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ON ALGEBRAS OF TWO DIMENSIONAL SINGULAR INTEGRAL OPERATORS WITH HOMOGENEOUS DISCONTINUITIES IN SYMBOLS

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We describe the Fredholm symbol algebra for the C^*-algebra generated by two dimensional singular integral operators, acting on $L_2(\mathbb{R}^2)$, and whose symbols admit homogeneous discontinuities. Locally these discontinuities are modeled by homogeneous functions having slowly oscillating (and, in particular, piecewise continuous) discontinuities on a system of rays outgoing from the origin.

These results extend the well-known Plamenevsky results for the two dimensional case. We present here an alternative and much clearer approach to the problem.

1 Introduction

The aim of this paper is to describe the Fredholm symbol algebra for the C^*-algebra of two dimensional singular integral operators whose symbols admit homogeneous discontinuities. That is, we consider the C^*-algebra which is generated by the operators

$$a(x)I \qquad \text{and} \qquad b(D) = F^{-1}b(\xi)F$$

acting on the space $L_2(\mathbb{R}^2)$. Here F is the Fourier transform, $b(\xi)$ is a homogeneous function of zero order whose restriction to the unit circle $\mathbb{T}$ is continuous, and $a(x)$ belongs to an algebra of functions admitting homogeneous discontinuities (for the exact definition see the beginning of Sections 4 and 5).

The principal contribution in this direction has been made by B. A. Plamenevsky (see, for example [9], and the references there). He developed the (multidimensional) theory for the case when the points of discontinuities are modeled by homogeneous of zero order functions with continuous restrictions onto the unit sphere. Applying the Mellin transform

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with respect to the radial variable Plamenevsky reduces a pseudodifferential operator with a smooth homogeneous symbol $a(x, \xi)$ to the family of operators

$$E(\lambda)^{-1} a(t, \omega) E(\lambda), \tag{1.1}$$

acting on L_2 over the unit sphere. Here $\lambda \in \mathbb{R}$, the operators $E(\lambda)$, $E(\lambda)^{-1}$ (for the case n=2) are given by (2.5), and (2.13). Then he constructs the calculus of operators of the form (1.1).

We extend the results of Plamenevsky in the two dimensional case to a much more general situation, allowing the mentioned restrictions to be piecewise continuous, or even to have discontinuities of the second kind.

In Section 2 we give a short and self-contained description of the necessary part of the Plamenevsky theory for the two dimensional case using an alternative approach. Our main result in this section is given in Theorem 2.9, and says that for any function $b(\omega) \in C(\mathbb{T})$ the following representation

$$E(\lambda)^{-1} b(\omega) E(\lambda) = b(it) P_{\mathbb{T}}^{+} + b(-it) P_{\mathbb{T}}^{-} + K(\lambda) \tag{1.2}$$

holds in $L_2(\mathbb{T})$. Here $P_{\mathbb{T}}^{\pm} = \frac{1}{2}(I \pm S_{\mathbb{T}})$ are projections defined by the one-dimensional singular integral operator

$$(S_{\mathbb{T}} \varphi)(t) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{\varphi(\tau)\, d\tau}{\tau - t},$$

$K(\lambda)$ is a compact operator. This result connects the essential ingredients of the Plamenevsky theory with the very classical objects, clarify their structure and leads immediately to the main statements of the theory for the two dimensional case. Let us mention, in particular, the following representation of the operator (1.1)

$$E(\lambda)^{-1} a(t, \omega) E(\lambda) = a(t, it) P_{\mathbb{T}}^{+} + a(t, -it) P_{\mathbb{T}}^{-} + K(\lambda), \tag{1.3}$$

where $a(t, \omega) \in C^{\infty}(\mathbb{T} \times \mathbb{T})$ and $K(\lambda)$ is a compact operator.

To obtain our main equality (1.2) we use specific features of the two-dimensional case. The discrete Fourier transform on the unit circle converts the operators $E(\lambda)^{-1} b(\omega) E(\lambda)$ and $b(it) P_{\mathbb{T}}^{+} + b(-it) P_{\mathbb{T}}^{-}$ to some discrete convolution type operators, and then we use the limit operator technique [6, 11].

The representations (1.2) and (1.3) allow us to extend the known model case of point discontinuities to much more general settings. We do this in Section 3 considering cases when functions $a(x)$ are homogeneous of degree zero but now their restrictions onto the unit circle $\mathbb{T}$ are either piecewise continuous, or admit slowly oscillating discontinuities. These results are made easily possible due to established by us direct connections with the theory of one-dimensional singular integral operators and corresponding results in this theory (see, for example, [4] for the case of piecewise continuous coefficients, and [10] for slowly oscillating coefficients).

Sections 4 and 5 are devoted to the description of our general cases of the algebras of two dimensional singular integral operators with piecewise-continuous coefficients and with the coefficients that admit slowly oscillating discontinuities respectively. We use here the standard local principle gathering together all model local descriptions.

We deliberately omit here various extensions of the theory (weighted spaces, bounded domains, two dimensional manifolds, more general symbols, etc) wishing to present new ideas and results in a more simple context. We plan to publish these extensions elsewhere.

2 Preliminaries

Denote by $\mathbb{T}$ the unit circle in $\mathbb{R}^2$. Let $\mathcal{A}$ be a C^*-subalgebra of $L_\infty(\mathbb{T})$ containing the algebra $C(\mathbb{T})$, and let $H(\mathcal{A})$ be the algebra of homogeneous of order zero functions on $\mathbb{R}^2$ whose restrictions onto $\mathbb{T}$ belong to the algebra $\mathcal{A}$.

Denote by $\mathcal{R} = \Psi(H(\mathcal{A}), H(\mathcal{A}))$ the algebra generated by the multiplication operators

$$A = a(x)I, \quad a \in H(\mathcal{A}), \tag{2.1}$$

and by the singular integral operators

$$b(D) = F^{-1}b(\xi)F, \quad b \in H(\mathcal{A}), \tag{2.2}$$

acting on $L_2(\mathbb{R}^2)$. Here F is the Fourier transform defined on $L_2(\mathbb{R}^2)$ by

$$(Fu)(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\cdot\xi}u(x)dx,$$

and

$$(F^{-1}u)(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\cdot\xi}u(\xi)d\xi$$

is the inverse Fourier transform.

For a function $v(r) \in L_2(\mathbb{R}_+, rdr)$ its Mellin transform:

$$M : L_2(\mathbb{R}_+, rdr) \longrightarrow L_2(\mathbb{R}, d\lambda)$$

is defined as

$$(Mv)(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_+} r^{-i\lambda}v(r)dr \tag{2.3}$$

The inverse Mellin transform:

$$M^{-1} = M^* : L_2(\mathbb{R}, d\lambda) \longrightarrow L_2(\mathbb{R}_+, rdr)$$

has the form

$$(M^{-1}u)(r) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} r^{i\lambda-1}u(\lambda)d\lambda \tag{2.4}$$

Following [9], for $u(t) \in C^\infty(\mathbb{T})$, $\lambda \in \mathbb{C}$, $\lambda \neq ik$, $k = 1, 2, \ldots$, $\operatorname{Im}\lambda > 0$, we define the operator

$$(E(\lambda)u)(t) = \gamma(\lambda)\int_{\mathbb{T}}(-t\cdot\omega + i0)^{-i\lambda-1}u(\omega)d\omega \tag{2.5}$$

where

$$\gamma(\lambda) = \frac{1}{2\pi}\Gamma(1 + i\lambda)e^{\frac{\pi}{2}(i-\lambda)} \tag{2.6}$$

and the construction $(t \pm i0)^{\mu}$ for $t \in \mathbb{R}$ and $\mu \in \mathbb{C}$ is understood in the sense of distributions (see, for example, [3]):

$$(t \pm i0)^{\mu} = t_+^{\mu} + e^{\pm i\pi\mu} t_-^{\mu}, \qquad \mu \neq -1, -2, \dots$$

$$(t \pm i0)^{k} = t^{k} \pm (-1)^{k} \frac{i\pi}{(k-1)!} \delta^{(k-1)}(t), \qquad k = -1, -2, \dots$$

For $\operatorname{Im} \lambda \leq 0$ we understand the integral (2.5) in the sense of analytic continuation, since the function

$$\lambda \longrightarrow E(\lambda) u(t), \qquad u(t) \in C^{\infty}(\mathbb{T})$$

admits analytic continuation in the complex plane minus the poles

$$\lambda = ik, \quad k = 1, 2, \dots$$

of Γ-function in (2.6) (see [9]).

Throughout the paper we assume λ to be real if the contrary is not stated. Sometimes, in order to avoid a misunderstanding we will write

$$(E_{\omega \to t}(\lambda) u)(t)$$

instead of the left-hand side of (2.5).

Proposition 2.1 ([9]) *For any spherical harmonic $Y_m(t)$ we have*

$$E(\lambda) Y_m(t) = \mu(m, \lambda) Y_m(t), \;\; \lambda \neq ik, \; k = 1, 2, \dots \tag{2.7}$$

with

$$\mu(m, \lambda) = (-i)^m 2^{i\lambda} \frac{\Gamma(\frac{m+i\lambda+1}{2})}{\Gamma(\frac{m-i\lambda+1}{2})} \tag{2.8}$$

PROOF. Let us prove (2.7) for $\lambda = i\delta$, $\delta \in (0,1)$. Our case (as well as the general case of $\lambda \in \mathbb{C}$, $\lambda \neq ik$, $k = 0, 1, 2, \dots$) will follow then by analytic continuation principle.

For the mentioned λ using the Funk-Hecke formula (see [8]) for the case $n = 2$:

$$\int_{\mathbb{T}} f(t\omega) Y_m(\omega) d\omega = c_m Y_m(t),$$

where $f(x)(1-x^2)^{-\frac{1}{2}} \in L_1([-1,1])$, and

$$c_m = 2 \int_{-1}^{1} f(x)(1-x^2)^{-\frac{1}{2}} \cos(m \arccos x) dx,$$

we have

$$\int_{\mathbb{T}} (-t \cdot \omega + i0)^{-i\lambda - 1} Y_m(\omega) d\omega = 2 Y_m(t) \int_{-1}^{1} (-x + i0)^{\delta - 1} (1 - x^2)^{-\frac{1}{2}} \cos(m \arccos x) dx$$

Now to obtain (2.7) for $\lambda = i\delta$ it is sufficient to apply the formula 7.346 from [5]:

$$\int_0^1 x^{s-1}\cos(m\arccos x)(1-x^2)^{-\frac{1}{2}}dx = \frac{\pi}{s2^s}\Big[B(\frac{1+s+m}{2}, \frac{1+s-m}{2})\Big]^{-1}, \quad \mathrm{Re}\, s > 0 \quad (2.9)$$

and then to make an easy computation using the B-function properties.

Finally, note that the both the left- and right-hand sides of (2.7) are analytic functions in $\mathbb{C} \setminus \{ik : \ k = 1, 2, \ldots\}$ (for the left side see [9]), coinciding in the interval $\{i\delta : \ 0 < \delta < 1\}$.

□

The same result is valid if instead of the spherical harmonic system we consider the following orthonormal basis in $L_2(\mathbb{T})$:

$$\Big\{\frac{1}{\sqrt{2\pi}}t^m\Big\}_{m\in\mathbb{Z}}, \quad t \in \mathbb{T}. \tag{2.10}$$

Proposition 2.2 *The relation*

$$E(\lambda)t^m = \mu(m,\lambda)t^m \ \ \lambda \neq ik, \quad k = 1, 2, \ldots \tag{2.11}$$

holds with the same $\mu(m,\lambda)$ as above.

Proof. The proof is straightforward: setting $t = e^{ix}$, $\omega = e^{iy}$ we have for $\lambda = i\delta$, $\delta \in (0,1)$:

$$\gamma(\lambda)\int_{\mathbb{T}}(-t\cdot\omega + i0)^{\delta-1}u(\omega)d\omega = e^{imx}\gamma(\lambda)\int_0^{2\pi}(-\cos y + i0)^{\delta-1}e^{imy}dy$$

$$= t^m\frac{\Gamma(-\delta+1)}{2\pi}(e^{i\frac{\pi}{2}(\delta+1)} + (-1)^m e^{i\frac{\pi}{2}(-\delta+1)})\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{\delta-1} y e^{imy}dy$$

$$= t^m 2\frac{\Gamma(-\delta+1))}{2\pi}(e^{i\frac{\pi}{2}(\delta+1)} + (-1)^m e^{i\frac{\pi}{2}(-\delta+1)})\int_0^1 y^{\delta-1}(1-x^2)^{-\frac{1}{2}}\cos(m\arccos y)dy.$$

Now, using the formula (2.9) and applying the analytic continuation principle, we come to the desired conclusion. □

Corollary 2.3 *The operator $E(\lambda)$ is unitary on the space $L_2(\mathbb{T})$.*

Proof. The operator $E(\lambda)$ is diagonal in the basis (2.10), and all its eigenvalues have modulus one. □

Let $V : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R})$ be a reflection operator:

$$(Vf)(\lambda) = f(-\lambda).$$

Passing to polar coordinates in the plane, we have

$$L_2(\mathbb{R}^2) = L_2(\mathbb{R}_+, rdr) \otimes L_2(\mathbb{T}, |dt|) = L_2(\mathbb{R}_+, rdr) \otimes L_2(\mathbb{T}),$$

where $|dt| = \frac{dt}{it}$, as well as the following representation of the Fourier transform:

Proposition 2.4 ([9]) *The Fourier transform admits the following decomposition:*

$$F = (M^{-1} \otimes I)(V \otimes I)(I \otimes E(\lambda))(M \otimes I). \tag{2.12}$$

PROOF. The proof is a rather straightforward calculation. $\square$

From (2.12) taking into account the fact that $F^{-1}f(x) = Ff(-x)$ one can easily obtain that $E^{-1}_{\omega\to t}(\lambda) = E_{\omega\to -t}(-\lambda)$, thus the inverse operator $E(\lambda)^{-1}$ has the form :

$$E(\lambda)^{-1}v(\omega) = \gamma(-\lambda)\int_{\mathbb{T}}(\omega\cdot t + i0)^{i\lambda-1}v(t)dt, \quad \lambda \neq -ik, \quad k = 1, 2, \ldots \tag{2.13}$$

where the integral in (2.13) is understood in the same manner as in (2.5)

It is also obvious that the formulae

$$\begin{aligned} E(\lambda)^{-1}Y_m(t) &= (-1)^m\mu(m,-\lambda)Y_m(t), \quad \lambda \neq -ik, \quad k = 1, 2, \ldots && (2.14)\\ E(\lambda)^{-1}t^m &= (-1)^m\mu(m,-\lambda)t^m, \qquad \lambda \neq -ik, \quad k = 1, 2, \ldots && (2.15)\end{aligned}$$

for the unitary operator $E(\lambda)^{-1}$ hold.

Given $\lambda \in \mathbb{R}$, introduce the algebra $\mathfrak{S}_\lambda = \Phi_\lambda(\mathcal{A}, \mathcal{A})$ generated by the following operators acting on the space $L_2(\mathbb{T})$:

$$a(t)I \quad \text{and} \quad b_\lambda(\Omega) = E(\lambda)^{-1}b(\omega)E(\lambda), \tag{2.16}$$

with a, $b \in \mathcal{A}$.

Let $\mathfrak{S}$ be the C^*-algebra of all norm-continuous bounded operator functions $U = U(\lambda)$ defined on $\mathbb{R}$ ($\lambda \in \mathbb{R}$) and valued on the algebra $\mathfrak{S}_\lambda = \Phi_\lambda(\mathcal{A}, \mathcal{A})$, for each λ, with the norm

$$||U|| = \sup_{\lambda\in\mathbb{R}} ||U(\lambda)||.$$

Proposition 2.5 *The algebra $\mathcal{R} = \Psi(H(\mathcal{A}), H(\mathcal{A}))$ is isomorphic and isometric to a subalgebra of $\mathfrak{S}$. The isomorphic imbedding*

$$\sigma : \mathcal{R} \longrightarrow \mathfrak{S} \tag{2.17}$$

is generated by the following mapping of generators:

$$\begin{aligned} \sigma &: a(x)I \longrightarrow a(t)I, \quad \lambda \in \mathbb{R},\\ \sigma &: F^{-1}b(\xi)F \longrightarrow b_\lambda(\Omega), \quad \lambda \in \mathbb{R}.\end{aligned}$$

PROOF. Under the isometrical isomorphism

$$\mathcal{R} \longrightarrow (M \otimes I)\mathcal{R}(M^{-1} \otimes I) \subset \mathfrak{S} \tag{2.18}$$

the generators (2.1), (2.2) of the algebra $\mathcal{R}$ take the form

$$\begin{aligned} (M \otimes I)A(M^{-1} \otimes I) &= I \otimes a(t)I,\\ (M \otimes I)b(D)(M^{-1} \otimes I) &= I \otimes E(\lambda)^{-1}b(\omega)E(\lambda) = I \otimes b_\lambda(\Omega).\end{aligned}$$

To finish the proof it is sufficient to show that the operator-functions $E(\lambda)$ and $E^{-1}(\lambda)$ are norm-continuous in $\mathbb{R}$. To do this observe that

$$\|E(\lambda)-E(\lambda_0)\|^2=\|(E(\lambda)-E(\lambda_0))(E(\lambda)-E(\lambda_0))^*\|=\sup_{m\in\mathbb{Z}}|\mu(m,\lambda)-\mu(m,\lambda_0)|^2,$$

here $\lambda,\ \lambda_0\neq ik,\ k\in\mathbb{N}$. Now, using the Γ - function properties ([5], 8.328.2), it is easy to show that

$$\sup_{m\in\mathbb{Z}}|\mu(m,\lambda)-\mu(m,\lambda_0)|\longrightarrow 0,$$

when $\lambda\longrightarrow\lambda_0$, and $\lambda,\ \lambda_0\in\mathbb{R}$. □

Let now $\mathcal{A}=C(\mathbb{T})$. Given $\lambda\in\mathbb{R}$, consider the "coordinate" algebra

$$\mathfrak{S}_\lambda=\Phi_\lambda(C(\mathbb{T}),C(\mathbb{T})), \tag{2.19}$$

i.e., from now on we assume that the functions $a(t),b(\omega)$ are continuous on the circle $\mathbb{T}$ (or which is the same, that the corresponding functions $a(x),b(\xi)$ are homogeneous of order zero and continuous in $\mathbb{R}^2\setminus\{0\}$).

It will be useful to outline the connection between the algebra $\mathfrak{S}_\lambda$ of operators on $L_2(\mathbb{T})$ and the algebra generated by convolution type operators acting on the space of two-sided sequences $l_2=l_2(\mathbb{Z})$:

$$l_2=\Big\{c=\{c(n)\}_{n\in\mathbb{Z}}:\ c(n)\in\mathbb{C},\ \ \|c\|_{l_2}=\Big(\sum_{n\in\mathbb{Z}}|c(n)|^2\Big)^{\frac{1}{2}}<\infty\Big\}.$$

To do this introduce the discrete Fourier transform

$$\mathcal{F}:L_2(\mathbb{T})\longrightarrow l_2 \tag{2.20}$$

as follows:

$$\mathcal{F}:f(t)\longrightarrow c=\{c(n)\}_{n\in\mathbb{Z}},$$

where

$$c(n)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}}f(t)\bar{t}^n\,dt, \tag{2.21}$$

Then its inverse

$$\mathcal{F}^{-1}=\mathcal{F}^*:l_2\longrightarrow L_2(\mathbb{T}) \tag{2.22}$$

is given by

$$\mathcal{F}^{-1}:c=\{c(n)\}\longrightarrow f(t)=\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}c(n)t^n.$$

Due to Proposition 2.2 and (2.15) the operators

$$\widehat{E}(\lambda)=\mathcal{F}E(\lambda)\mathcal{F}^{-1}\quad\text{and}\quad\widehat{E}^{-1}(\lambda)=\mathcal{F}E^{-1}(\lambda)\mathcal{F}^{-1}$$

acting on the space l_2 are nothing but the multiplication operators on the sequences

$$\mu(\lambda)=\{\mu(m,\lambda)\}_{m\in\mathbb{Z}}\quad\text{and}\quad\mu^{-1}(\lambda)=\{(-1)^m\mu(m,-\lambda)\}_{m\in\mathbb{Z}},$$

respectively. The image $\widehat{b} = \mathcal{F}b(t)\mathcal{F}^{-1}$ of the multiplication operator $b(t)I$, $b(t) \in C(\mathbb{T})$, under the isometrical isomorphism (2.20) is obviously the following convolution operator:

$$(\widehat{b}\,c)(k) = \sum_{j\in\mathbb{Z}} \widehat{b}(j-k)c(j), \quad k \in \mathbb{Z},$$

where $\widehat{b}(k)$ are Fourier coefficients of $b(t)$ in the ortonormal basis (2.10).

Let $\mathcal{G}$ be the algebra of the operators on l_2 generated by the multiplication operators aI, where $a = \{a(k)\}_{k\in\mathbb{Z}} \in l_\infty$ and by the shift operators: $(V_h c)(k) = c(k-h)$, $h \in \mathbb{Z}$.

As it was proved in [6, 11] for an operator A from the algebra $\mathcal{G}$ and for any sequence $h = \{h_m\}$, $h_m \in \mathbb{Z}$ and $h_m \longrightarrow \infty$ there exists a subsequence h_{m_k} such that there exists a strong limit:

$$s-\lim_{k\to\infty} V^{-1}_{h_{m_k}} A V_{h_{m_k}} = A_h.$$

This limit A_h is called the *limit operator* for the operator A, defined by the sequence $h = \{h_{m_k}\}$.

Proposition 2.6 ([6, 11]) *The operator $A \in \mathcal{G}$ is compact if and only if all its limit operators A_h are equal to zero.*

Denote by $\widehat{b}_+$ and $\widehat{b}_-$ the convolution operators corresponding to the functions $b(it)$ and $b(-it)$ as above. Finally, let

$$\widehat{b}_\lambda(\Omega) = \mathcal{F}b_\lambda(\Omega)\mathcal{F}^{-1} = \widehat{E}(\lambda)^{-1}\widehat{b}\widehat{E}(\lambda), \tag{2.23}$$
$$\widehat{N} = \widehat{b}_+\chi_+(k)I + \widehat{b}_-\chi_-(k)I, \tag{2.24}$$

where $\chi_+(k)$ is a characteristic function of the set $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$, and $\chi_-(k) = 1 - \chi_+(k)$. We note, that

$$(\widehat{b}_\pm c)(k) = \sum_{j\in\mathbb{Z}} (\pm i)^{k-j}\widehat{b}(k-j)c(j). \tag{2.25}$$

Note that all the limit operators for the operators (2.23), (2.24) are obtained by taking the sequences $h_m = m$ and $h_m = -m$, $m = 0, 1, \ldots$ without passing to subsequences.

Proposition 2.7 *The sets of limit operators for the operators $\widehat{b}_\lambda(\Omega)$ and $\widehat{N}$ are the same, and are of the form $\{\widehat{b}_+, \widehat{b}_-\}$.*

Proof. Let us consider the operator $\widehat{b}_\lambda(\Omega)$ first. We obviously have

$$\begin{aligned} V_{-m}\widehat{E}(\lambda)V_m &= \{\mu(m+k,\lambda)\}_{k\in\mathbb{Z}}I, \\ V_{-m}\widehat{E}(\lambda)^{-1}V_m &= \{(-1)^{m+k}\mu(m+k,-\lambda)\}_{k\in\mathbb{Z}}I, \\ V_{-m}\widehat{b}V_m &= \widehat{b}. \end{aligned}$$

Thus,

$$V_{-m}\widehat{E}(\lambda)^{-1}\widehat{b}\widehat{E}(\lambda)V_m = \{(-1)^{m+k}\mu(m+k,-\lambda)\}_{k\in\mathbb{Z}}\widehat{b}\{\mu(m+j,\lambda)\}_{j\in\mathbb{Z}}I.$$

More precisely, for the sequence $c = \{c(k)\}_{k\in\mathbb{Z}} \in l_2$ we have

$$(V_{-m}\widehat{b}_\lambda(\Omega)V_m c)(k) = \sum_{j\in\mathbb{Z}} (-i)^{k-j}\mu(m,k,j)\widehat{b}(k-j)c(j).$$

with

$$\mu(m,k,j) = \frac{\Gamma(\frac{k+m-i\lambda+1}{2})}{\Gamma(\frac{k+m+i\lambda+1}{2})}\frac{\Gamma(\frac{j+m+i\lambda+1}{2})}{\Gamma(\frac{j+m-i\lambda+1}{2})}$$

From the above it follows that

$$s-\lim_{m\to+\infty} V_{-m}\widehat{b}_\lambda(\Omega)V_m = \widehat{b}_+, \quad s-\lim_{m\to-\infty} V_{-m}\widehat{b}_\lambda(\Omega)V_m = \widehat{b}_-. \tag{2.26}$$

Let us comment on the first relation from (2.26). We note that the operator norms are uniformly bounded:

$$\|V_{-m}\widehat{b}_\lambda(\Omega)V_m c\|_{l_2} \le C\|c\|_{l_2}, \qquad c \in l_2,$$

thus due to the Banach-Steinhaus theorem we have only to prove the convergence on the basis elements $e^k = (\ldots,0,\ldots,1,0,\ldots) \in l_2$, whose linear combinations form a dense set in l_2. Since $\mu(m,k,j) \longrightarrow 1$, when $m \longrightarrow \infty$ for all fixed k, j and $|\mu(m,k,j)-1| \le 2$, applying the Lebesgue dominated convergence theorem we have:

$$\|(V_{-m}\widehat{b}_\lambda(\Omega)V_m - \widehat{b}_+)e^k\|_{l_2}^2 = \sum_{j=-\infty}^{\infty} |(\mu(m,k,j)-1)\widehat{b}(k-j)|^2 \longrightarrow 0, \quad m\to\infty.$$

Now the second relation from (2.26) follows from the above due to the formula $\mu(m,\lambda) = \mu(-m,\lambda)$.
Turning to the operator $\widehat{N}$, we have obviously

$$\begin{aligned} V_{-m}\chi_+(k)V_m &= \chi_+^m(k)I, \\ V_{-m}\chi_-(k)V_m &= \chi_-^m(k)I, \end{aligned}$$

where by $\chi_+^m(k) = \chi_+(m+k)$, and $\chi_-^m(k) = 1-\chi_+^m(k)$. Now from the relations

$$\begin{aligned} \lim_{m\to-\infty} \|\chi_+^m(k)c\|_{l_2} &= \lim_{m\to+\infty} \|\chi_-^m(k)c\|_{l_2} = 0, \\ \lim_{m\to+\infty} \|(1-\chi_+^m(k))c\|_{l_2} &= \lim_{m\to-\infty} \|(1-\chi_-^m(k))c\|_{l_2} = 0, \end{aligned}$$

for each $c \in l_2$, we have

$$s-\lim_{m\to\pm\infty} V_{-m}NV_m = \widehat{b}_\pm,$$

which proves the proposition. □

Corollary 2.8 *The difference between the operators $\widehat{b}_\lambda(\Omega)$ and $\widehat{N}$ is compact, i.e.,*

$$\widehat{b}_\lambda(\Omega) = \widehat{N} + K(\lambda),$$

where $K(\lambda)$ is a compact operator.

PROOF. Follows from Propositions 2.6 and 2.7. $\square$

Return now to the operator $b_\lambda(\Omega) \in \mathfrak{S}_\lambda$. Let

$$(S_{\mathbb{T}}u)(t) = \frac{1}{\pi i}\int_{\mathbb{T}} \frac{u(\tau)}{\tau - t} d\tau \tag{2.27}$$

be the one-dimensional singular integral operator, and $P_{\mathbb{T}}^{\pm} = \frac{1}{2}(I \pm S_{\mathbb{T}})$ be as usual the complementary projections. Since

$$\mathcal{F}(b(it)P_{\mathbb{T}}^{+} + b(-it)P_{\mathbb{T}}^{-})\mathcal{F}^{-1} = \widehat{N},$$

we have the following important result.

Theorem 2.9 *The operator $b_\lambda(\Omega) = E^{-1}(\lambda)b(\omega)E(\lambda)$, where $b(t) \in C(\mathbb{T})$, admits the representation*

$$b_\lambda(\Omega) = b(it)P_{\mathbb{T}}^{+} + b(-it)P_{\mathbb{T}}^{-} + K(\lambda) \tag{2.28}$$

where $K(\lambda)$ is a compact operator.

Remark 2.10 Theorem 2.9 connects the algebras $\mathfrak{S}_\lambda$, $\lambda \in \mathbb{R}$, with very classical objects – algebras of one-dimensional singular integral operators, clarifies the structure of algebras $\mathfrak{S}_\lambda$, and leads immediately to the following results.

Corollary 2.11 *The algebras $\mathfrak{S}_\lambda$ are the same for each $\lambda \in \mathbb{R}$, and coincide with the algebra*

$$\mathfrak{S}_0 = \mathcal{R}(C(\mathbb{T}), S_{\mathbb{T}})$$

of one-dimensional singular integral operators with continuous coefficients.

The operators $a(t)I$ and $b_\lambda(\Omega)$, with $a,\, b \in C(\mathbb{T})$, commute modulo a compact operator.

The algebra $\mathfrak{S}_0$ contains the ideal $\mathcal{K}$ of compact in $L_2(\mathbb{T})$ operators, is irreducible, and every element has the form

$$A = c(t)P_{\mathbb{T}}^{+} + d(t)P_{\mathbb{T}}^{-} + K,$$

where $c(t),\, d(t) \in C(\mathbb{T})$, K is a compact operator.

The quotient algebra $\widehat{\mathfrak{S}}_0 = \mathfrak{S}_0/\mathcal{K}$ is isomorphic and isometric to the algebra $C(\mathbb{T}) \oplus C(\mathbb{T})$. The quotient homomorphism

$$\sigma : \mathfrak{S}_0 \longrightarrow C(\mathbb{T}) \oplus C(\mathbb{T})$$

is given by the following mapping:

$$\sigma : A = c(t)P_{\mathbb{T}}^{+} + d(t)P_{\mathbb{T}}^{-} + K \longmapsto (c(t),\, d(t)).$$

Following [9] introduce the set $V_{2,2}$ of all mutually orthogonal unit vectors of $\mathbb{T}$, and represent it as follows:

$$V_{2,2} = \{(t, \omega_\pm(t)) \in \mathbb{T} \times \mathbb{T} : \omega_+(t) = it,\ \omega_-(t) = -it\} \cong \mathbb{T} \sqcup \mathbb{T}. \tag{2.29}$$

Now, the operator $E(\lambda)^{-1}a(t,\omega)E(\lambda)$ with $a(t,\omega) \in C^\infty(\mathbb{T}\times\mathbb{T})$, studied in [9], admits the following representation

$$E(\lambda)^{-1}a(t,\omega)E(\lambda) = a(t,it)P_{\mathbb{T}}^+ + a(t,-it)P_{\mathbb{T}}^- + K(\lambda),$$

where K is compact. Thus due to Corollary 2.11 we have the following statements from [9].

Proposition 2.12 *The commutator of two operators of the form $E(\lambda)^{-1}a(t,\omega)E(\lambda)$ is compact. The operator $E(\lambda)^{-1}a(t,\omega)E(\lambda)$ is compact if and only if $a(t,\omega)|_{V_{2,2}} \equiv 0$. The following composition formula holds:*

$$E(\lambda)^{-1}a_1(t,\omega)E(\lambda)\cdot E(\lambda)^{-1}a_2(t,\omega)E(\lambda) = E(\lambda)^{-1}a_1(t,\omega)a_2(t,\omega)E(\lambda) + K(\lambda),$$

where $K(\lambda)$ is compact. The essential norm of the operator $E(\lambda)^{-1}a(t,\omega)E(\lambda)$ is given by

$$|||E(\lambda)^{-1}a(t,\omega)E(\lambda)||| = \sup_{(t,\omega)\in V_{2,2}} |a(t,\omega)|.$$

Now Proposition 2.5 for the case $\mathcal{A} = C(\mathbb{T})$ can be reformulated as follows.

Proposition 2.13 *The algebra $\mathcal{R}_0 = \Psi(H(C(\mathbb{T})), H(C(\mathbb{T})))$ is isomorphic and isometric to a subalgebra of $C_b(\mathbb{R}, \mathcal{R}(C(\mathbb{T}), S_{\mathbb{T}}))$ (bounded norm-continuous operator-functions on $\mathbb{R}$ valued in the algebra $\mathcal{R}(C(\mathbb{T}), S_{\mathbb{T}})$). The isomorphic imbedding*

$$\sigma : \mathcal{R}_0 \longrightarrow C_b(\mathbb{R},\ \mathcal{R}(C(\mathbb{T}), S_{\mathbb{T}}))$$

is generated by the following mapping of generators of the algebra $\mathcal{R}_0$:

$$\begin{array}{lll} \sigma & : & a(x)I \longmapsto a(t)I, \quad \lambda \in \mathbb{R}, \\ \sigma & : & F^{-1}b(\xi)F \longmapsto b(it)P_{\mathbb{T}}^+ + b(-it)P_{\mathbb{T}}^- + K(\lambda), \quad \lambda \in \mathbb{R}, \end{array}$$

where $K(\lambda) = E^{-1}(\lambda)b(\omega)E(\lambda) - b(it)P_{\mathbb{T}}^+ - b(-it)P_{\mathbb{T}}^-$ is a compact operator, and $a(x)$, $b(\xi) \in H(C(\mathbb{T}))$.

Corollary 2.14 *An operator A from the algebra $\mathcal{R}_0$ is invertible if and only if the one-dimensional singular integral operators $\sigma(A)(\lambda) = c(t)P_{\mathbb{T}}^+ + d(t)P_{\mathbb{T}}^- + K(\lambda)$ are invertible for all $\lambda \in \mathbb{R}$. The conditions $c(t) \neq 0$, $d(t) \neq 0$ for all $t \in \mathbb{T}$, and*

$$\{\arg \frac{d(t)}{c(t)}\}_{\mathbb{T}} = 0$$

are necessary for the invertibility of the operator A.

Example 1. Consider the two-dimensional pseudodifferential operator

$$(Au)(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)d\xi$$

with a smooth symbol $a(x,\xi) \in H(C(\mathbb{T})) \otimes H(C(\mathbb{T}))$, acting on $L_2(\mathbb{R}^2)$. Here $\widehat{u}(\xi)$ is the Fourier transform of a function u. Then the conditions $a(t,it) \neq 0$, $a(t,-it) \neq 0$ for all $t \in \mathbb{T}$, and

$$\{\arg \frac{a(t,-it)}{a(t,it)}\}_{\mathbb{T}} = 0$$

are necessary for the invertibility of the operator A.

3 Model algebras

Theorem 2.9 allows us to treat easily more complicated cases of algebras of type $\mathcal{R}$. We will start with the case of the algebra with piece-wise continuous coefficients. Next we will treat the case when the coefficients admit slowly varying discontinuities.

Let $\Lambda = \{t_1, t_2, \ldots, t_k\}$ be a finite set of different points on $\mathbb{T}$. Denote by $PC(\mathbb{T},\Lambda)$ the algebra of piecewise continuous functions on $\mathbb{T}$, continuous in $\mathbb{T}\backslash\Lambda$, and having one-sided limit values at the points of Λ. Introduce the C^*-algebra $\mathcal{R}_1 = \Psi(H(PC(\mathbb{T},\Lambda)), H(C(\mathbb{T})))$, which is generated by all operators of the form

$$a(x)I, \quad a \in H(PC(\mathbb{T},\Lambda)), \quad \text{and} \quad b(D) = F^{-1}b(\xi)F, \quad b \in H(C(\mathbb{T})),$$

acting on the space $L_2(\mathbb{R}^2)$.

Denote by $\mathfrak{S}_1 = \mathcal{R}(PC(\mathbb{T},\Lambda), S_{\mathbb{T}})$ the C^*-algebra generated by one-dimensional singular integral operators with piecewise continuous coefficients, acting on $L_2(\mathbb{T})$, i.e., generated by all operators of the form

$$c(t)P_{\mathbb{T}}^{+} + d(t)P_{\mathbb{T}}^{-}, \qquad c(t),\, d(t) \in PC(\mathbb{T},\Lambda).$$

The algebra $\mathfrak{S}_1$ contains the ideal $\mathcal{K}$ of compact operators and is irreducible. To describe the quotient algebra $\widehat{\mathfrak{S}}_1 = \mathfrak{S}_1/\mathcal{K}$ we introduce some notations. Denote by $\widehat{\mathbb{T}}_{\pm}$ two disjoint copies of the circle, cut by the points t_p, $1 \le p \le k$, of the set Λ. Under this, each point t_p of the set Λ gives rise to the points $t_p^{\pm}+0$ and $t_p^{\pm}-0$ from $\widehat{\mathbb{T}}_{\pm}$. Next, let $Y = \widehat{\mathbb{T}}_{+}\bigcup\widehat{\mathbb{T}}_{-}$, and let $\overline{X} = \bigcup_{p=1}^{k}\Delta_p$ be the disjoint union of segments $\Delta_p = [0,1]$. Denote by $\widetilde{\mathfrak{S}}_1$ the algebra of all pairs $\sigma = (\sigma_1,\sigma_2)$, where $\sigma_1 \in C(Y)$ and $\sigma_2 \in C(\overline{X}, \mathrm{Mat}_2(\mathbb{C}))$ ($\mathrm{Mat}_2(\mathbb{C})$ is the set of all 2×2 complex matrices), which satisfy the following compatibility conditions

$$\lim_{z\to 0_p}\sigma_2(z) = \begin{pmatrix} \sigma_1(t_p^{+}+0) & 0 \\ 0 & \sigma_1(t_p^{-}-0) \end{pmatrix}, \quad \lim_{z\to 1_p}\sigma_2(z) = \begin{pmatrix} \sigma_1(t_p^{+}-0) & 0 \\ 0 & \sigma_1(t_p^{-}+0) \end{pmatrix}$$

Now by [4] we have

Theorem 3.1 *The (Fredholm) symbol algebra* $\widehat{\mathfrak{S}}_1 = \mathrm{Sym}\,\mathfrak{S}_1$ *is isometric and isomorphic to the algebra* $\widetilde{\mathfrak{S}}_1$. *Under their identification the symbol homomorphism*

$$\mathrm{sym}\,:\mathfrak{S}_1 \longrightarrow \mathrm{Sym}\,\mathfrak{S}_1 = \widetilde{\mathfrak{S}}_1$$

is generated by the following mapping of generators of the algebra $\mathfrak{S}_1$*: if* $B = c(t)P_{\mathbb{T}}^{+} + d(t)P_{\mathbb{T}}^{-}$, *then*

$$\mathrm{sym}\,B = \begin{cases} c(t), & t \in \widehat{\mathbb{T}}_+, \\ d(t), & t \in \widehat{\mathbb{T}}_-, \\ \begin{pmatrix} c(t_p+0)(1-x) + c(t_p-0)x & (d(t_p+0) - d(t_p-0))\sqrt{x(1-x)} \\ (c(t_p+0) - c(t_p-0))\sqrt{x(1-x)} & d(t_p+0)x + d(t_p-0)(1-x) \end{pmatrix}, & t_p \in \Lambda, \end{cases}$$

where $x \in [0,1]$.

Any operator B *in* $\mathfrak{S}_1$ *can be uniquely represented in the form*

$$B = c(t)P_{\mathbb{T}}^{+} + d(t)P_{\mathbb{T}}^{-} + B_0 + K,$$

where K *is compact operator,* $\mathrm{sym}\,B_0|_Y \equiv 0$, $c(t) = \mathrm{sym}\,B|_{\widehat{\mathbb{T}}_+}$ *and* $d(t) = \mathrm{sym}\,B|_{\widehat{\mathbb{T}}_-}$.

Using results of [1, 15] we give the construction of the operator B_0 by the symbol $\mathrm{sym}\,B$ of the operator B. This construction allows us to recover any operator B of the algebra $\mathfrak{S}_1$ by its symbol $\mathrm{sym}\,B$, up to a compact summand.

Introduce the operator $\widetilde{M} : L_2^2(\mathbb{R}_+) \longrightarrow L_2^2(\mathbb{R})$ and its inverse $\widetilde{M}^{-1} = \widetilde{M}^* : L_2^2(\mathbb{R}) \longrightarrow L_2^2(\mathbb{R}_+)$ as follows:

$$\widetilde{M} = (M\frac{1}{\sqrt{r}}I, M\frac{1}{\sqrt{r}}I), \qquad \widetilde{M}^{-1} = (\sqrt{r}M^{-1}, \sqrt{r}M^{-1}),$$

where M and M^{-1} are the operators (2.3) and (2.4). Following [15] define the unitary operator $H_1 : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{T})$ by the rule:

$$(H_1 f)(t) = \frac{\sqrt{2}}{t+1} f\Big(i\frac{1-t}{1+t}\Big),$$

then the inverse operator $H_1^{-1} : L_2(\mathbb{T}) \longrightarrow L_2(\mathbb{R})$ is given by

$$(H_1^{-1} f)(\eta) = \frac{i\sqrt{2}}{\eta + i} f\Big(\frac{i-\eta}{i+\eta}\Big).$$

Next, let $H_2 : L_2(\mathbb{R}) \longrightarrow L_2^2(\mathbb{R}_+)$ is defined by:

$$(H_2\varphi)(\eta) = (\varphi(\eta), \varphi(-\eta)), \quad \eta \in \mathbb{R}_+,$$

then $H_2^{-1} = H_2^* : L_2^2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})$ is given by

$$(H_2^{-1}(\psi_1, \psi_2))(\eta) = \begin{cases} \psi_1(\eta), & \eta \geq 0, \\ \psi_2(-\eta), & \eta < 0. \end{cases}$$

Let also $\widetilde{H}_3 : L_2^2(\mathbb{R}) \longrightarrow L_2^2([0,1])$ is given by $\widetilde{H}_3 = (H_3, H_3)$, where

$$(H_3\varphi)(z)\frac{1}{\sqrt{2\pi}\sqrt{z(1-z)}}\varphi\Big(\frac{1}{2\pi}\ln\frac{z}{1-z}\Big),$$

then the inverse operator is $\widetilde{H}_3^{-1} = \widetilde{H}_3^{*} = (H_3^{-1}, H_3^{-1}) : L_2^2([0,1]) \longrightarrow L_2^2(\mathbb{R})$ with

$$(H_3^{-1}\psi)(\lambda) = \frac{\sqrt{2\pi}e^{\pi\lambda}}{e^{2\pi\lambda}+1}\psi\Big(\frac{e^{2\pi\lambda}}{e^{2\pi\lambda}+1}\Big).$$

Let $H_4 : L_2^2([0,1]) \longrightarrow L_2^2([0,1])$ be the operator of multiplication by the matrix:

$$\begin{pmatrix} \sqrt{z} & i\sqrt{1-z} \\ \sqrt{1-z} & -i\sqrt{z} \end{pmatrix}, \qquad z \in [0,1],$$

then the inverse operator $H_4^{-1} = H_4^{*} : L_2^2([0,1]) \longrightarrow L_2^2([0,1])$ is the operator of multiplication by the matrix

$$\begin{pmatrix} \sqrt{z} & \sqrt{1-z} \\ -i\sqrt{1-z} & i\sqrt{z} \end{pmatrix}, \quad z \in [0,1].$$

Finally, let $\widetilde{\Omega}_p$ be the rotation operator on $L_2(\mathbb{T}) : (\widetilde{\Omega}_p f)(t) = f(t_p t)$, $t_p \in \Lambda$ and χ_δ be the characteristic function of the set $\{t \in \mathbb{T} : |\arg t| \le \delta\}$ with a fixed positive δ, such that

$$\delta < \frac{1}{2}|t_n - t_m|, \quad t_n,\, t_m \in \Lambda, \quad m \neq n. \tag{3.1}$$

Define the operator $H_0 : L_2(\mathbb{T}) \longrightarrow L_2^2([0,1])$ by the rule:

$$H_0 = H_4 H_3 \widetilde{M} H_2 H_1^{-1}.$$

Now from [15] we have: if $c(t) = \operatorname{sym} B|_{\widehat{\mathbb{T}}_+}$ and $d(t) = \operatorname{sym} B|_{\widehat{\mathbb{T}}_-}$ then

$$B_0 = \sum_{p=1}^{k} \Omega_p^{-1}\chi_+ H_0^{-1}[\operatorname{sym} B - \operatorname{sym}(c(t)P_{\mathbb{T}}^+ + d(t)P_{\mathbb{T}}^-)]\Big|_{\Delta_p} H_0\chi_+\Omega_p. \tag{3.2}$$

Remark 3.2 In what follows we will need the description of one-dimensional representations of the algebra $\mathfrak{S}_1 = \mathcal{R}(PC(\mathbb{T},\Lambda), S_{\mathbb{T}})$. These representations are parameterized by points of the set $Y = \widehat{\mathbb{T}}_+ \cup \widehat{\mathbb{T}}_-$, and are as follows:

$$\pi(y) : B = c(t)P_{\mathbb{T}}^+ + d(t)P_{\mathbb{T}}^- + B_0 + K \in \mathfrak{S}_1 \longmapsto (\operatorname{sym} B)(y) = \begin{cases} c(y), & \text{if } \ y \in \widehat{\mathbb{T}}_+, \\ d(y), & \text{if } \ y \in \widehat{\mathbb{T}}_-. \end{cases} \tag{3.3}$$

Note that besides these one-dimensional representations the algebra $\mathfrak{S}_1 = \mathcal{R}(PC(\mathbb{T},\Lambda), S_{\mathbb{T}})$ has one infinite dimensional irreducible representation – the identical representation, as well as, the family of two-dimensional irreducible representations parameterized by points of $\operatorname{Int}\overline{X} = \bigcup_{p=1}^{k}(0,1)_p$:

$$\pi_{t_p} \ : \ B \in \mathfrak{S}_1 \longmapsto (\operatorname{sym} B)(t_p) = \tag{3.4}$$

$$\begin{pmatrix} c(t_p+0)(1-x) + c(t_p-0)x & (d(t_p+0) - d(t_p-0))\sqrt{x(1-x)} \\ (c(t_p+0) - c(t_p-0))\sqrt{x(1-x)} & d(t_p+0)x + d(t_p-0)(1-x) \end{pmatrix}, \ x \in (0,1)_p.$$

By Proposition 2.5 the algebra $\mathcal{R}_1 = \Psi(H(PC(\mathbb{T},\Lambda)), H(C(\mathbb{T})))$ is described in terms of operator-functions valued in "coordinate" algebras $\mathfrak{S}_\lambda = \Phi_\lambda(PC(\mathbb{T},\Lambda), C(\mathbb{T}))$, $\lambda \in \mathbb{R}$. Each such algebra $\mathfrak{S}_\lambda$ is generated either by the operators

$$a(t)I, \quad a(t) \in PC(\mathbb{T},\Lambda),$$

$$b_\lambda(\Omega) = E(\lambda)^{-1} b(\omega) E(\lambda) = b(it)P_{\mathbb{T}}^{+} + b(-it)P_{\mathbb{T}}^{-} + K(\lambda), \qquad b(\omega) \in C(\mathbb{T}),$$

or by the operators

$$E(\lambda)^{-1} a(t,\omega) E(\lambda) = a(t,it)P_{\mathbb{T}}^{+} + a(t,-it)P_{\mathbb{T}}^{-} + K(\lambda)$$

with piecewise smooth functions $a(t,\omega) \in PC(\mathbb{T},\Lambda) \otimes C(\mathbb{T})$, acting on $L_2(\mathbb{T})$. Thus Theorem 3.1 leads directly to the following corollary.

Corollary 3.3 *The algebras $\mathfrak{S}_\lambda$ are the same for each $\lambda \in \mathbb{R}$, and coincide with the algebra*

$$\mathfrak{S}_1 = \mathcal{R}(PC(\mathbb{T},\Lambda), S_{\mathbb{T}})$$

of one-dimensional singular integral operators with piecewise continuous coefficients.

The algebra $\mathfrak{S}_1$ contains the ideal $\mathcal{K}$ of compact in $L_2(\mathbb{T})$ operators, is irreducible, and its quotient algebra $\widehat{\mathfrak{S}}_1 = \mathcal{R}(PC(\mathbb{T},\Lambda), S_{\mathbb{T}})/\mathcal{K}$ is noncommutative. In particular the commutator of two operators of the form $E(\lambda)^{-1}a(t,\omega)E(\lambda)$ is not compact in general.

The quotient homomorphism

$$\mathrm{sym}\,:\mathcal{R}(PC(\mathbb{T},\Lambda), S_{\mathbb{T}}) \longrightarrow \mathrm{Sym}\,\mathfrak{S}_1$$

is generated by the following mappings of generators of the algebra $\mathfrak{S}_1$:

$$\mathrm{sym}\,: a(t) \longrightarrow \begin{cases} a(t), & t \in \widehat{\mathbb{T}}_+, \\ a(t), & t \in \widehat{\mathbb{T}}_-, \\ \begin{pmatrix} a(t_p+0)(1-x) + a(t_p-0)x & (a(t_p+0)-a(t_p-0))\sqrt{x(1-x)} \\ (a(t_p+0)-a(t_p-0))\sqrt{x(1-x)} & a(t_p+0)x + a(t_p-0)(1-x) \end{pmatrix}, \end{cases}$$

where $t_p \in \Lambda$, $x \in [0,1]$, and

$$\mathrm{sym}\,: b_\lambda(\Omega) \longrightarrow \begin{cases} b(it), & t \in \widehat{\mathbb{T}}_+, \\ b(-it), & t \in \widehat{\mathbb{T}}_-, \\ \begin{pmatrix} b(it_p) & 0 \\ 0 & b(-it_p) \end{pmatrix}, & t_p \in \Lambda, \end{cases}$$

or

$$\mathrm{sym}\,: E(\lambda)^{-1}a(t,\omega)E(\lambda) =$$

$$\begin{cases} a(t,it), & t \in \widehat{\mathbb{T}}_+, \\ a(t,-it), & t \in \widehat{\mathbb{T}}_-, \\ \begin{pmatrix} a(t_p+0,it_p)(1-x) + a(t_p-0,it_p)x & (a(t_p+0,-it_p)-a(t_p-0,-it_p))\sqrt{x(1-x)} \\ (a(t_p+0,it_p)-a(t_p-0,it_p))\sqrt{x(1-x)} & a(t_p+0,-it_p)x + a(t_p-0,-it_p)(1-x) \end{pmatrix}, \end{cases}$$

where $t_p \in \Lambda$ *and* $x \in [0,1]$.

Each operator B *in the algebra* $\mathfrak{S}_1 = \mathcal{R}(PC(\mathbb{T},\Lambda), S_{\mathbb{T}})$ *can be uniquely represented in the form*

$$B = c(t)P_{\mathbb{T}}^{+} + d(t)P_{\mathbb{T}}^{-} + B_0 + \mathcal{K},$$

where $\mathcal{K}$ *is compact operator,* $c(t) = \operatorname{sym} B|_{\widehat{\mathbb{T}}_+}$ *and* $d(t) = \operatorname{sym} B|_{\widehat{\mathbb{T}}_-}$ *and the operator* B_0 *is given by (3.2).*

An operator B *of the algebra* $\mathfrak{S}_1 = \mathcal{R}(PC(\mathbb{T},\Lambda), S_{\mathbb{T}})$ *is compact if and only if* $\operatorname{sym} B \equiv 0$. *In particular the operator* $E(\lambda)^{-1}a(t,\omega)E(\lambda)$ *is compact if and only if* $a(t,\omega)|_{V_{2,2}} \equiv 0$.

We have the following analogue of Proposition 2.13.

Theorem 3.4 *The algebra* $\mathcal{R}_1 = \Psi(H(PC(\mathbb{T},\Lambda)), H(C(\mathbb{T})))$ *is isomorphic and isometric to a subalgebra of* $C_b(\mathbb{R},\ \mathcal{R}(PC(\mathbb{T},\Lambda), S_{\mathbb{T}}))$. *The isomorphic imbedding*

$$\sigma : \mathcal{R}_1 \longrightarrow C_b(\mathbb{R},\ \mathcal{R}(PC(\mathbb{T},\Lambda), S_{\mathbb{T}}))$$

is generated by the following mapping of generators of the algebra $\mathcal{R}_1$*:*

$$\begin{aligned} \sigma &: a(x)I \longmapsto a(t)I, \quad \lambda \in \mathbb{R}, \\ \sigma &: F^{-1}b(\xi)F \longmapsto b(it)P_{\mathbb{T}}^{+} + b(-it)P_{\mathbb{T}}^{-} + K(\lambda), \quad \lambda \in \mathbb{R}, \end{aligned}$$

where $K(\lambda) = E^{-1}(\lambda)b(\omega)E(\lambda) - b(it)P_{\mathbb{T}}^{+} - b(-it)P_{\mathbb{T}}^{-}$ *is a compact operator,* $a(x) \in H(PC(\mathbb{T},\Lambda))$, *and* $b(\xi) \in H(C(\mathbb{T}))$.

Example 2. Consider the two-dimensional pseudodifferential operator

$$(Au)(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)d\xi$$

with a piece-wise smooth symbol $a(x,\xi) \in H(PC(\mathbb{T},\Lambda)) \otimes H(C(\mathbb{T}))$, acting on $L_2(\mathbb{R}^2)$. Here again $\widehat{u}(\xi)$ is the Fourier transform of a function u. Then the conditions

$$a(t,it) \neq 0, \quad t \in \widehat{\mathbb{T}}_+, \qquad a(t,-it) \neq 0, \quad t \in \widehat{\mathbb{T}}_-,$$

$$a(t_p+0, it_p)a(t_p-0, -it_p)(1-x) + a(t_p-0, it_p)a(t_p+0, -it_p)x \neq 0,$$

for each $t_p \in \Lambda$ and $x \in [0,1]$, as well as

$$\operatorname{Ind}\left(a(t,it)P_{\mathbb{T}}^{+} + a(t,-it)P_{\mathbb{T}}^{-}\right) = 0$$

are necessary for the invertibility of the operator A.

Now consider the case of slowly oscillating discontinuities in coefficients. We will say that a function $a(t) \in L_\infty(\mathbb{T})$ has a slowly oscillating discontinuity at the point t_0 if $a(t)$ is a differentiable function in a neighborhood $U_{t_0}^{\delta} = \{t \in \mathbb{T} : 0 < |t-t_0| < \delta\}$ and

$$\lim_{t\to t_0,\ t\in\mathbb{T}} |t-t_0|a'(t) = 0$$

Denote by $SO(\mathbb{T},\Lambda)$ the C^*-algebra of continuous in $\mathbb{T}\setminus\Lambda$ (Λ is defined as above) functions with slowly oscillating discontinuities at the points $t_p\in\Lambda$, and with respect to the $L_\infty(\mathbb{T})$-norm. Note, that $PC(\mathbb{T},\Lambda)\subset SO(\mathbb{T},\Lambda)$.

Introduce the C^*-algebra $\mathcal{R}_2=\Psi(H(SO(\mathbb{T},\Lambda)),H(C(\mathbb{T})))$ which is generated by all operators of the form

$$a(x)I,\quad a\in H(SO(\mathbb{T},\Lambda))\quad \text{and}\quad b(D)=F^{-1}b(\xi)F,\quad b\in H(C(\mathbb{T})),$$

acting on $L_2(\mathbb{R}^2)$. And denote by $\mathfrak{S}_2=\mathcal{R}(SO(\mathbb{T},\Lambda),S_{\mathbb{T}})$ the C^*-algebra generated by one-dimensional singular integral operator with coefficients from $SO(\mathbb{T},\Lambda)$, that is by all the operators of the form:

$$c(t)P^+_{\mathbb{T}}+d(t)P^-_{\mathbb{T}},$$

where $c(t),\ d(t)\in SO(\mathbb{T},\Lambda)$.

The algebra $\mathfrak{S}_2$ contains the ideal $\mathcal{K}$ of compact operators, and is irreducible. To describe the (Fredholm) symbol algebra $\operatorname{Sym}\mathfrak{S}_2=\widehat{\mathfrak{S}}_2=\mathfrak{S}_2/\mathcal{K}$ of the algebra $\mathfrak{S}_2$ we introduce some notations. Let $\widetilde{\mathbb{T}}_\pm=\operatorname{Int}\widehat{\mathbb{T}}_\pm=\widehat{\mathbb{T}}_\pm\setminus(\{t_1^\pm+0,\dots,t_k^\pm+0\}\cup\{t_1^\pm-0,\dots,t_k^\pm-0\})$, $\overline{X}=\bigcup_{p=1}^k\Delta_p$ as above, and $\widetilde{Y}=\widetilde{\mathbb{T}}_+\cup\widetilde{\mathbb{T}}_-$. Let $U_p^+,\ U_p^-$ be a "left" $(\arg t_p\le\arg t<\arg t_p+\delta)$ and a "right" $(\arg t_p-\delta<\arg t\le\arg t_p)$ neighborhood of the point t_p, respectively, and for $U=\{t\in\mathbb{T}:0\le\arg t<\delta\}$ let $J(U)$ be the maximal ideal of continuous in U functions $a(t)\in SO(U,1)$, such that $a(1)=0$ (the definition of $SO(U,1)$ is quite obvious). Denote by $a_p^+(t)$ and $a_p^-(t)$ the restrictions of $a(t)$ on U_p^+ and U_p^-, respectively. Fix now δ according to (3.1). To each point $t_p\in\Lambda$ and to each $a(t)\in SO(\mathbb{T},\Lambda)$ assign the cosets $\widehat{a}_p^\pm\in SO(U,1)/J(U)$ as follows. Assuming that $t\in U$, we set

$$\widehat{a}_p^+=a_p^+(t_pt)+J(U),\quad \widehat{a}_p^-=a_p^-(t_p\overline{t})+J(U).$$

Denote by $\widetilde{\mathfrak{S}}_2$ the algebra of all pairs $\sigma=(\sigma_1,\sigma_2)$, where

$$\sigma_1\in C(\widetilde{Y})\quad\text{and}\quad \sigma_2\in C\Big(\overline{X},\operatorname{Mat}_2(SO(U,1)/J(U))\Big),$$

which satisfy the following compatibility conditions:

$$\lim_{z\to 0_p}\sigma_2(z)=\begin{pmatrix}\widehat{(\sigma_1|_{\widetilde{\mathbb{T}}_+})}^+_p & 0\\ 0 & \widehat{(\sigma_1|_{\widetilde{\mathbb{T}}_-})}^-_p\end{pmatrix},\quad \lim_{z\to 1_p}\sigma_2(z)=\begin{pmatrix}\widehat{(\sigma_1|_{\widetilde{\mathbb{T}}_+})}^-_p & 0\\ 0 & \widehat{(\sigma_1|_{\widetilde{\mathbb{T}}_-})}^+_p\end{pmatrix}$$

Theorem 3.5 *The (Fredholm) symbol algebra* $\operatorname{Sym}\mathfrak{S}_2=\widehat{\mathfrak{S}}_2$ *is isometric and isomorphic to the algebra* $\widetilde{\mathfrak{S}}_2$. *Under their identification the symbol homomorphism*

$$\operatorname{sym}\,:\mathfrak{S}_2\longrightarrow\operatorname{Sym}\mathfrak{S}_2=\widetilde{\mathfrak{S}}_2$$

is generated by the following mapping of generators of the algebra $\mathfrak{S}_2$: *if* $B=c(t)P^+_{\mathbb{T}}+d(t)P^-_{\mathbb{T}}$, *then*

$$\operatorname{sym}B=\begin{cases}c(t), & t\in\widetilde{\mathbb{T}}_+,\\ d(t), & t\in\widetilde{\mathbb{T}}_-,\\ \begin{pmatrix}\widehat{c}_p^+(1-x)+\widehat{c}_p^-x & (\widehat{d}_p^+-\widehat{d}_p^-)\sqrt{x(1-x)}\\ (\widehat{c}_p^+-\widehat{c}_p^-)\sqrt{x(1-x)} & \widehat{d}_p^+x+\widehat{d}_p^-(1-x)\end{pmatrix}, & p=1,\dots,k,\quad x\in[0,1].\end{cases}$$

PROOF. The description of the symbol algebra $\operatorname{Sym}\mathfrak{S}_2 = \widehat{\mathfrak{S}}_2$ of the algebra $\mathfrak{S}_2$ is based on the standard local principle ([2, 14]). Let $\mathcal{Z} = \{aI,\ a(t) \in C(\mathbb{T})\}$, the algebra $\widehat{\mathcal{Z}} = \mathcal{Z}/\mathcal{K} \cong \mathcal{Z}$ is a central commutative subalgebra of $\widehat{\mathfrak{S}}_2 = \mathfrak{S}_2/\mathcal{K}$. For each $t_0 \in \mathbb{T}$ denote by J_{t_0} the maximal ideal of the algebra $\widehat{\mathcal{Z}}$, which corresponds to the point t_0, and denote by $J(t_0)$ the closed two-sided ideal of $\widehat{\mathfrak{S}}_2$ generated by J_{t_0}:

$$J(t_0) = J_{t_0}\widehat{\mathfrak{S}}_2.$$

Then the algebra $\mathfrak{S}_2(t_0) = \widehat{\mathfrak{S}}_2/J(t_0)$ is called the local algebra corresponding to the point $t_0 \in \mathbb{T}$, and the natural projection

$$\pi_{t_0} : \mathfrak{S}_2 \longrightarrow \widehat{\mathfrak{S}}_2 \longrightarrow \mathfrak{S}_2(t_0) \tag{3.5}$$

identifies the locally equivalent in the point t_0 elements of the algebra $\mathfrak{S}_2$.

For the points $t_0 \in \mathbb{T} \setminus \Lambda$ of continuity of coefficients the description of the local algebra $\mathfrak{S}_2(t_0)$ is the same as in case of the algebra $\mathfrak{S}_1$ of singular integral operators with piece wise continuous coefficients. That is, in this case $\mathfrak{S}_2(t) \cong \mathfrak{S}_1(t) \cong \mathbb{C}^2$, and the homomorphism (3.5) is generated by the following mapping

$$\pi_{t_0} : B = c(t)P_{\mathbb{T}}^+ + d(t)P_{\mathbb{T}}^- \longmapsto (c(t_0), d(t_0)).$$

Let now $t_0 = t_p \in \Lambda$. Then by [10] the local algebra $\mathfrak{S}_2(t_p)$ is isometric and isomorphic to the subalgebra of the algebra $C\Big(\mathbb{R}, \mathrm{Mat}_2(SO(U,1)/J(U))\Big)$, and the homomorphism (3.5) is generated by the following mapping

$$\begin{aligned}
\pi_{t_0} &: S_{\mathbb{T}} \longmapsto \begin{pmatrix} -\tanh\pi\lambda & (\cosh\pi\lambda)^{-1} \\ (\cosh\pi\lambda)^{-1} & \tanh\pi\lambda \end{pmatrix}, \qquad \lambda \in \mathbb{R}, \\
\pi_{t_0} &: a(t)I \longmapsto \begin{pmatrix} \widehat{a}_p^+ & 0 \\ 0 & \widehat{a}_p^- \end{pmatrix}, \qquad \lambda \in \mathbb{R},
\end{aligned}$$

where $a(t) \in SO(\mathbb{T}, \Lambda)$. Following [12] introduce the isomorphism $U : L_2([0,1]) \longrightarrow L_2(\mathbb{R})$ by the rule

$$U\varphi(\lambda) = \frac{\sqrt{2\pi}}{e^{-\pi\lambda} + e^{\pi\lambda}}\varphi\Big(\frac{e^{\pi\lambda}}{e^{-\pi\lambda} + e^{\pi\lambda}}\Big).$$

Then

$$U^{-1}(-\tanh\pi\lambda)U = (1-2x)I, \quad U^{-1}(\cosh\pi\lambda)^{-1}U = 2\sqrt{x(1-x)}I,$$

and

$$U^{-1}C\Big(\mathbb{R}, \mathrm{Mat}_2(SO(U,1)/J(U))\Big)U = C\Big([0,1], \mathrm{Mat}_2(SO(U,1)/J(U))\Big).$$

We have now

$$\begin{aligned}
P_{\mathbb{T}}^+ = \frac{1}{2}(I + S_{\mathbb{T}}) &\longmapsto \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix}, \qquad x \in [0,1], \\
a(t)I &\longmapsto \begin{pmatrix} \widehat{a}_p^+ & 0 \\ 0 & \widehat{a}_p^- \end{pmatrix}.
\end{aligned}$$

Introduce the unitary matrix

$$M = \begin{pmatrix} \sqrt{1-x} & \sqrt{x} \\ -\sqrt{x} & \sqrt{1-x} \end{pmatrix},$$

and consider the isometrical isomorphism:

$$M^{-1}C\Big([0,1], \mathrm{Mat}_2(SO(U,1)/J(U))\Big)M = C\Big([0,1], \mathrm{Mat}_2(SO(U,1)/J(U))\Big).$$

Now the homomorphism

$$\pi_{t_0} : \mathfrak{S}_2 \longrightarrow \widehat{\mathfrak{S}}_2 \longrightarrow \mathfrak{S}_2(t_0) \subset C\Big([0,1], \mathrm{Mat}_2(SO(U,1)/J(U))\Big)$$

is generated by the following mapping

$$\pi_{t_p} : B = c(t)P_{\mathbb{T}}^{+} + d(t)P_{\mathbb{T}}^{-} \longmapsto \begin{pmatrix} \widehat{c}_p^{+}(1-x) + \widehat{c}_p^{-}x & (\widehat{d}_p^{+} - \widehat{d}_p^{-})\sqrt{x(1-x)} \\ (\widehat{c}_p^{+} - \widehat{c}_p^{-})\sqrt{x(1-x)} & \widehat{d}_p^{+}x + \widehat{d}_p^{-}(1-x) \end{pmatrix}, \quad x \in [0,1].$$

To finish the proof one needs to paste together all the local descriptions. $\square$

The algebra $\mathcal{R}_2 = \Psi(H(SO(\mathbb{T},\Lambda)), H(C(\mathbb{T})))$ is described in terms of operator-functions valued in "coordinate" algebras $\mathfrak{S}_\lambda = \Phi_\lambda(SO(\mathbb{T},\Lambda), C(\mathbb{T}))$, $\lambda \in \mathbb{R}$, which is generated by the operators

$$a(t)I, \qquad a(t) \in SO(\mathbb{T},\Lambda),$$

and

$$b_\lambda(\Omega) = E(\lambda)^{-1}b(\omega)E(\lambda) = b(it)P_{\mathbb{T}}^{+} + b(-it)P_{\mathbb{T}}^{-} + K(\lambda), \qquad b(\omega) \in C(\mathbb{T}),$$

or by the operators

$$E(\lambda)^{-1}a(t,\omega)E(\lambda) = a(t,it)P_{\mathbb{T}}^{+} + a(t,-it)P_{\mathbb{T}}^{-} + K(\lambda),$$

with $a(t,\omega) \in SO(\mathbb{T},\Lambda) \otimes C(\mathbb{T})$, acting on $L_2(\mathbb{T})$. As above, we have

Corollary 3.6 *The algebras $\mathfrak{S}_\lambda$ are the same for each $\lambda \in \mathbb{R}$, and coincide with the algebra*

$$\mathfrak{S}_2 = \mathcal{R}(SO(\mathbb{T},\Lambda), S_{\mathbb{T}})$$

of one-dimensional singular integral operators with slowly oscillating coefficients. The algebra $\mathfrak{S}_2$ contains the ideal $\mathcal{K}$ of compact on $L_2(\mathbb{T})$ operators, is irreducible, and its quotient algebra $\widehat{\mathfrak{S}}_2 = \mathcal{R}(SO(\mathbb{T},\Lambda), S_{\mathbb{T}})/\mathcal{K}$ is noncommutative.

The quotient homomorphism

$$\mathrm{sym} : \mathcal{R}(SO(\mathbb{T},\Lambda), S_{\mathbb{T}}) \longrightarrow \mathrm{Sym}\,\mathfrak{S}_2$$

is generated by the following mappings of generators of the algebra $\mathfrak{S}_2$:

$$\mathrm{sym} : a(t) \longrightarrow \begin{cases} a(t), & t \in \widetilde{\mathbb{T}}_{+}, \\ a(t), & t \in \widetilde{\mathbb{T}}_{-}, \\ \begin{pmatrix} \widehat{a}_p^{+}(1-x) + \widehat{a}_p^{-}x & (\widehat{a}_p^{+} - \widehat{a}_p^{-})\sqrt{x(1-x)} \\ (\widehat{a}_p^{+} - \widehat{a}_p^{-})\sqrt{x(1-x)} & \widehat{a}_p^{+}x + \widehat{a}_p^{-}(1-x) \end{pmatrix}, \end{cases}$$

where $t_p \in \Lambda$, $x \in [0,1]$, *and*

$$\mathrm{sym}\ : b_\lambda(\Omega) \longrightarrow \begin{cases} b(it), & t \in \widetilde{\mathbb{T}}_+, \\ b(-it), & t \in \widetilde{\mathbb{T}}_-, \\ \begin{pmatrix} b(it_p) & 0 \\ 0 & b(-it_p) \end{pmatrix}, & t_p \in \Lambda, \end{cases}$$

or

$$\mathrm{sym}\ : E(\lambda)^{-1} a(t,\omega) E(\lambda) =$$

$$\begin{cases} a(t,it), \quad t \in \widetilde{\mathbb{T}}_+, \\ a(t,-it), \quad t \in \widetilde{\mathbb{T}}_-, \\ \begin{pmatrix} \widehat{a}_p^+(+)(1-x) + \widehat{a}_p^-(+)x & (\widehat{a}_p^+(-) - \widehat{a}_p^-(-))\sqrt{x(1-x)} \\ (\widehat{a}_p^+(+) - \widehat{a}_p^-(+))\sqrt{x(1-x)} & \widehat{a}_p^+(-)x + \widehat{a}_p^-(-)(1-x) \end{pmatrix}, \end{cases}$$

where $t_p \in \Lambda$ *and* $x \in [0,1]$. *Here, as above,* $\widehat{a}_p^+ = a_p^+(t_p t) + J(U)$, $\widehat{a}_p^- = a_p^-(t_p \overline{t}) + J(U)$ *and* $\widehat{a}_p^+(\pm) = a_p^+(t_p t, \pm it_p) + J(U)$, $\widehat{a}_p^-(\pm) = a_p^-(t_p \overline{t}, \pm it_p) + J(U)$, $\ t \in U$.

An operator B *of the algebra* $\mathfrak{S}_2 = \mathcal{R}(SO(\mathbb{T},\Lambda), S_{\mathbb{T}})$ *is compact if and only if* $\mathrm{sym}\, B \equiv 0$. *In particular the operator* $E(\lambda)^{-1} a(t,\omega) E(\lambda)$ *is compact if and only if* $a(t,\omega)|_{V_{2,2}} \equiv 0$.

Theorem 3.7 *The algebra* $\mathcal{R}_2 = \Psi(H(SO(\mathbb{T},\Lambda)), H(C(\mathbb{T})))$ *is isomorphic and isometric to a subalgebra of* $C_b(\mathbb{R},\ \mathcal{R}(SO(\mathbb{T},\Lambda), S_{\mathbb{T}}))$. *The isomorphic imbedding*

$$\sigma : \mathcal{R}_2 \longrightarrow C_b(\mathbb{R},\ \mathcal{R}(SO(\mathbb{T},\Lambda), S_{\mathbb{T}}))$$

is generated by the following mapping of generators of the algebra $\mathcal{R}_2$*:*

$$\begin{aligned} \sigma \ &: \ a(x)I \longmapsto a(t)I, \quad \lambda \in \mathbb{R}, \\ \sigma \ &: \ F^{-1} b(\xi) F \longmapsto b(it)P_{\mathbb{T}}^+ + b(-it)P_{\mathbb{T}}^- + K(\lambda), \quad \lambda \in \mathbb{R}, \end{aligned}$$

where $K(\lambda) = E^{-1}(\lambda) b(\omega) E(\lambda) - b(it)P_{\mathbb{T}}^+ - b(-it)P_{\mathbb{T}}^-$ *is a compact operator,* $a(x) \in H(SO(\mathbb{T},\Lambda))$, *and* $b(\xi) \in H(C(\mathbb{T}))$.

Example 3. Consider the two-dimensional pseudodifferential operator

$$(Au)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix\cdot\xi} a(x,\xi) \widehat{u}(\xi) d\xi$$

with a slowly oscillating symbol $a(x,\xi) \in H(SO(\mathbb{T},\Lambda)) \otimes H(C(\mathbb{T}))$, acting on $L_2(\mathbb{R}^2)$. Then the conditions

$$a(t,it) \neq 0, \quad t \in \widetilde{\mathbb{T}}_+, \qquad a(t,-it) \neq 0, \quad t \in \widetilde{\mathbb{T}}_-, \tag{3.6}$$

$$\lim_{\varepsilon\to 0} \inf_{0<\arg t<\varepsilon} \left| a_p^+(t_p t, it_p) a_p^-(t_p \overline{t}, -it_p)(1-x) + a_p^-(t_p \overline{t}, it_p) a_p^+(t_p t, -it_p) x \right| > 0 \tag{3.7}$$

for each $p = 1\ldots, k$ and $x \in [0,1]$, as well as

$$\mathrm{Ind}\,(a(t,it)P_{\mathbb{T}}^+ + a(t,-it)P_{\mathbb{T}}^-) = 0$$

are necessary for the invertibility of the operator A.

Remark 3.8 If both $c(t)$ and $d(t)$, or one of them have the same oscillation on the right and left hand side side of t_p (which means, that, for example, for $c(t)$ we have: $\widehat{c}_p^+ = \widehat{c}_p^-$), then the above statements can be simplified. If, for example, $\widehat{c}_p^+ = \widehat{c}_p^- = \widehat{c}_p$ and $\widehat{d}_p^+ = \widehat{d}_p^- = \widehat{d}_p$ for each $1 \le p \le k$, then the conditions (3.6), (3.7) take the form: for each $p = 1, \dots, k$

$$\inf_{t \in \widetilde{\mathbb{T}}_+} |a(t, it_p)| > 0 \qquad \text{and} \qquad \inf_{t \in \widetilde{\mathbb{T}}_-} |a(t, -it_p)| > 0.$$

4 Description of the algebra $\mathcal{R} = \Psi(PC(\dot{\mathbb{R}}^2, \mathcal{L}), H(C(\mathbb{T})))$

Let $\dot{\mathbb{R}}^2$ be, as usual, the one-point compactification of $\mathbb{R}^2$ by the point at ∞, and let $\mathcal{L} = \bigcup_{p=1}^k \mathcal{L}_p$ be the union of open line segments $\mathcal{L}_p = \{x \in \mathbb{R}^2 : \arg x = \alpha_k,\ 0 < |x| < r_k\}$, outgoing from the origin. Denote by $PC(\dot{\mathbb{R}}^2, \mathcal{L})$ the C^*-algebra of functions $a(x)$, satisfying the following conditions:

i) The functions $a(x)$ are continuous in $\dot{\mathbb{R}}^2 \setminus (\mathcal{L} \cup \{0\})$.

ii) The functions $a(x)$ have finite one-sided limit values at each point $x \in \mathcal{L}_p$, $p = 1, ..., k$, which we will denote by $a_+(x)$ and $a_-(x)$, with respect to the (outgoing) orientation of segments $\mathcal{L}_p$.

iii) For each $a(x) \in PC(\dot{\mathbb{R}}^2, \mathcal{L})$ there exists a homogeneous function of order zero $h_a(x)$, such that the difference $a(x) - h_a(x)$ has the limit zero at the origin.

Denote by $t_1, t_2, \dots, t_k$ the points of intersection with the unit circle of the rays containing the segments $\mathcal{L}_p$, and let $\Lambda = \{t_1, t_2, \dots, t_k\}$. It is evident that the above mentioned function $h_a(x)$ is unique, and its restriction to the unit circle belongs to $PC(\mathbb{T}, \Lambda)$.

Finally introduce the C^*-algebra $\mathcal{R} = \Psi(PC(\dot{\mathbb{R}}^2, \mathcal{L}), H(C(\mathbb{T})))$ generated by the operators

$$A = a(x)I, \quad a(x) \in PC(\dot{\mathbb{R}}^2, \mathcal{L}), \tag{4.1}$$

$$b(D) = F^{-1}b(\xi)F, \quad b \in H(C(\mathbb{T})), \tag{4.2}$$

acting on $L_2(\mathbb{R}^n)$. To describe the (Fredholm) symbol algebra $\operatorname{Sym}\mathcal{R} = \widehat{\mathcal{R}} = \mathcal{R}/\mathcal{K}$ of the algebra $\mathcal{R}$ we will use the standard local principle ([2, 14]). Let $\mathcal{Z} = \{aI,\ a(x) \in C(\dot{\mathbb{R}}^2)\}$. Each operator $aI \in \mathcal{Z}$ commutes with operators of the form (4.1), and commutes with operators of the form (4.2) up to a compact summand [7]. Thus the algebra $\widehat{\mathcal{Z}} = \mathcal{Z}/\mathcal{K} \cong \mathcal{Z}$ is a central commutative subalgebra of $\widehat{\mathcal{R}} = \mathcal{R}/\mathcal{K}$. For each $x_0 \in \dot{\mathbb{R}}^2$ denote by J_{x_0} the maximal ideal of the algebra $\widehat{\mathcal{Z}}$ corresponding to the point x_0, and denote by $J(x_0)$ the two-sided closed ideal of $\widehat{\mathcal{R}}$, generated by J_{x_0}:

$$J(x_0) = J_{x_0}\widehat{\mathcal{R}}.$$

Then the algebra $\mathcal{R}(x_0) = \widehat{\mathcal{R}}/J(x_0)$ is the local algebra at the point $x_0 \in \dot{\mathbb{R}}^2$, and the natural projection

$$\pi_{x_0} : \mathcal{R} \longrightarrow \widehat{\mathcal{R}} \longrightarrow \mathcal{R}(x_0) \tag{4.3}$$

identifies the locally equivalent at the point x_0 elements of the algebra $\mathcal{R}$.

We have three different types of local algebras, which correspond to points of continuity, to points of discontinuity different from the origin, and to the origin.

Consider first points of continuity; i.e., let $x_0 \in \dot{\mathbb{R}}^2 \setminus \mathcal{L}$, $x_0 \neq 0$.

Proposition 4.1 *Let $x_0 \in \dot{\mathbb{R}}^2 \setminus \mathcal{L}$, $x_0 \neq 0$. Then the local algebra $\mathcal{R}(x_0)$ is isometric and isomorphic to the algebra $C(\mathbb{T})$. The homeomorphism*

$$\pi_{x_0} : \mathcal{R} = \Psi(PC(\dot{\mathbb{R}}^2, \mathcal{L}), H(C(\mathbb{T}))) \longrightarrow \mathcal{R}(x_0) \cong C(\mathbb{T})$$

is generated by the following mappings of generators of the algebra $\mathcal{R}$:

$$\begin{aligned} \pi_{x_0} \quad &: \quad aI \longrightarrow a(x_0), \\ \pi_{x_0} \quad &: \quad b(D) \longrightarrow b(\omega), \quad \omega \in \mathbb{T}. \end{aligned}$$

Proof. Since a function $a(x)$ is continuous at the point x_0, the operator aI is locally equivalent at this point to the scalar operator $a(x_0)I$. Each operator (4.2) is locally equivalent at any point x_0 to itself [13]. Thus the local algebra $\mathcal{R}(x_0)$ is generated by constants and the elements of the form $\pi_{x_0}(b(D))$, and that is why is isomorphic to the algebra of operators (4.2). The last algebra is isometrically isomorphic to the algebra $C(\mathbb{T})$. Under these arguments, the homomorphism π_{x_0} maps the operator $a(x)b(D)$ to the function $a(x_0)b(\omega) \in C(\mathbb{T})$. □

Let now $x_0 \in \mathcal{L}$. Introduce the algebra $C(\{-1,1\}, \mathcal{R}(C(\overline{\mathbb{R}}), S_{\mathbb{R}}))$ of all operator-functions defined on $\{-1,1\}$ and valued in the algebra $\mathcal{R}(C(\overline{\mathbb{R}}), S_{\mathbb{R}})$, which is generated by the one-dimensional singular integral operator $S_{\mathbb{R}} = F^{-1}\text{sign}\xi F$ and by the continuous on $\overline{\mathbb{R}} = [-\infty, +\infty]$ functions.

Fix now k counter-clockwise rotations in $\mathbb{R}^2$ with respect to the origin: $x \in \mathbb{R}^2 \longrightarrow \omega_p(x)$, $p = 1, \dots, k$, defined by the rule: if $x_0 \in \mathcal{L}_p$ then $\omega_p(x_0) = (0, |x_0|)$. Denote by

$$\Omega_p f(x) = f(\omega_p^{-1}(x)) \tag{4.4}$$

corresponding rotation unitary operators on $L_2(\mathbb{R}^2)$. And let $b_p(\xi) = \Omega_p b(\xi) = b(\omega_p^{-1}(\xi))$.

Proposition 4.2 ([15]) *Let $x_0 \in \mathcal{L}$. Then the local algebra $\mathcal{R}(x_0)$ is isometric and isomorphic to a subalgebra of the algebra $C(\{-1,1\}, \mathcal{R}(C(\overline{\mathbb{R}}), S_{\mathbb{R}}))$. The homomorphism*

$$\pi_{x_0} : \mathcal{R} = \Psi(PC(\dot{\mathbb{R}}^2, \mathcal{L}), H(C(\mathbb{T}))) \longrightarrow \mathcal{R}(x_0) \subset C(\{-1,1\}, \mathcal{R}(C(\overline{\mathbb{R}}), S_{\mathbb{R}}))$$

is generated by the following mappings of generators of the algebra $\mathcal{R}$:

$$\begin{aligned} \pi_{x_0} \quad &: \quad aI \longrightarrow a_+(x_0)P_{\mathbb{R}}^+ + a_-(x_0)P_{\mathbb{R}}^-, \qquad P_{\mathbb{R}}^{\pm} = \frac{1}{2}(I \pm S_{\mathbb{R}}), \\ \pi_{x_0} \quad &: \quad b(D) \longrightarrow b_p\Big(\frac{\xi_1}{\sqrt{1+\xi_1^2}}, \frac{\gamma}{\sqrt{1+\xi_1^2}}\Big)I, \qquad \gamma = \text{sign}\xi_2 \in \{-1,1\}, \quad x_0 \in \mathcal{L}_p. \end{aligned} \tag{4.5}$$

Sketch of the proof. At the point $x_0 \in \mathcal{L}_p$ the operator aI is locally equivalent to the operator $a_+(x_0)\mathcal{X}_+I + a_-(x_0)\mathcal{X}_-I$, where $\mathcal{X}_+$ is the characteristic function of the left-hand-side half-plane determined by the (outgoing) orientation of $\mathcal{L}_p$. As above, the operator $b(D)$ is locally equivalent to itself. Then the local algebra $\mathcal{R}(x_0)$ is isomorphic and isometric to the operator algebra $\mathcal{R}(\mathcal{X}_+I, b(D))$, which is generated by multiplication on $\mathcal{X}_+$ operator, and all operators of the form $b(D)$, with $b \in H(C(\mathbb{T}))$. Under the corresponding rotation in $\mathbb{R}^2$ we have the following isometrical isomorphism:

$$\Omega_p \mathcal{R}(\mathcal{X}_+ I, b(D))\Omega_p^{-1} = \mathcal{R}(\frac{1}{2}(1 - \mathrm{sign} x_1) I \otimes I, b_p(D)) \cong \mathcal{R}(\mathrm{sign} x_1 I \otimes I, b_p(D)),$$

where $b_p(D) = F^{-1} b(\omega_p^{-1}(\xi))F$. By [15] the algebra $\mathcal{R}(\mathrm{sign} x_1 I \otimes I, b_p(D))$ is isomorphic and isometric to a subalgebra of the algebra $C(\{-1, 1\}, \mathcal{R}(C(\overline{\mathbb{R}}), S_{\mathbb{R}}))$. To show this introduce the unitary operator

$$U = VF : L_2(\mathbb{R}^2) \longrightarrow L_2(\mathbb{R}^2)$$

where F is the Fourier transform on $L_2(\mathbb{R}^2)$ and V is the unitary operator on $L_2(\mathbb{R}^2)$ defined by the rule:

$$(Vf)(x_1, x_2) = |x_2|^{\frac{1}{2}} f(|x_2|x_1, x_2).$$

Now, under the isometrical imbedding

$$U\mathcal{R}(\mathrm{sign} x_1 I \otimes I, b_{x_0}(D))U^{-1} \longrightarrow C(\{-1, 1\}, \mathcal{R}(C(\overline{\mathbb{R}}), S_{\mathbb{R}}))$$

the generators take the form

$$U(\mathrm{sign} x_1 I \otimes I)U^{-1} = -S_{\mathbb{R}} \otimes I, \qquad \text{where} \quad S_{\mathbb{R}} = F^{-1} \mathrm{sign} x_1 F,$$

and

$$U b_p(D) U^{-1} = b_p\Big(\frac{\xi_1}{\sqrt{1+\xi_1^2}}, \frac{\mathrm{sign}\xi_2}{\sqrt{1+\xi_1^2}}\Big) I.$$

□

Remark 4.3 The coefficients of the algebra $\mathcal{R}(C(\overline{\mathbb{R}}), S_{\mathbb{R}}))$ have only one point of discontinuity – the infinity, thus an arbitrary element of the algebra $C(\{-1, 1\}, \mathcal{R}(C(\overline{\mathbb{R}}), S_{\mathbb{R}}))$ has (see, Theorem 3.1) a form $(B(+),\ B(-))$, where

$$B(\pm) = c_\pm(\xi_1)P_{\mathbb{R}}^+ + d_\pm(\xi_1)P_{\mathbb{R}}^- + B_0^\pm + K_\pm.$$

The generators (4.5) of the local algebra $\mathcal{R}(x_0)$ have a special feature:

$$\lim_{\xi_1 \to +\infty} b_p\Big(\frac{\xi_1}{\sqrt{1+\xi_1^2}}, \frac{+1}{\sqrt{1+\xi_1^2}}\Big) = \lim_{\xi_1 \to +\infty} b_p\Big(\frac{\xi_1}{\sqrt{1+\xi_1^2}}, \frac{-1}{\sqrt{1+\xi_1^2}}\Big)$$

and

$$\lim_{\xi_1 \to -\infty} b_p\Big(\frac{\xi_1}{\sqrt{1+\xi_1^2}}, \frac{+1}{\sqrt{1+\xi_1^2}}\Big) = \lim_{\xi_1 \to -\infty} b_p\Big(\frac{\xi_1}{\sqrt{1+\xi_1^2}}, \frac{-1}{\sqrt{1+\xi_1^2}}\Big).$$

This implies that an arbitrary element of the algebra $\mathcal{R}(x_0)$ has a form $(B(+),\ B(-))$, with

$$B(\pm) = c_\pm(\xi_1)P_{\mathbb{R}}^+ + d_\pm(\xi_1)P_{\mathbb{R}}^- + B_0 + K_\pm,$$

and

$$\lim_{\xi_1 \to +\infty} c_+(\xi_1) = \lim_{\xi_1 \to +\infty} c_-(\xi_1), \qquad \lim_{\xi_1 \to -\infty} c_+(\xi_1) = \lim_{\xi_1 \to -\infty} c_-(\xi_1).$$

Thus, in particular, the local algebra $\mathcal{R}(x_0)$ has two series π^+_ω and π^-_ω of one-dimensional representations parameterized by points of $\omega \in \mathbb{T}$ and defined as follows: let $\omega_\pm = \left(\frac{\xi_1}{\sqrt{1+\xi_1^2}}, \frac{\pm 1}{\sqrt{1+\xi_1^2}}\right) \in \mathbb{T}$, then

$$\pi^+_{\omega_\pm} \ : \ (B(+),\, B(-)) \longmapsto c_\pm(\xi_1), \tag{4.6}$$
$$\pi^-_{\omega_\pm} \ : \ (B(+),\, B(-)) \longmapsto d_\pm(\xi_1). \tag{4.7}$$

Finally consider the point $x_0 = 0$. For each function $a \in PC(\dot{\mathbb{R}}^2, \mathcal{L})$ there exists the (unique) homogeneous of zero degree function $h_a(x) \in H(PC(\mathbb{T}, \Lambda))$, such that

$$\lim_{x\to 0}[a(x) - h_a(x)] = 0.$$

Proposition 4.4 *Let $x_0 = 0$. The local algebra $\mathcal{R}(x_0)$ is isometric and isomorphic to the subalgebra of the algebra $C_b(\mathbb{R},\, \mathcal{R}(PC(\mathbb{T}, \Lambda), S_{\mathbb{T}}))$. The homeomorphism*

$$\pi_{x_0} : \mathcal{R} = \Psi(PC(\dot{\mathbb{R}}^2, \mathcal{L}), H(C(\mathbb{T}))) \longrightarrow \mathcal{R}(x_0) \cong C_b(\mathbb{R},\, \mathcal{R}(PC(\mathbb{T}, \Lambda), S_{\mathbb{T}}))$$

is generated by the following mappings of generators of the algebra $\mathcal{R}$:

$$\begin{aligned} \pi_{x_0} \ &: \ a(x)I \longmapsto h_a(t)I, \quad \lambda \in \mathbb{R}, \\ \pi_{x_0} \ &: \ F^{-1}b(\xi)F \longmapsto b(it)P^+_{\mathbb{T}} + b(-it)P^-_{\mathbb{T}} + K(\lambda), \quad \lambda \in \mathbb{R}, \end{aligned}$$

where $K(\lambda) = E^{-1}(\lambda)b(\omega)E(\lambda) - b(it)P^+_{\mathbb{T}} - b(-it)P^-_{\mathbb{T}}$ is a compact operator.

PROOF. The operators aI and h_aI are locally equivalent at the point 0. As above, the operator $b(D)$ is locally equivalent at the origin to itself. Thus, the local algebra $\mathcal{R}(x_0)$ is isometric and isomorphic to the considered algebra $\Psi(H(PC(\mathbb{T}, \Lambda)), H(C(\mathbb{T})))$, and the statement follows directly from Theorem 3.4. □

Denote by $\widehat{\mathbb{R}}^2$ the compact of maximal ideals of the algebra $PC(\widehat{\mathbb{R}}^2, \mathcal{L})$, which can be considered as a compactification of $\dot{\mathbb{R}}^2$ cut by $\mathcal{L} \cup \{0\}$. Under this procedure each point $x \in \mathcal{L}_p \subset \mathcal{L}$ passes to the two points x_+ and x_- in $\widehat{\mathbb{R}}^2$, which correspond to one-sided limit values of functions $a \in PC(\widehat{\mathbb{R}}^2, \mathcal{L})$ at the point x:

$$a(x_+) = a_+(x) \qquad \text{and} \qquad a(x_-) = a_-(x);$$

the point 0 passes to the family of points 0_t, parameterized by $t \in \widehat{\mathbb{T}} = \overline{\mathbb{T} \setminus \Lambda}$ (the compact of maximal ideals of the algebra $PC(\mathbb{T}, \Lambda)$), which correspond to the radial limit values of functions $a \in PC(\widehat{\mathbb{R}}^2, \mathcal{L})$ at the origin:

$$a(0_t) = a_t(0) = \lim_{\substack{x\to 0 \\ \arg x = t}} a(x).$$

Introduce $\mathfrak{M} = [\widehat{\mathbb{R}}^2 \times \mathbb{T}] \bigcup [\mathcal{L} \times \{-1, 1\}] \bigcup [\{0\} \times \mathbb{R}]$, and denote by $\mathfrak{S}$ the set of all bounded operator-functions σ of the form:

$$\sigma = \begin{cases} \sigma_1(x, \omega) \in \mathbb{C}, & (x, \omega) \in \widehat{\mathbb{R}}^2 \times \mathbb{T}, \\ \sigma_2(x, \omega) \in \mathcal{R}(C(\overline{\mathbb{R}}), S_{\mathbb{R}}), & (x, \omega) \in \mathcal{L} \times \{-1, 1\}, \\ \sigma_3(x, \omega) \in \mathcal{R}(PC(\mathbb{T}, \Lambda), S_{\mathbb{T}}), & (x, \omega) \in \{0\} \times \mathbb{R}, \end{cases}$$

which satisfy the following conditions:

1. Each operator-function $\sigma_j(x,\omega)$, $j=1,2,3$, is norm continuous, being considered on its domain.

2. For each $\sigma_1(x,\omega)$ there exists a (unique) homogeneous function $h_{\sigma_1}(x,\omega) \in H(PC(\mathbb{T},\Lambda)) \otimes H(C(\mathbb{T}))$ such that

$$\lim_{x\to 0} \sup_{\omega\in\mathbb{T}} |\sigma_1(x,\omega) - h_{\sigma_1}(x,\omega)| = 0.$$

3. Let $x \in \mathcal{L}_p$, and x_+, x_- are the corresponding points of $\widehat{\mathbb{R}}^2$. Let furthermore $\omega_\pm = \left(\frac{\xi_1}{\sqrt{1+\xi_1^2}}, \frac{\pm 1}{\sqrt{1+\xi_1^2}}\right) \in \mathbb{T}$. Then

$$\begin{aligned} \pi^+_{\omega_\pm}(\sigma_2(x,+1),\, \sigma_2(x,-1)) &= \Omega_p(\sigma_1)(x_+,\omega_\pm), \\ \pi^-_{\omega_\pm}(\sigma_2(x,+1),\, \sigma_2(x,-1)) &= \Omega_p(\sigma_1)(x_-,\omega_\pm), \end{aligned}$$

where Ω_p is the rotation operator (4.4), and $\pi^+_{\omega_\pm}$, $\pi^-_{\omega_\pm}$ are one-dimensional representations (4.6), (4.7).

4. For each $\lambda \in \mathbb{R}$, and points $0_t \in \widehat{\mathbb{R}}^2$:

$$\begin{aligned} \pi(t)(\sigma_3(0,\lambda)) &= \sigma_1(0_t, it), & t \in \widehat{\mathbb{T}}_+ , \\ \pi(t)(\sigma_3(0,\lambda)) &= \sigma_1(0_t, -it), & t \in \widehat{\mathbb{T}}_- , \end{aligned}$$

and

$$\pi_{t_p}(\sigma_3(0,\lambda)) =$$
$$\begin{pmatrix} \sigma_1(0_{t_p+0}, it_p)(1-x) + \sigma_1(0_{t_p-0}, it_p)x & (\sigma_1(0_{t_p+0}, -it_p) - \sigma_1(0_{t_p-0}, -it_p))\sqrt{x(1-x)} \\ (\sigma_1(0_{t_p+0}, it_p) - \sigma_1(0_{t_p-0}, it_p))\sqrt{x(1-x)} & \sigma_1(0_{t_p+0}, -it_p)x + \sigma_1(0_{t_p-0}, -it_p)(1-x) \end{pmatrix},$$
$x \in (0,1)_p$,

where $\pi(\cdot)$ are one-dimensional representations (3.3) and π_{t_p} are two dimensional representations (3.4) of the algebra $\mathcal{R}(PC(\mathbb{T},\Lambda), S_{\mathbb{T}})$.

The set $\mathfrak{S}$ is a C^*-algebra with respect to the point-wise operations and the norm

$$\|\sigma\| = \sup_{(x,\omega)\in\mathfrak{M}} \|\sigma(x,\omega)\|.$$

Pasting together all the local descriptions we come to the following theorem.

Theorem 4.5 *The algebra* $\operatorname{Sym}\mathcal{R} = \Psi(PC(\dot{\mathbb{R}}^2,\mathcal{L}), H(C(\mathbb{T})))/\mathcal{K}$ *is isometric and isomorphic to the algebra* $\mathfrak{S}$. *Under this identification, the homeomorphism*

$$\operatorname{sym} : \mathcal{R} \longrightarrow \operatorname{Sym}\mathcal{R}$$

is generated by the following mapping of generators of the algebra $\mathcal{R}$*:*

$$\operatorname{sym}(a(x)b(D)) = \begin{cases} a(x)b(\omega), \\ \left(a_+(x)P^+_{\mathbb{R}} + a_-(x)P^-_{\mathbb{R}}\right) b_p\left(\frac{\xi_1}{\sqrt{1+\xi_1^2}}, \frac{\omega}{\sqrt{1+\xi_1^2}}\right) I, \\ h_a(t)\Big(b(it)P^+_{\mathbb{T}} + b(-it)P^-_{\mathbb{T}} + K(\omega)\Big), \end{cases}$$

where $(x,\omega) \in (\mathbb{R}^2 \setminus (\mathcal{L} \cup \{0\})) \times \mathbb{T}$ *at the first line,* $(x,\omega) \in \mathcal{L}_p \times \{-1,1\}$, $p = 1,\ldots,k$ *at the second line, and* $(x,\omega) \in \{0\} \times \mathbb{R}$ *at the third line respectively, furthermore* $b_p(\xi) = b(\omega_p^{-1}(\xi))$, $P_{\mathbb{R}}^{\pm} = \frac{1}{2}(I \pm S_{\mathbb{R}})$, $P_{\mathbb{T}}^{\pm} = \frac{1}{2}(I \pm S_{\mathbb{T}})$, $h_a(x)$ *is the homogeneous function, corresponding to* $a(x)$, $K(\lambda) = E^{-1}(\lambda)b(\varphi)E(\lambda) - b(it)P_{\mathbb{T}}^{+} - b(-it)P_{\mathbb{T}}^{-}$ *is a compact operator.*

5 Description of the algebra $\mathcal{R} = \Psi(SO(\dot{\mathbb{R}}^2,\mathcal{L}), H(C(\mathbb{T})))$

We shall use the notation of the previous section. For a function $a(x)$ set $a_p(x) = a(\omega_p^{-1}(x))$. We say that the function $a(x)$ has a slowly oscillating discontinuity at the point $x_0 = (x_1^0, x_2^0) \in \mathcal{L}_p$ in the normal direction, if the function $h(x_1) = a_p(x_1, x_2^0)$ has a slowly oscillating discontinuity at the point $x_1 = 0$ (i.e. if $h(x_1)$ is differentiable in a neighbourhood of the origin and $x_1 h'(x_1) \longrightarrow 0$, when $x_1 \to 0$).

Denote by $SO(\dot{\mathbb{R}}^2,\mathcal{L})$ the C^*-algebra generated by functions $a(x)$ satisfying the following conditions:

i) The functions $a(x)$ are continuous in $\dot{\mathbb{R}}^2 \setminus (\mathcal{L} \cup \{0\})$.

ii) The functions $a(x)$ are slowly oscillating at the point of each segment $\mathcal{L}_p$ in the normal direction.

iii) For each $a(x) \in SO(\dot{\mathbb{R}}^2,\mathcal{L})$ there exists a homogeneous function of order zero $h_a(x)$, such that the difference $a(x) - h_a(x)$ has limit zero at the origin.

The above mentioned function $h_a(x)$ is unique (for each fixed $a(x)$), and its restriction onto the unit circle belongs to $SO(\mathbb{T},\Lambda)$.

Introduce the C^*-algebra $\mathcal{R} = \Psi(SO(\dot{\mathbb{R}}^2,\mathcal{L}), H(C(\mathbb{T})))$ generated by the operators

$$A = aI, \quad a(x) \in SO(\dot{\mathbb{R}}^2,\mathcal{L}), \tag{5.1}$$

$$b(D) = F^{-1}b(\xi)F, \quad b(\xi) \in H(C(\mathbb{T})), \tag{5.2}$$

acting on $L_2(\mathbb{R}^n)$.

To describe the (Fredholm) symbol algebra $\operatorname{Sym}\mathcal{R} = \widehat{\mathcal{R}} = \mathcal{R}/\mathcal{K}$ of the algebra $\mathcal{R}$ we will use the local principle ([2, 14]) mentioned in the previous section; i.e., considering the natural projections $\pi_{x_0} : \mathcal{R} \longrightarrow \widehat{\mathcal{R}} \longrightarrow \mathcal{R}(x_0) = \widehat{\mathcal{R}}/J(x_0)$, which identify the locally equivalent in the point x_0 elements of the algebra $\mathcal{R}$. For the definitions see previous section.

Considering the point $x_0 \in \dot{\mathbb{R}}^2 \setminus \mathcal{L}$ or $x_0 = 0$ we refer to the corresponding result of the previous section: the proof is the same.

Proposition 5.1 *Let* $x_0 \in \dot{\mathbb{R}}^2 \setminus \mathcal{L},\ x_0 \neq 0$. *Then the local algebra* $\mathcal{R}(x_0)$ *is isometric and isomorphic to the algebra* $C(\mathbb{T})$. *The homeomorphism*

$$\pi_{x_0} : \mathcal{R} = \Psi(SO(\dot{\mathbb{R}}^2,\mathcal{L}), H(C(\mathbb{T}))) \longrightarrow \mathcal{R}(x_0) \cong C(\mathbb{T})$$

is generated by the following mappings of generators of the algebra $\mathcal{R}$:

$$\begin{aligned} \pi_{x_0} &: aI \longrightarrow a(x_0), \\ \pi_{x_0} &: b(D) \longrightarrow b(\omega), \quad \omega \in \mathbb{T}. \end{aligned}$$

Let now $x_0 \in \mathcal{L} \setminus \{0\}$. Following [10] we introduce some definitions. We will say that the operator-function $m(r,\lambda)$ belongs to the class $\mathcal{E}(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T})))$, if $m(r,\lambda) \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$, valued in $B(L_2(\mathbb{T}))$ (the space of bounded on $L_2(\mathbb{T})$ operators) and

$$\sup_{\substack{\lambda\in\mathbb{R}\\ r\in\mathbb{R}_+}} \|(r\partial_r)^\beta \partial_\lambda^\alpha m(r,\lambda)\|_{B(L_2(\mathbb{T}))} < \infty, \quad \alpha, \beta = 1, 2, 3, \ldots.$$

Here, as usual, $\partial_r = \frac{\partial}{\partial r}$. We say that $m(r,\lambda)$ ($\in \mathcal{E}(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T})))$ has slowly oscillating discontinuity at the point 0 if

$$\lim_{r\to 0+} \sup_{\lambda\in\mathbb{R}} \|r\partial_r m(r,\lambda)\|_{B(L_2(\mathbb{T}))} = 0$$

and write $m(r,\lambda) \in \widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T}))$. If moreover

$$\lim_{r\to 0+} \sup_{\lambda\in\mathbb{R}} \|m(r,\lambda)\|_{B(L_2(\mathbb{T}))} = 0$$

then we write $m(r,\lambda) \in \mathcal{E}^0 = \mathcal{E}^0(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T}))$.

Let the symbol $m(r,\lambda)(\in \mathcal{E}(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T})))$ be assigned to the Mellin pseudodifferential operator defined on $u \in C^\infty(\mathbb{R}_+, L_2(\mathbb{T}))$ by

$$\mathbb{M}u(r) = (M^{-1} m(r,\lambda) M u)(r) = (2\pi r)^{-1} \int\int_{\mathbb{R}_+\times\mathbb{R}} m(r,\lambda)(r\rho^{-1})^{i\lambda} u(\rho) d\rho d\lambda. \tag{5.3}$$

By $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}(L_2(\mathbb{R}_+))$ we denote the C^*-algebra of operators (5.3) (acting on $L_2(\mathbb{R}_+)$) with the symbols $m(r,\lambda) \in \widetilde{\mathcal{E}}(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T}))$ and by $\mathcal{B}^0 = \mathcal{B}^0(L_2(\mathbb{R}_+))$ the subalgebra of $\widetilde{\mathcal{B}}(L_2(\mathbb{R}_+))$ of operators with the symbols from $\mathcal{E}^0(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T}))$. Let also $\widetilde{C} = \widetilde{C}(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T}))$ and $C^0 = C^0(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T}))$ be the C^* - algebras generated by classes $\widetilde{\mathcal{E}}$ and $\mathcal{E}^0$ (with respect to the supremum norm $\sup_{r,\lambda} \|\cdot\|_{B(L_2(\mathbb{T}))}$), respectively.

Rewrite the function $a(x - x_0)$ in the polar coordinates with the center at the point x_0:

$$a(x - x_0) = \widetilde{a}(r,t), \qquad x - x_0 = (r\cos t, r\sin t).$$

Proposition 5.2 *Let $x_0 \in \mathcal{L}$. Then the local algebra $\mathcal{R}(x_0)$ is isometric and isomorphic to a subalgebra of the algebra $\widehat{C} = \widetilde{C}/C^0$. The homomorphism*

$$\pi_{x_0} : \mathcal{R} = \Psi(SO(\dot{\mathbb{R}}^2, \mathcal{L}), H(C(\mathbb{T}))) \longrightarrow \mathcal{R}(x_0) \subset \widehat{C}$$

is generated by the following mappings of generators of the algebra $\mathcal{R}$:

$$\begin{aligned} \pi_{x_0} &: aI \longrightarrow m_1(r) + C^0, \\ \pi_{x_0} &: b(D) \longrightarrow m_2(\lambda), \end{aligned}$$

where

$$m_1(r) = \widetilde{a}^t(r) \equiv \widetilde{a}(r,t)$$

for each fixed r is the operator of multiplication on $L_2(\mathbb{T})$ by $\widetilde{a}(r,t)$ and

$$m_2(\lambda) = b_\lambda(\Omega) = E^{-1}(\lambda) b(\omega) E(\lambda) = b(it)P_{\mathbb{T}}^+ + b(-it)P_{\mathbb{T}}^- + K(\lambda).$$

PROOF. Since $a(x) \in SO(\dot{\mathbb{R}}^2, \mathcal{L})$ this obviously implies that for any fixed t the function $\widetilde{a}(r,t)$ is slowly oscillating when $r \to 0$; i.e.,

$$\lim_{r\to 0} r\partial_r \widetilde{a}(r,t) = 0$$

(while the inverse is not always true). Represent $L_2(\mathbb{R}^2)$ in the form

$$L_2(\mathbb{R}^2) = L_2\Big((\mathbb{R}_+, rdr), L_2(\mathbb{T})\Big) \equiv \mathcal{H},$$

i.e., as a set of functions from $L_2(\mathbb{R}_+, rdr)$ valued in $L_2(\mathbb{T})$ and with the natural norm

$$\|u\|_{\mathcal{H}} = \Big(\int_{\mathbb{R}_+} r\|u(r)\|^2_{L_2(\mathbb{T})} dr\Big)^{\frac{1}{2}}$$

The above with the decomposition (2.12) allow us to identify the algebra $\mathcal{R} = \Psi(SO(\dot{\mathbb{R}}^2, \mathcal{L}), H(C(\mathbb{T})))$ with the subalgebra $\widetilde{\mathcal{R}}$ of the algebra $\widetilde{\mathcal{B}}$:

$$\mathcal{R} \cong \widetilde{\mathcal{R}} \subset \widetilde{\mathcal{B}}. \tag{5.4}$$

Under this identification, to the generator $a(x)I$ of the algebra $\mathcal{R}$ we assign the operator

$$m_1(r)I$$

and to the generator $b(D)$ we assign the operator

$$M^{-1}m_2(\lambda)M.$$

Let also $\widetilde{I} = \widetilde{\mathcal{R}} \cap \mathcal{B}^0$. We have

$$\mathcal{R}(x_0) \cong \widetilde{\mathcal{R}}/\widetilde{I} = (\widetilde{\mathcal{R}} + \mathcal{B}^0)/\mathcal{B}^0 \subset \widetilde{\mathcal{B}}/\mathcal{B}^0$$

By [10], Theorem 2.3 we have the isometrical isomorphism

$$\widetilde{\mathcal{B}}/\mathcal{B}^0 \cong \widehat{C},$$

generated by the following mapping of generators of the algebra $\widetilde{\mathcal{B}}$:

$$M^{-1}m(r,\lambda)M \longrightarrow m(r,\lambda) + C^0, \quad m(r,\lambda) \in \widetilde{\mathcal{E}}(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T})).$$

This finishes the proof. □

Consider finally the point $x_0 = 0$ and take the (unique) homogeneous function of zero order $h_a(x) \in H(SO(\mathbb{T}, \Lambda))$ such that

$$\lim_{x\to 0}[a(x) - h_a(x)] = 0.$$

Proposition 5.3 *Let $x_0 = 0$. The local algebra $\mathcal{R}(x_0)$ is isometric and isomorphic to the subalgebra of the algebra $C_b(\mathbb{R},\ \mathcal{R}(SO(\mathbb{T},\Lambda), S_{\mathbb{T}}))$. The homeomorphism*

$$\pi_{x_0} : \mathcal{R} = \Psi(SO(\dot{\mathbb{R}}^2, \mathcal{L}), H(C(\mathbb{T}))) \longrightarrow \mathcal{R}(x_0) \cong C_b(\mathbb{R},\ \mathcal{R}(SO(\mathbb{T},\Lambda), S_{\mathbb{T}}))$$

is generated by the following mappings of generators of the algebra $\mathcal{R}$:

$$\begin{array}{lcl} \pi_{x_0} & : & a(x)I \longmapsto h_a(t)I, \quad \lambda \in \mathbb{R}, \\ \pi_{x_0} & : & F^{-1}b(\xi)F \longmapsto b(it)P_{\mathbb{T}}^{+} + b(-it)P_{\mathbb{T}}^{-} + K(\lambda), \quad \lambda \in \mathbb{R}, \end{array}$$

where $K(\lambda) = E^{-1}(\lambda)b(\omega)E(\lambda) - b(it)P_{\mathbb{T}}^{+} - b(-it)P_{\mathbb{T}}^{-}$ is a compact operator.

PROOF. Follows from Theorem 3.7. □

Denote $\widetilde{\mathbb{R}}^2 = [\dot{\mathbb{R}}^2 \setminus (\mathcal{L} \bigcup \{0\})] \bigsqcup [\mathbb{T} \setminus \Lambda]$. Here the point $0_t \in \mathbb{T} \setminus \Lambda$ corresponds to the radial limit value of functions $a \in SO(\mathbb{R}^2, \mathcal{L})$ at the origin:

$$a(0_t) = a_t(0) = \lim_{\substack{x \to 0 \\ \arg x = t}} a(x), \quad t \in \mathbb{T} \setminus \Lambda.$$

Set $\mathfrak{M} = [(\widetilde{\mathbb{R}}^2 \times \mathbb{T}] \bigcup \mathcal{L} \bigcup [\{0\} \times \mathbb{R}]$ and denote by $\mathfrak{S}$ the C^*-algebra with respect to the point-wise operations and the supremum norm of all bounded operator-functions σ of the form:

$$\sigma = \begin{cases} \sigma_1(x,\omega) \in \mathbb{C}, & (x,\omega) \in \widetilde{\mathbb{R}}^2 \times \mathbb{T}, \\ \sigma_2(x) \in \widehat{C} = \widetilde{C}(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T}))/C^0(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T})), & x \in \mathcal{L}, \\ \sigma_3(x,\omega) \in \mathcal{R}(SO(\mathbb{T},\Lambda), S_{\mathbb{T}}), & (x,\omega) \in \{0\} \times \mathbb{R}, \end{cases}$$

which satisfy the following conditions:

1. Each operator-function σ_j, $j = 1, 2, 3$, is norm continuous, being considered on its domain.

2. For each $\sigma_1(x,\omega)$ there exists the unique function $h_{\sigma_1}(x,\omega) \in H(SO(\mathbb{T},\Lambda)) \otimes H(C(\mathbb{T}))$ such that

$$\lim_{x \to 0} \sup_{\omega \in \mathbb{T}} |\sigma_1(x,\omega) - h_{\sigma_1}(x,\omega)| = 0.$$

3. Let $x_0 \in \mathcal{L}$. Set

$$\widetilde{\sigma}_1(r,t,\omega) = \sigma_1(x - x_0, \omega), \quad x - x_0 = (r\cos t, r\sin t).$$

Then the conditions are

$$m^t(r,\lambda) \equiv E^{-1}_{\omega \to t}(\lambda)\widetilde{\sigma}_1(r,t,\omega)E_{\psi \to \omega}(\lambda) \in \sigma_2(x_0).$$

4. For each $\lambda \in \mathbb{R}$, and points $0_t \in \widetilde{\mathbb{R}}^2$:

$$\begin{array}{rcll} \pi(t)(\sigma_3(0,\lambda)) & = & \sigma_1(0_t, it), & t \in \mathbb{T} \setminus \Lambda\,, \\ \pi(t)(\sigma_3(0,\lambda)) & = & \sigma_1(0_t, -it), & t \in \mathbb{T} \setminus \Lambda\,, \end{array}$$

where $\pi(\cdot)$ are one-dimensional representations of the algebra $\mathfrak{S}_2 = \mathcal{R}(SO(\mathbb{T},\Lambda), S_{\mathbb{T}})$ parameterized by the points of $\widetilde{Y} = \widetilde{\mathbb{T}}_+ \cup \widetilde{\mathbb{T}}_-$ and defined as follows (see Theorem 3.5)

$$\pi(y) : B \in \mathfrak{S}_2 \longmapsto (\mathrm{sym}\, B)(y) = \begin{cases} c(y), & \text{if } \ y \in \widetilde{\mathbb{T}}_+\,, \\ d(y), & \text{if } \ y \in \widetilde{\mathbb{T}}_-\,, \end{cases}$$

where

$$c(t) = \mathrm{sym}\, B|_{\widetilde{\mathbb{T}}_+}, \quad d(t) = \mathrm{sym}\, B|_{\widetilde{\mathbb{T}}_-}.$$

5. In the same way, consider for each $t_p \in \Lambda$ the symbol homomorphism of the algebra $\mathfrak{S}_2 = \mathcal{R}(SO(\mathbb{T},\Lambda), S_{\mathbb{T}})$,

$$\pi_{t_p} : B \in \mathfrak{S}_2 \longmapsto (\mathrm{sym}\, B)(t_p) = \begin{pmatrix} \widehat{c}_p^{\,+}(1-x) + \widehat{c}_p^{\,-}x & (\widehat{d}_p^{\,+} - \widehat{d}_p^{\,-})\sqrt{x(1-x)} \\ (\widehat{c}_p^{\,+} - \widehat{c}_p^{\,-})\sqrt{x(1-x)} & \widehat{d}_p^{\,+}x + \widehat{d}_p^{\,-}(1-x) \end{pmatrix}, \ x \in [0,1].$$

Here $\widehat{c}_p^{\,\pm}$, $\widehat{d}_p^{\,\pm}$ are defined by $c(t)$, $d(t)$ as in Theorem 3.5. Take the unique homogeneous function $h_{\sigma_1}(x,\omega)$, corresponding to $\sigma_1(x,\omega)$ as above. Obviously, $E(\lambda)^{-1}h_{\sigma_1}(x,\omega)E(\lambda) \in \mathfrak{S}_2$. Then the conditions are: for each $t_p \in \Lambda$ and each $\lambda \in \mathbb{R}$,

$$\pi_{t_p}(\sigma_3(0,\lambda)) = \pi_{t_p}(E(\lambda)^{-1}h_{\sigma_1}(x,\omega)E(\lambda)).$$

Theorem 5.4 *The algebra* $\mathrm{Sym}\,\mathcal{R} = \Psi(SO(\dot{\mathbb{R}}^2,\mathcal{L}), H(C(\mathbb{T})))/\mathcal{K}$ *is isometric and isomorphic to a subalgebra of the algebra* $\mathfrak{S}$. *Under this identification, the homeomorphism*

$$\mathrm{sym} : \mathcal{R} \longrightarrow \mathrm{Sym}\,\mathcal{R}$$

is generated by the mapping of generators of the algebra $\mathcal{R}$*:*

$$\mathrm{sym}\,(a(x)b(D)) = \begin{cases} a(x)b(\omega), & (x,\omega) \in \widetilde{\mathbb{R}}^2 \times \mathbb{T}, \\ m^t(r,\lambda) + C^0(\mathbb{R}_+ \times \mathbb{R}, B(L_2(\mathbb{T})), & x \in \mathcal{L}, \\ h_a(t)\Big(b(it)P_{\mathbb{T}}^+ + b(-it)P_{\mathbb{T}}^- + K(\omega)\Big), & (x,\omega) \in \{0\} \times \mathbb{R}. \end{cases}$$

where as above: $h_a(x)$ *is the homogeneous function corresponding to* $a(x)$, $K(\lambda) = E_{\varphi\to t}(\lambda)^{-1}b(\varphi)E_{\eta\to\varphi}(\lambda) - b(it)P_{\mathbb{T}}^+ - b(-it)P_{\mathbb{T}}^-$ *is a compact operator.*

References

[1] M. Costabel. An inverse for the Gohberg-Krupnik symbol map. *Proc. Roy. Soc. Edinburg*, 1980.

[2] Ronald G. Douglas. *Banach Algebra Techniques in Operator Theory.* Academic Press, New York, 1972.

[3] I. M. Gelfand and G. E. Shilov. *Generalized Functions. Volume 1: Properties and Operations.* Academic Press, New York, 1964.

[4] I. Gohberg and N. Krupnik. On the algebra generated by one–dimensional singular integral operators with piece–wise continuous coefficients. *Funct. Analysis Appl.*, 4(3):193–201, 1970.

[5] I. S. Gradshteyn and I. M. Ryzhik. *Tables of Integrals, Series, and Products.* Academic Press, New York, 1980.

[6] B. V. Lange and V. S. Rabinovich. Pseudodifferential operators on $\mathbb{R}^n$ and limit operators. *Matem. Sbornik*, 129(2):175–185, 1986.

[7] S. G. Mikhlin and S. Proessdorf. *Singular Integral Operators.* Springer Verlag, Berlin, Heidelberg, 1986.

[8] C. Müller. *Spherical Harmonics*, volume 17 of *Lect. Notes Math.* Springer Verlag, Berlin, 1966.

[9] B. A. Plamenevsky. *Algebras of Pseudodifferential Operators.* Kluwer Academic Publishers, 1989.

[10] V. S. Rabinovich. Singular integral operators on composite contours, and pseudodifferential operators. *Math. Notes*, 58(1):772–734, 1995.

[11] V. S. Rabinovich, S. Roch, and B. Silbermann. Fredholm theory and finite section method for band-dominated operators. *Integr. Eq. Oper. Theory*, 30:452–495, 1998.

[12] E. Ramírez de Arellano and N. L. Vasilevski. Algebras of singular integral operators generated by three orthogonal projections. *Integr. Equat. Oper. Th.*, 25:277–288, 1996.

[13] I. B. Simonenko. A new general method for studing linear operator equations of singular integral eguation type, I, II. *Izv. Akad. Nauk SSSR, Ser. Math.*, 29:567–586, 757–782, 1965.

[14] J. Varela. *Duality of C^*-algebras, Memories Amer. Math. Soc.*, volume 148, pages 97–108. AMS, Providence, Rhode Island, 1974.

[15] N. L. Vasilevski. Algebras generated by multidimensional singular integral operators and by coefficients admitting discontinuities of homogeneous type. *Math. USSR. Sbornik*, 57(1):1–19, 1987.

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