcal{D}_\psi \left[\mathcal{D}_\psi[f]\right].$$

The operator $\mathcal{D}_\psi \circ \mathcal{D}_\psi$ acts on V–valued functions and in this sense $\mathcal{D}_\psi * \mathcal{D}_\psi^{\#} \neq {}^{\psi}\mathfrak{D} * {}^{\psi}\mathfrak{D}^{\#}$.

3.6 For a fixed set ψ introduce the differential l–form

$$\sigma_{\psi,x}^{(l)} := \sum_{\alpha=0}^{l} (-1)^\alpha \cdot \psi^\alpha dx_{[\alpha]}$$

where $dx_{[\alpha]}$ is the differential l–form $dx := dx_0 \wedge \ldots \wedge dx_l$ with dx_α omitted. The operator of exterior differentiation d acts on such $\widetilde{S}$–valued differential forms as a $\widetilde{S}$–linear mapping. Then, if $g \in \mathcal{C}^1(\Omega, \widetilde{S})$, $f \in \mathcal{C}^1(\Omega, V)$, an easy calculation gives:

$$d\left(g * \sigma_{\psi,x}^{(l)} \circ f\right) = \left(\mathfrak{D}^{\psi}[g] \circ f + g * \mathcal{D}_\psi[f]\right) dx.$$

And analogously, for $g, h \subset \mathfrak{C}^1(\Omega, \widetilde{S})$

$$d\left(g * \sigma^{(l)}_{\psi,x} * h\right) = \left(\mathfrak{D}^{\psi}[g] * h + g * {}^{\psi}\mathfrak{D}[h]\right) dx.$$

3.7 If now we assume that Ω is a bounded domain with smooth enough boundary $\Gamma = \partial\Omega$, then application of the Stokes formula results immediately in the following equalities:

$$\int_{\Gamma} g * \sigma^{(l)}_{\psi,x} \circ f = \int_{\Omega} \left(\mathfrak{D}^{\psi}[g] \circ f + g * \mathcal{D}_{\psi}[f]\right) dx, \tag{2}$$

$$\int_{\Gamma} g * \sigma^{(l)}_{\psi,x} * h = \int_{\Omega} \left(\mathfrak{D}^{\psi}[g] * h + g * {}^{\psi}\mathfrak{D}[h]\right) dx. \tag{3}$$

3.8 Up to now we assumed no restrictions on η. But if we want to have a good function theory, we should limit ourselves to the cases where $\eta = I_{p+1}$ or $\eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -I_p \end{smallmatrix}\right)$. Let one of these conditions be fulfilled. Denote by $\Delta_{l+1}(\widetilde{S})$ and $\Delta_{l+1}(V)$ the usual $(l+1)$–dimensional Laplace operator acting on $\mathfrak{C}^2(\Omega, \widetilde{S})$ and $\mathfrak{C}^2(\Omega, V)$, respectively.

Then

$${}^{\psi}\mathfrak{D}^{\#} * {}^{\psi}\mathfrak{D} = {}^{\psi}\mathfrak{D} * {}^{\psi}\mathfrak{D}^{\#} = \mathfrak{D}^{\psi\#} * \mathfrak{D}^{\psi} = \mathfrak{D}^{\psi} * \mathfrak{D}^{\psi\#} = \Delta_{l+1}(\widetilde{S}), \tag{4}$$

$$\mathcal{D}_{\psi} * \mathcal{D}^{\#}_{\psi} = \mathcal{D}^{\#}_{\psi} * \mathcal{D}_{\psi} = \Delta_{l+1}(V). \tag{5}$$

It is easy to describe all ψ's with the properties (4)–(5).

3.9 It is well–know that most part of the usual one–dimensional complex analysis (i.e. the theory of holomorphic functions of one complex variable) can be constructed starting from only two facts: a) factorization of the Laplace operator by the conjugate Cauchy–Riemann operators and b) the Green's (or 2–dimensional Stokes) formula. Some multidimensional generalizations, such as the quaternionic and the Clifford analysis, are based on these two facts (one can find the detailed substantiation of this point of view in [S], for example).

Formulas (2) and (4), as well as (3) and (5) show that we can develop the corresponding theories for V–valued and $\widetilde{S}$–valued functions in the same way. We will show the initial part of this procedure just to illustrate the idea.

We shall use the notations

$$\ker\ \mathcal{D}_{\psi} =: \mathfrak{N}_{\psi}(\Omega, V);\ \ker\ {}^{\psi}\mathfrak{D} =: {}^{\psi}\mathfrak{M}(\Omega, \widetilde{S});\ \ker\ \mathfrak{D}^{\psi} =: \mathfrak{M}^{\psi}(\Omega, \widetilde{S}),$$

and call the elements of these sets V–valued and $\widetilde{S}$–valued hyperholomorphic functions, respectively (in the latter case adding sometimes the word "left" of "right").

3.10 Let θ_{l+1} denote the fundamental solution of Δ_{l+1} in R^{l+1}, i.e. $\Delta_{l+1}(\theta_{l+1}) = \delta$,

$$\theta_{l+1} : x \in R^{l+1} \setminus \{0\} \longrightarrow \frac{1}{(1-l)|S^l|} \cdot |x|^{1-l},$$

where $|S^l|$ is the area of the unit sphere in R^{l+1}. We cannot identify θ_{l+1} in a natural way with a V–valued function, but we can identify it with the $\widetilde{S}$–valued function

$\theta_{l+1} \cdot \varepsilon_0$. Hence we can introduce the function

$$\mathcal{K}_\psi(x) := {}^{\psi}\mathfrak{D}^{\#}[\theta_{l+1}](x) = \mathfrak{D}^{\psi\#}[\theta_{l+1}](x) = \\ = \frac{1}{|S^l| \cdot |x|^{l+1}} \cdot \sum_{\alpha=0}^{l} \psi^{\alpha\#} \cdot x_\alpha,$$

which will play the role of the Cauchy kernel for both theories. It has the following important properties:

a) $\mathcal{K}_\psi \in \mathfrak{C}^\infty(R^{l+1} \setminus \{0\}, \widetilde{S})$

b) $\mathcal{K}_\psi \in {}^{\psi}\mathfrak{M}(R^{l+1} \setminus \{0\}, \widetilde{S}) \cap \mathfrak{M}^{\psi}(R^{l+1} \setminus \{0\}, \widetilde{S})$,

c) Let $y \in R^{l+1}$, $\mathcal{K}_{\psi,y}(x) := \mathcal{K}_\psi(y - x)$; then $\mathcal{K}_{\psi,y} \in {}^{\psi}\mathfrak{M}(R^{l+1} \setminus \{y\}) \cap \mathfrak{M}^{\psi}(R^{l+1} \setminus \{y\})$.

3.11 Theorem (Borel–Pompeiu formula). *Let $f \in \mathfrak{C}^1(\overline{\Omega}, V)$, $g \in \mathfrak{C}^1(\overline{\Omega}, \widetilde{S})$, then for $\forall x \in \Omega$,*

$$f(x) = \int_\Gamma \mathcal{K}_\psi(\tau - x) * \sigma^{(l)}_{\psi,\tau} \circ f(\tau) - \int_\Omega \mathcal{K}_\psi(\tau - x) * \mathcal{D}_\psi[f](\tau) d\tau,$$

$$g(x) = \int_\Gamma \mathcal{K}_\psi(\tau - x) * \sigma^{(l)}_{\psi,\tau} * g(\tau) - \int_\Omega \mathcal{K}_\psi(\tau - x) * {}^{\psi}\mathfrak{D}[g](\tau) d\tau = \\ = \int_\Gamma g(\tau) * \sigma^{(l)}_{\psi,\tau} * \mathcal{K}_\psi(\tau - x) - \int_\Omega \mathfrak{D}^{\psi}[g] * \mathcal{K}_\psi(\tau - x) d\tau.$$

Proof. Cut out a small ball centered in x, apply (2) and (3) to the rest of Ω; substitute $\mathcal{K}_\psi$ instead of g or f. Standard routine calculations give the answer.

3.12 Theorem (Cauchy integral theorem). *Let $f \in \mathfrak{N}_\psi(\overline{\Omega}, V)$, $g \in \mathfrak{M}^{\psi}(\overline{\Omega}, \widetilde{S})$, $h \in {}^{\psi}\mathfrak{M}(\overline{\Omega}, \widetilde{S})$, then*

$$\int_\Gamma g * \sigma^{(l)}_{\psi,\tau} \circ f = 0,$$

$$\int_\Gamma g * \sigma^{(l)}_{\psi,\tau} * h = 0.$$

Proof. Directly from (2)–(3).

3.13 Theorem (Cauchy integral formula). *Let $f \in \mathfrak{N}_\psi(\overline{\Omega}, V)$, $g \in \mathfrak{M}^{\psi}(\overline{\Omega}, \widetilde{S})$, $h \in {}^{\psi}\mathfrak{M}(\overline{\Omega}, \widetilde{S})$, then for $\forall x \in \Omega$*

$$f(x) = \int_\Gamma \mathcal{K}_\psi(\tau - x) * \sigma^{(l)}_{\psi,\tau} \circ f(\tau),$$

$$h(x) = \int_\Gamma \mathcal{K}_\psi(\tau - x) * \sigma^{(l)}_{\psi,\tau} * h(\tau),$$

$$g(x) = \int_\Gamma g(\tau) * \sigma^{(l)}_{\psi,\tau} * \mathcal{K}_\psi(\tau - x).$$

Proof. Directly from Theorem 3.11.

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