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Toeplitz Operators on the Unit Ball in $\mathbb{C}^n$ with Radial Symbols

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Abstract

The paper is devoted to the study of Toeplitz operators with radial symbols on the weighted Bergman spaces on the unit ball in $\mathbb{C}^n$. Admitting "badly" behaved unbounded symbols we get new qualitative features. In particular, contrary to known results, a Toeplitz operator with the same (unbounded) symbol now can be bounded in one weighted Bergman space and unbounded in another, compact in one weighted Bergman space and bounded but not compact in another, compact in one weighted Bergman space and unbounded in another.

In our case of radial symbols, the Wick (or covariant) symbol of a Toeplitz operator gives complete information about the operator, providing its spectral decomposition.

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1 Introduction

We consider the weighted Bergman space $\mathcal{A}^2_\mu(\mathbb{B}^n)$ of holomorphic functions in unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ which belong to the weighted space $L_2^\mu(\mathbb{B}^n)$, and Toeplitz operators with radial symbols acting on $\mathcal{A}^2_\mu(\mathbb{B}^n)$.

The theory of Bergman type spaces and problems of boundedness and compactness of the Toeplitz operators acting on these spaces have been studied intensively in recent years. Without claiming completeness we refer to [1, 7, 11, 13], and also [14] for references. The methods in mentioned works are mainly based on Berezin transform techniques and Tauberian type theorems, which do not work well or at all when the symbols of the Toeplitz operators may have singular behaviour near the boundary (the sphere S^{2n-1}, for us).

On the other hand, in the recent work [12] (see also [6]) a new approach has been proposed, which allows handling radial symbols having a "bad" behaviour. In particular, it has been shown that Toeplitz operators can be bounded and even compact for badly behaved symbols (for example, unbounded near the boundary).

The papers [6, 12] are devoted to the case of the unit disk, while here we apply the methods of [6, 12] to the study of the n - dimensional case. It turns out that there is no qualitative difference between the one-dimensional and multi-dimensional cases when studying global properties such as commutative algebra structure etc.; this is why we are emphasize questions concerning the properties of concrete Toeplitz operators. For example, for weighted Bergman spaces on the unit ball with the weights $\mu_\lambda(|z|) = (1 - |z|^2)^{\lambda-1}$, $\lambda > 0$, boundedness (compactness) of the Toeplitz operator T_a with a positive (and even unbounded) symbol a on some weighted space ($\lambda = \lambda_0$) implies boundedness (compactness) on all the spaces ($\lambda > 0$) (see [14] for the case of the unit disk). Nevertheless we give an example of a symbol for which the corresponding Toeplitz operator is bounded when $\lambda = 1$ (weightless case) and unbounded for $\lambda = 2$, compact for $\lambda = 1$ and bounded but not compact for $\lambda = 2$, compact for $\lambda = 1$, and unbounded for $\lambda = 2$. Such examples draw attention to qualitative new features and reflect the very singular nature of the symbols under consideration.

We also use the Berezin concept of Wick and anti-Wick symbols. It turns out that in our particular (radial symbol) case the Wick (or covariant) symbol of a Toeplitz operator gives complete information about the operator, providing its spectral decomposition.

All that can be obtained from the results of [6, 12] with slight changes, we present here without the proofs, referring to those papers.

2 Preliminaries

We will identify $\mathbb{C}^n = \mathbb{R}^{2n}$ writing $z_k = x_k + iy_k$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Let $z,\ \xi \in \mathbb{C}^n$, we will use the following standard notation: $z \cdot \xi = \sum_{j=1}^n z_j \xi_j$, $|z| = \sqrt{z \cdot \overline{z}}$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, is a multiindex and $|\alpha| = \alpha_1 + \ldots + \alpha_n$ is its length, $\alpha! = \alpha_1! \cdots \alpha_n!$.

Consider a non negative measurable function (weight) $\mu(r)$, $r \in (0,1)$, such that $\text{mes}\,\{r \in (0,1) : \mu(r) > 0\} = 1$, and

$$\int_{\mathbb{B}^n} \mu(|z|) dxdy = |S^{2n-1}| \int_0^1 \mu(r) r^{2n-1} dr < \infty,$$

where $|S^{2n-1}| = 2\pi^{n-\frac{1}{2}}\Gamma^{-1}(n-\frac{1}{2})$ is the surface area of the unit sphere S^{2n-1}, and $\Gamma(z)$ is the Gamma function.

Introduce the weighted space

$$L_2^\mu(\mathbb{B}^n) = \{f : \|f\|^2_{L_2^\mu(\mathbb{B}^n)} = \int_{\mathbb{B}^n} |f(z)|^2 \mu(|z|) d\nu(z) < \infty\},$$

where $d\nu(z) = dxdy$ is the usual Lebesgue volume measure, and the space $L_2(S^{2n-1})$ with the usual Lebesgue surface measure.

Let $\mathcal{H}_k$ be the space of spherical harmonics of order k (see, for example, [8]). The space $L_2(S^{2n-1})$ is the direct sum of mutually orthogonal spaces $\mathcal{H}_k$, i.e.,

$$L_2(S^{2n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.$$

Each space $\mathcal{H}_k$ is the direct sum (under the identification $\mathbb{C}^n = \mathbb{R}^{2n}$) of the mutually orthogonal spaces $H_{p,q}$ (see, for example, [9]):

$$\mathcal{H}_k = \bigoplus_{\substack{p+q=k \\ p,q \in \mathbb{Z}_+}} H_{p,q}, \qquad k \in \mathbb{Z}_+,$$

where $H_{p,q}$, for each $p, q = 0, 1, \ldots$, is the space of harmonic polynomials (their restrictions to the unit sphere, more precisely) of complete order p in the variable z and complete order q in the conjugate variable $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$. Thus

$$L_2(S^{2n-1}) = \bigoplus_{p,q \in \mathbb{Z}_+} H_{p,q}.$$

The Hardy space $H^2(\mathbb{B}^n)$ in the unit ball $\mathbb{B}^n$ is a closed subspace of $L_2(S^{2n-1})$. Denote by $P_{S^{2n-1}}$ the Szegö orthogonal projection of $L_2(S^{2n-1})$ onto the Hardy space $H^2(\mathbb{B}^n)$. It is well known that

$$H^2(\mathbb{B}^n) = \bigoplus_{p=0}^{\infty} H_{p,0}.$$

The standard orthonormal base in $H^2(\mathbb{B}^n)$ has the form (see, for example, [9])

$$e_\alpha(\omega) = d_{n,\alpha}\ \omega^\alpha, \qquad d_{n,\alpha} = \sqrt{\frac{(n-1+|\alpha|)!}{|S^{2n-1}|(n-1)!\alpha!}}\ , \qquad |\alpha| = 0, 1, \ldots .$$

Fix now and in all that follows an ortonormal basis $\{e_{\alpha,\beta}(\omega)\}_{\alpha,\beta}$, $\alpha,\ \beta \in \mathbb{Z}_+^n$, in the space $L_2(S^{2n-1})$ so that $e_{\alpha,0}(\omega) \equiv e_\alpha(\omega),\ |\alpha| = 0, 1, \ldots$.

Passing to the spherical coordinates we have

$$L_2^\mu(\mathbb{B}^n) = L_2((0,1), \mu(r)r^{2n-1}dr) \otimes L_2(S^{2n-1}). \tag{2.1}$$

Now each function $f(z) \in L_2^\mu(\mathbb{B}^n)$ admits the decomposition

$$f(z) = \sum_{|\alpha|+|\beta|=0}^{\infty} c_{\alpha,\beta}(r)\ e_{\alpha,\beta}(\omega), \qquad r = |z|, \quad \omega = \frac{z}{r}\ . \tag{2.2}$$

with the coefficients $c_{\alpha,\beta}(r)$ satisfying the condition

$$\|f\|^2_{L_2^\mu(\mathbb{B}^n)} = \sum_{|\alpha|+|\beta|=0}^{\infty} \int_0^1 |c_{\alpha,\beta}(r)|^2\, \mu(r)\, r^{2n-1}dr < \infty.$$

Thus the decomposition (2.1), (2.2) together with the Parseval's equality give rise to the unitary operator

$$\begin{aligned} U_1 : L_2((0,1), \mu(r)r^{2n-1}dr) \otimes L_2(S^{2n-1}) &\longrightarrow L_2((0,1), \mu(r)r^{2n-1}dr) \otimes l_2 \\ &\equiv l_2(L_2((0,1), \mu(r)r^{2n-1}dr)), \end{aligned}$$

defined as

$$U_1 : f(z) \longrightarrow \{c_{\alpha,\beta}(r)\},$$

with

$$\|f\|^2_{L_2^\mu(\mathbb{B}^n)} = \|c_{\alpha,\beta}(r)\|^2_{l_2(L_2((0,1),\,\mu(r)r^{2n-1}dr)} = \sum_{|\alpha|+|\beta|=0}^{\infty} \|c_{\alpha,\beta}(r)\|^2_{L_2((0,1),\,\mu(r)r^{2n-1}dr)}.$$

Let $f(z)$ be a holomorphic function in the unit ball $\mathbb{B}^n$, and let

$$f(z) = \sum_{|\alpha|=0}^{\infty} c_\alpha z^\alpha$$

be its Taylor series (which converges uniformly on each compact subset of $\mathbb{B}^n$, see [9]). We have

$$f(z) = \sum_{|\alpha|=0}^{\infty} c_\alpha z^\alpha = \sum_{|\alpha|=0}^{\infty} c_\alpha r^{|\alpha|}\omega^\alpha = \sum_{|\alpha|=0}^{\infty} c_\alpha(r) e_\alpha(\omega), \tag{2.3}$$

where $c_\alpha(r) = c_\alpha \; d_{n,\alpha}^{-1} \; r^{|\alpha|}, \quad r = |z|, \; \omega = z/r.$

Let $\mathcal{A}_\mu^2(\mathbb{B}^n)$ be the Bergman space of holomorphic in $\mathbb{B}^n$ functions from $L_2^\mu(\mathbb{B}^n)$. Denote by $B_{\mathbb{B}^n}^\mu$ the Bergman orthogonal projection of $L_2^\mu(\mathbb{B}^n)$ onto the Bergman space $\mathcal{A}_\mu^2(\mathbb{B}^n)$. From the above it follows that to characterize a function $f(z) \in \mathcal{A}_\mu^2(\mathbb{B}^2)$ and considering its decomposition according to (2.3), one can restrict to the functions having the representation

$$f(z) = \sum_{|\alpha|=0}^{\infty} c_{\alpha,0}(r) \; e_{\alpha,0}(\omega). \tag{2.4}$$

Now let us take an arbitrary function $f(z)$ from $\mathcal{A}_\mu^2(\mathbb{B}^n)$ in the form (2.4). It has to satisfy the Cauchy-Riemann equations, i.e.,

$$\frac{\partial}{\partial \overline{z}_k} f(z) \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\Big) f(z) = 0, \quad k = 1, \dots, n, \quad z \in \mathbb{B}^n. \tag{2.5}$$

Applying $\frac{\partial}{\partial \overline{z}_k}$ to (2.4) we have

$$\frac{\partial}{\partial \overline{z}_k} \sum_{|\alpha|=0}^{\infty} c_{\alpha,0}(r) \; e_{\alpha,0}(\omega) = \frac{z_k}{r} \sum_{|\alpha|=0}^{\infty} \Big(\frac{d}{dr} c_{\alpha,0}(r) - \frac{|\alpha|}{r} c_{\alpha,0}(r)\Big) e_{\alpha,0}(\omega), \quad k = 1, \dots, n, \tag{2.6}$$

and we come to the infinite system of ordinary linear differential equations

$$\frac{d}{dr} c_{\alpha,0}(r) - \frac{|\alpha|}{r} c_{\alpha,0}(r) = 0, \quad |\alpha| = 0, 1, \dots.$$

Their general solution has the form

$$c_{\alpha,0}(r) = b_\alpha r^{|\alpha|} = \lambda(n, |\alpha|) \; c_{\alpha,0} r^{|\alpha|},$$

with

$$\lambda(n, m) = \Big(\int_0^1 t^{2m+2n-1} \mu(t) dt\Big)^{-\frac{1}{2}}.$$

Hence, for any $f(z) \in \mathcal{A}_\mu^2(\mathbb{B}^n)$ we have

$$f(z) = \sum_{|\alpha|=0}^{\infty} c_{\alpha,0} \; \lambda(n, |\alpha|) \; r^{|\alpha|} \; e_{\alpha,0}(\omega)$$

and, as is easy to verify,

$$\|f\|^2_{L_{2,\mu}(\mathbb{B}^n)} = \sum_{|\alpha|=0}^{\infty} |c_{\alpha,0}|^2.$$

Thus the image $\mathcal{A}^2_{1,\mu}(\mathbb{B}^n) = U_1(\mathcal{A}^2_\mu(\mathbb{B}^n))$ is characterized as the (closed) subspace of

$$L_2((0,1), \mu(r)r^{2n-1}dr) \otimes l_2 = l_2(L_2((0,1), \mu(r)r^{2n-1}dr))$$

which consists of all sequences $c_{\alpha,\beta}(r)$ of the form

$$c_{\alpha,\beta}(r) = \begin{cases} \lambda(n,|\alpha|)\ c_{\alpha,0}\ r^{|\alpha|}, & |\beta| = 0, \\ 0, & |\beta| \neq 0, \end{cases}$$

and, in addition,

$$\|f\|_{L_2^\mu(\mathbb{B}^n)} = \|\{c_{\alpha,\beta}(r)\}\|_{l_2(L_2((0,1),\mu(r)r^{2n-1}dr))} = \left(\sum_{|\alpha|=0}^{\infty} |c_{\alpha,0}|^2\right)^{\frac{1}{2}}.$$

For each $m \in \mathbb{Z}_+$ introduce the function

$$\varphi_m(\rho) = \lambda(n,m)^{\frac{1}{n}} \left(\int_0^\rho r^{2m+2n-1}\mu(r)dr\right)^{\frac{1}{2n}}, \qquad \rho \in [0,1]. \tag{2.7}$$

Obviously, there exists the inverse function for the function $\varphi_m(\rho)$ on $[0,1]$, which we will denote by $\phi_m(r)$. Introduce the operator

$$(u_m f)(r) = \frac{\sqrt{2n}}{\lambda(n,m)}\ \phi_m^{-m}(r)\ f(\phi_m(r)). \tag{2.8}$$

Proposition 2.1 *The operator u_m maps unitary the space $L_2((0,1), \mu(r)r^{2n-1}dr)$ onto the space $L_2((0,1), r^{2n-1}dr)$ in such a way that*

$$u_m(\lambda(n,m)r^m) = \sqrt{2n}, \qquad m \in \mathbb{Z}_+. \tag{2.9}$$

Proof. Consider the operator $u_m : L_2((0,1), \mu(r)r^{2n-1}dr) \longrightarrow L_2((0,1), r^{2n-1}dr)$ of the form

$$(u_m f)(r) = \psi_m(r) f(\phi_m(r)).$$

Here we assume that $\psi_m(r) \geq 0,\ r \in (0,1)$, and that $\phi_m(r)$ is bijective and continuous on $[0,1]$. Let $r = \varphi_m(y)$ be the inverse of ϕ_m on $(0,1)$. Since we assume that u_m is unitary, we have the following condition

$$[\psi_m(\varphi_m(\rho))]^2 \varphi_m^{2n-1}(\rho)\varphi_m'(\rho) = \rho^{2n-1}\mu(\rho). \tag{2.10}$$

Now condition (2.9) implies

$$\sqrt{2n} = \psi_m(r)\lambda(n,m)\phi_m^m(r), \tag{2.11}$$

or

$$2n = [\psi_m(\varphi_m(\rho))]^2 \lambda(n,m)^2 \rho^{2m}.$$

Combining this with (2.10) we have

$$2n\, \varphi_m^{2n-1}(\rho)\varphi_m'(\rho) = \lambda^2(n,m)\rho^{2n+2m-1}\mu(\rho),$$

or

$$\varphi_m^{2n}(\rho) = \lambda^2(n,m)\int_0^\rho r^{2n+2m-1}\mu(r)dr$$

which gives (2.7).

Finally calculating $\psi_m(r)$ from (2.11) we arrive to (2.8). $\square$

Introduce the unitary operator

$$U_2 : l_2(L_2((0,1), \mu(r)r^{2n-1}dr)) \to l_2(L_2((0,1), r^{2n-1}dr)) \equiv L_2((0,1), r^{2n-1}dr) \otimes l_2,$$

where

$$U_2 : \{c_{\alpha,\beta}(r)\} \longrightarrow \{(u_{|\alpha|+|\beta|}c_{\alpha,\beta})(r)\}.$$

Then, the space $\mathcal{A}^2_{2,\mu} = U_2(\mathcal{A}^2_{1,\mu})$ coincides with the space of all sequences $b_{\alpha,\beta}$ for which

$$b_{\alpha,\beta} = \begin{cases} \sqrt{2n}\ b_\alpha, & \text{for} \quad |\beta| = 0 \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \sum_{|\alpha|=0}^{\infty} |b_\alpha|^2 < \infty.$$

Let $l_0(r) = \sqrt{2n}$; we have $l_0(r) \in L_2((0,1), r^{2n-1}dr)$ and $\|l_0\|_{L_2((0,1),r^{2n-1}dr)} = 1$. Denote by L_0 the one-dimensional subspace of $L_2((0,1), r^{2n-1}dr)$ generated by $l_0(r)$. The orthogonal projection P_0 of $L_2((0,1), r^{2n-1}dr)$ onto L_0 has obviously the form

$$(P_0 f)(r) = \langle f, l_0\rangle\, l_0 = \sqrt{2n}\int_0^1 f(\rho)\,\sqrt{2n}\,\rho^{2n-1}\,d\rho. \tag{2.12}$$

Denote by l_2^+ the subspace of l_2 consisting of all sequences $\{b_{\alpha,\beta}\}$, such that $b_{\alpha,\beta} = 0$ for all β with $|\beta| > 0$. And let p^+ be the orthogonal projections of l_2 onto l_2^+, then $p^+ = \chi_+(\alpha,\beta)I$, where $\chi_+(\alpha,\beta) = 1$, if $|\beta| = 0$ and $\chi_+(\alpha,\beta) = 0$, if $|\beta| \neq 0$.

Observe that $\mathcal{A}^2_{2,\mu} = L_0 \otimes l_2^+$, and the orthogonal projection B_2 of $l_2(L_2((0,1), r^{2n-1}dr)) \equiv L_2((0,1), r^{2n-1}dr) \otimes l_2$ onto $\mathcal{A}^2_{2,\mu}$ has the form

$$B_2 = P_0 \otimes p^+.$$

Thus we arrive at the following theorem.

Theorem 2.2 *The unitary operator $U = U_2 U_1$ gives an isometric isomorphism of the space $L_2^\mu(\mathbb{B}^n)$ onto $l_2(L_2((0,1), r^{2n-1}dr)) \equiv L_2((0,1), r^{2n-1}dr) \otimes l_2$ under which*

1. *The Bergman space $\mathcal{A}^2_\mu(\mathbb{B}^n)$ is mapped onto $L_0 \otimes l_2^+$,*

$$U : \mathcal{A}^2_\mu(\mathbb{B}^n) \longrightarrow L_0 \otimes l_2^+,$$

where L_0 is the one-dimensional subspace of $L_2((0,1), r^{2n-1}dr)$, generated by the function $l_0(r) = \sqrt{2n}$.

2. *The Bergman projection $B^\mu_{\mathbb{B}^n}$ is unitary equivalent to*

$$U B^\mu_{\mathbb{B}^n} U^{-1} = P_0 \otimes p^+,$$

where P_0 is the one-dimensional projection (2.12) of $L_2((0,1), r^{2n-1}dr)$ onto L_0.

Introduce the operator

$$R_0 : l_2^+ \longrightarrow L_2((0,1), r^{2n-1}dr) \otimes l_2$$

by the rule

$$R_0 : \{c_{\alpha,\beta}\} \longrightarrow l_0(r)\ \{\chi_+(\alpha,\beta)\ c_{\alpha,\beta}\},$$

that is, we extend a sequence $\{c_{\alpha,\beta}\} \in l_2^+$ to all of l_2 putting zero values on $l_2 \ominus l_2^+$, and then multiply this sequence by $l_0(r)$.

The mapping R_0 is obviously an isometric embedding, and the image of R_0 coincides with the space $\mathcal{A}^2_{2,\mu}$. The adjoint operator

$$R_0^* : L_2((0,1), r^{2n-1}dr) \otimes l_2 \to l_2^+$$

is given by

$$R_0^* : \{c_{\alpha,\beta}(r)\} \longrightarrow \Big\{\chi_+(\alpha,\beta)\ \int_0^1 c_{\alpha,\beta}(\rho)\ \sqrt{2n}\ \rho^{2n-1}\ d\rho\Big\},$$

and

$$\begin{aligned} R_0^* R_0 = I &\quad : \quad l_2^+ \to l_2^+, \\ R_0 R_0^* = B_2 &\quad : \quad L_2((0,1), r^{2n-1}dr) \otimes l_2 \to \mathcal{A}^2_{2,\mu} = L_0 \otimes l_2^+ \end{aligned}$$

Now the operator $R = R_0^* U$ maps the space $L_2^\mu(\mathbb{B}^n)$ onto l_2^+, and its restriction

$$R|_{\mathcal{A}^2_\mu(\mathbb{B}^n)} : \mathcal{A}^2_\mu(\mathbb{B}^n) \longrightarrow l_2^+$$

is an isometric isomorphism. The adjoint operator is given by

$$R^* = U^* R_0 : l_2 \longrightarrow \mathcal{A}^2_\mu(\mathbb{B}^n) \subset L_2^\mu(\mathbb{B}^n),$$

and its restriction $R^*|_{l_2^+}$ is an isometric isomorphism of l_2^+ onto $\mathcal{A}^2_\mu(\mathbb{B}^n)$.

Remark 2.3 We have

$$\begin{aligned} R\,R^* = I &\quad : \quad l_2^+ \longrightarrow l_2^+ \\ R^* R = B^\mu_{\mathbb{B}^n} &\quad : \quad L_2^\mu(\mathbb{B}^n) \longrightarrow \mathcal{A}^2_\mu(\mathbb{B}^n). \end{aligned}$$

Theorem 2.4 *The isometric isomorphism*

$$R^* = U^* R_0 : l_2^+ \longrightarrow \mathcal{A}^2_\mu(\mathbb{B}^n)$$

is given by

$$R^* : \{c_{\alpha,\beta}\} \longmapsto \sum_{|\alpha|=0}^{\infty} \lambda(n, |\alpha|)\; c_{\alpha,0}\; r^{|\alpha|}\; e_{\alpha,0}(\omega).$$

Proof. Let $\{c_{\alpha,\beta}\} \in l_2^+$, we have

$$\begin{aligned} R^* = U_1^* U_2^* R_0 \quad : \quad & \{c_{\alpha,\beta}\} \longmapsto U_1^* U_2^*(\{\sqrt{2n}\; c_{\alpha,\beta}\}) = U_1^*(\{\sqrt{2n}\; \lambda(n,|\alpha|)\; c_{\alpha,\beta}\; r^{|\alpha|}\}) \\ = \; & \sum_{|\alpha|=0}^{\infty} \lambda(n,|\alpha|)\; c_{\alpha,0}\; r^{|\alpha|} e_{\alpha,0}(\omega). \end{aligned}$$

□

Corollary 2.5 *The inverse isomorphism*

$$R : \mathcal{A}^2_\mu(\mathbb{B}^n) \longrightarrow l_2^+$$

is given by

$$R : \varphi(z) \ \longmapsto \{c_{\alpha,\beta}\},$$

where

$$c_{\alpha,0} = \langle \varphi, \widetilde{e}^\mu_\alpha \rangle = \lambda(n,|\alpha|)\ d_{n,\alpha} \int_{\mathbb{B}^n} \varphi(z)\ \overline{z}^\alpha\ d\nu(z), \qquad |\alpha| \in \mathbb{Z}_+,$$

and $\widetilde{e}^\mu_\alpha(z)$, $|\alpha| \in \mathbb{Z}_+$, *are the elements of the standard ortonormal base in* $\mathcal{A}^2_\mu(\mathbb{B}^n)$*; i.e.,*

$$\widetilde{e}^\mu_\alpha(z) = l_{n,\alpha}\ z^\alpha,$$

with

$$l_{n,\alpha} = \left(\int_{\mathbb{B}^n} z^\alpha \overline{z}^\alpha \mu(|z|) dxdy \right)^{-\frac{1}{2}} = d_{n,\alpha}\, \lambda(n,|\alpha|).$$

For $\varphi \in L_2(S^{2n-1})$ let $\varphi(\omega) = \sum_{|\alpha|+|\beta|=0}^{\infty} b_{\alpha,\beta}\ e_{\alpha,\beta}(\omega)$ be its decomposition in $L_2(S^{2n-1})$. Let also $\mathcal{F}$ be the (unitary discrete Fourier) transform

$$\mathcal{F} : \varphi \longrightarrow \{b_{\alpha,\beta}\} \in l_2, \quad \|\varphi\|_{L_2(S^{2n-1})} = \|\{b_{\alpha,\beta}\}\|_{l_2}.$$

Introduce the operator $\widetilde{R} : L_2^\mu(\mathbb{B}^n) \to L_2(S^{2n-1})$ as follows,

$$\widetilde{R} = \mathcal{F}^{-1} R.$$

Corollary 2.6 *We have the following isometric isomorphisms between the Bergman* $\mathcal{A}^2_\mu(\mathbb{B}^n)$ *and the Hardy* $H^2(\mathbb{B}^n)$ *spaces:*

$$\begin{aligned} \widetilde{R}|_{\mathcal{A}^2_\mu(\mathbb{B}^n)} &: \ \mathcal{A}^2_\mu(\mathbb{B}^n) \longrightarrow H^2(\mathbb{B}^n), \\ \widetilde{R}^*|_{H^2(\mathbb{B}^n)} &: \ H^2(\mathbb{B}^n) \longrightarrow \mathcal{A}^2_\mu(\mathbb{B}^n). \end{aligned}$$

The operators $\widetilde{R}$ *and* $\widetilde{R}^*$ *provide the following decomposition of the Bergman* $B^\mu_{\mathbb{B}^n}$ *and the Szegö* $P_{S^{2n-1}}$ *projections:*

$$\begin{aligned} \widetilde{R}^* \widetilde{R} = B^\mu_{\mathbb{B}^n} &: \ L_2^\mu(\mathbb{B}^n) \longrightarrow \mathcal{A}^2_\mu(\mathbb{B}^n), \\ \widetilde{R}\, \widetilde{R}^* = P_{S^{2n-1}} &: \ L_2(S^{2n-1}) \longrightarrow H^2(\mathbb{B}^n), \end{aligned}$$

Another connection between the Bergman and the Hardy spaces, as well as between the corresponding projections is given by the following

Theorem 2.7 *The unitary operator $V = (I \otimes \mathcal{F}^{-1})U_2(I \otimes \mathcal{F})$ gives an isometric isomorphism of the spaces $L_2^{\mu}(\mathbb{B}^n)$ and $L_2(\mathbb{B}^n)$ under which*

1. *the Bergman $\mathcal{A}_{\mu}^2(\mathbb{B}^n)$ and the Hardy $H^2(\mathbb{B}^n)$ spaces are connected by the formula*

$$V(\mathcal{A}_{\mu}^2(\mathbb{B}^n)) = L_0 \otimes H^2(\mathbb{B}^n),$$

2. *the Bergman $B_{\mathbb{B}^n}^{\mu}$ and the Szegö $P_{S^{2n-1}}$ projections are connected by the formula*

$$V\, B_{\mathbb{B}^n}^{\mu} V^{-1} = P_0 \otimes P_{S^{2n-1}},$$

where P_0 is the one-dimensional projection (2.12) of $L_2([0,1), r^{2n-1}dr)$ onto one-dimensional space L_0 generated by $l_0(r) = \sqrt{2n} \in L_2([0,1), r^{2n-1}dr)$.

3 Toeplitz operators with radial symbols

We study here the Toeplitz operators

$$T_a = B_{\mathbb{B}^n}^{\mu} a : \varphi \in \mathcal{A}_{\mu}^2(\mathbb{B}^n) \longmapsto B_{\mathbb{B}^n}^{\mu} a\varphi \in \mathcal{A}_{\mu}^2(\mathbb{B}^n)$$

with radial symbols $a = a(r)$.

Theorem 3.1 *Let $a = a(r)$ be a measurable function on the segment $[0,1]$. Then the Toeplitz operator T_a acting on $\mathcal{A}_{\mu}^2(\mathbb{B}^n)$ is unitary equivalent to the multiplication operator $\gamma_{a,\mu}I$ acting on l_2^+. The sequence $\gamma_{a,\mu} = \{\gamma_{a,\mu}(|\alpha|)\}$ is given by*

$$\gamma_{a,\mu}(m) = \frac{1}{2}\lambda^2(n,m) \int_0^1 a(\sqrt{r})\ \mu(\sqrt{r})\ r^{m+n-1}\, dr, \quad m \in \mathbb{Z}_+. \tag{3.1}$$

Proof. The operator T_a is unitary equivalent to the operator

$$\begin{aligned} R\, T_a\, R^* &= R\, B_{\mathbb{B}^n}^{\mu} a B_{\mathbb{B}^n}^{\mu} R^* = R(R^*R)a(R^*R)R^* = (RR^*)RaR^*(RR^*) = RaR^* \\ &= R_0^* U_2\, U_1\, a(r)\, U_1^{-1} U_2^{-1} R_0 = R_0^* U_2 \{a(r)\} U_2^{-1} R_0 \\ &= R_0^* \{\chi_+(\alpha,\beta)\ a(\phi_{|\alpha|}(r))\} R_0. \end{aligned}$$

Further, let $\{c_{\alpha,\beta}\}$ be a sequence from l_2^+. Then

$$\begin{aligned} R_0^*\{\chi_+(\alpha,\beta)\; a(\phi_{|\alpha|}(r))\}R_0\{c_{\alpha,\beta}\} &= \left\{\int_0^1 a(\phi_{|\alpha|}(r))\; 2n\; c_{\alpha,\beta}\; r^{2n-1}\; dr\right\} \\ &= \{\gamma_{a,\mu}(|\alpha|)\; c_{\alpha,\beta}\}. \end{aligned}$$

Here we use that

$$\int_0^1 a(\phi_m(r))\; 2n\; r^{2n-1}\; dr = \int_0^1 a(y)\; d\varphi_m^{2n}(y) = \lambda^2(n,m)\int_0^1 a(y)\; y^{2m+2n-1}\; \mu(y)\; dy.$$

□

Corollary 3.2 *The Toeplitz operator T_a with measurable radial symbol $a = a(r)$ is bounded on $\mathcal{A}^2_\mu(\mathbb{B}^n)$ if and only if*

$$\sup_{m\in\mathbb{Z}_+} |\gamma_{a,\mu}(m)| < \infty.$$

Moreover,

$$\|T_a\| = \sup_{m\in\mathbb{Z}_+} |\gamma_{a,\mu}(m)|. \tag{3.2}$$

The Toeplitz operator T_a is compact if and only if

$$\lim_{m\to\infty} \gamma_{a,\mu}(m) = 0.$$

The spectrum of the bounded Toeplitz operator T_a is given by

$$\mathrm{sp}\, T_a = \overline{\{\gamma_{a,\mu}(m) :\; m \in \mathbb{Z}_+\}},$$

and its essential spectrum ess-sp T_a *coincides with the set of all limit points of the sequence* $\{\gamma_{a,\mu}(m)\}_{m\in\mathbb{Z}_+}$.

Recall now the essential ingredients of the Berezin's theory (see, for example, [2, 3, 4]). Let H be a Hilbert space, and $\{\varphi_g\}_{g\in G}$ be a subset of elements of H parameterized by elements g of some set G with a measure $d\mu$. Then $\{\varphi_g\}_{g\in G}$ is a *system of coherent states* if for all $\varphi \in H$

$$\|\varphi\|^2 = (\varphi,\varphi) = \int_G |(\varphi,\varphi_g)|^2 d\mu,$$

or, equivalently, if for all $\varphi_1, \varphi_2 \in H$

$$(\varphi_1,\varphi_2) = \int_G (\varphi_1,\varphi_g)\overline{(\varphi_2,\varphi_g)} d\mu. \tag{3.3}$$

Define an isomorphic inclusion $V : H \longrightarrow L_2(G)$ by the rule

$$V : \varphi \in H \longmapsto f = f(g) = (\varphi, \varphi_g) \in L_2(G).$$

By (3.3) we have $(\varphi_1, \varphi_2) = \langle f_1, f_2 \rangle$, where $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ are the scalar products on H and $L_2(G)$, respectively, and $f_h(g) = \overline{f_g(h)}$.

Let $H_2(G) = V(H) \subset L_2(G)$. A function $f \in L_2(G)$ is an element of $H_2(G)$ if and only if for all $h \in G$

$$\langle f, f_h \rangle = f(h).$$

The operator

$$(Pf)(g) = \int_G (\varphi_t, \varphi_g) f(t) d\mu(t)$$

is the orthogonal projection of $L_2(G)$ onto $H_2(G)$.

Let $f_g(t) = V\varphi_g = (\varphi_g, \varphi_t)$, $g \in G$, be the image in $H_2(G)$ of the system of coherent states $\{\varphi_g\}_{g \in G}$, then

$$(Pf)(g) = \langle f, f_g \rangle = \int_G f(t) f_t(g) d\mu(t).$$

The function $a(g)$, $g \in G$, is called the *anti–Wick* (or *contravariant*) symbol of an operator $T : H \to H$ if

$$VTV^{-1}|_{H_2(G)} = Pa(g)P = Pa(g)I|_{H_2(G)} : H_2(G) \longrightarrow H_2(G),$$

or if the operator $VTV^{-1}|_{H_2(G)}$ is the Toeplitz operator

$$T_{a(g)} = Pa(g)I|_{H_2(G)} : H_2(G) \longrightarrow H_2(G)$$

with the symbol $a(g)$.

Given an operator $T : H \to H$, introduce the (Wick) function

$$\widetilde{a}(g, h) = \frac{(T\varphi_h, \varphi_g)}{(\varphi_h, \varphi_g)}, \qquad g,\, h \in G. \tag{3.4}$$

If the operator T has an anti-Wick symbol, that is $VTV^{-1} = T_{a(g)}$ for some function $a = a(g)$, then

$$\widetilde{a}(g, h) = \frac{\langle T_a f_h, f_g \rangle}{\langle f_h, f_g \rangle}, \qquad g,\, h \in G,$$

and

$$\begin{aligned}(T_a f)(g) &= \int_G a(t)f(t)f_t(g)d\mu(t) = \int_G a(t)f_t(g)d\mu(t)\int_G f(h)f_h(t)d\mu(h)\\ &= \int_G f(h)d\mu(h)\int_G a(t)f_t(g)f_h(t)d\mu(t)\\ &= \int_G f(h)d\mu(h)\,\frac{f_h(g)}{\langle f_h, f_g\rangle}\,\int_G a(t)f_h(t)\overline{f_g(t)}d\mu(t)\\ &= \int_G \widetilde{a}(g,h)f(h)f_h(g)d\mu(h). \qquad (3.5)\end{aligned}$$

Interchanging the integrals above, we understand them in a weak sense.

The restriction of the function $\widetilde{a}(g,h)$ onto the diagonal

$$\widetilde{a}(g) = \widetilde{a}(g,g) = \frac{(T\varphi_g, \varphi_g)}{(\varphi_g, \varphi_g)}, \qquad g \in G,$$

is called the *Wick* (or *covariant,* or *Berezin*) symbol of the operator $T : H \to H$, and the formula (3.5) gives the representation of the operator T_a in terms of the Wick symbol.

The Wick and anti–Wick symbols of an operator $T : H \to H$ are connected by the Berezin transform

$$\widetilde{a}(g) = \int_G a(t)\frac{(\varphi_g, \varphi_t)(\varphi_t, \varphi_g)}{(\varphi_g, \varphi_g)}d\mu(t) = \frac{\int_G a(t)|f_g(t)|^2 d\mu(t)}{\int_G |f_g(t)|^2 d\mu(t)}.$$

Recall that the Bergman kernel in the space $\mathcal{A}^2_\mu(\mathbb{B}^n)$ has the form

$$K(z,\overline{w}) = \sum_{|\alpha|=0}^{\infty} \widetilde{e}^{\mu}_{\alpha}(z)\,\overline{\widetilde{e}^{\mu}_{\alpha}(w)}.$$

The reproducing property

$$f(z) = (\mathbb{B}^{\mu}_{\mathbb{B}^n} f)(z) = \int_{\mathbb{B}^n} f(w)K(z,\overline{w})\mu(|w|)d\nu(w) = \langle f, \overline{K(z,\overline{w})}\rangle = \langle f, K(w,\overline{z})\rangle \qquad (3.6)$$

shows that the system of functions $k_w(z) = K(z,\overline{w})$, $w \in \mathbb{B}^n$, forms a system of coherent states in the space $\mathcal{A}^2_\mu(\mathbb{B}^n)$. That is, in our context, we have $G = \mathbb{B}^n$, $d\mu = \mu(|z|)dxdy$, $H = H_2(G) = \mathcal{A}^2_\mu(\mathbb{B}^n)$, $L_2(G) = L^{\mu}_2(\mathbb{B}^n)$, $\varphi_g = f_g = k_g$, where $g = w \in \mathbb{B}^n$.

Now the operator T having the anti-Wick symbol a is nothing but the Toeplitz operator T_a with symbol a.

Theorem 3.3 *Let T_a be the Toeplitz operator with a radial symbol $a = a(r)$. Then the corresponding Wick function (3.4) has the form*

$$\widetilde{a}(z,w) = K^{-1}(z,\overline{w}) \sum_{|\alpha|=0}^{\infty} \widetilde{e}^{\mu}_{\alpha}(z)\, \overline{\widetilde{e}^{\mu}_{\alpha}(w)}\ \gamma_{a,\mu}(|\alpha|). \tag{3.7}$$

PROOF. Calculate

$$\begin{aligned}
\widetilde{a}(z,w) &= \frac{\langle ak_w, k_z\rangle}{\langle k_w, k_z\rangle} = k_w^{-1}(z)\ \langle ak_w, k_z\rangle \\
&= K^{-1}(z,\overline{w}) \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} \overline{\widetilde{e}^{\mu}_{\alpha}(w)}\ \widetilde{e}^{\mu}_{\beta}(z)\ \langle a\widetilde{e}^{\mu}_{\alpha}, \widetilde{e}^{\mu}_{\beta}\rangle \\
&= K^{-1}(z,\overline{w}) \sum_{|\alpha|=0}^{\infty} \widetilde{e}^{\mu}_{\alpha}(z)\ \overline{\widetilde{e}^{\mu}_{\alpha}(w)}\ \langle a\widetilde{e}^{\mu}_{\alpha}, \widetilde{e}^{\mu}_{\alpha}\rangle \\
&= K^{-1}(z,\overline{w}) \sum_{|\alpha|=0}^{\infty} \widetilde{e}^{\mu}_{\alpha}(z)\ \overline{\widetilde{e}^{\mu}_{\alpha}(w)}\ \gamma_{a}(|\alpha|).
\end{aligned}$$

$\square$

The Wick function (3.7) depends in fact on z and $\overline{w}$, thus we will write $\widetilde{a}(z,\overline{w})$ in what follows.

Denote by L^{μ}_{α} the one-dimensional subspace of $\mathcal{A}^2_{\mu}(\mathbb{B}^n)$ generated by the base element $\widetilde{e}^{\mu}_{\alpha}(z)$, $|\alpha| \in Z_+$. Then the one-dimensional projection P^{μ}_{α} of $\mathcal{A}^2_{\mu}(\mathbb{B}^n)$ onto L^{μ}_{α} has obviously the form

$$P^{\mu}_{\alpha} f = \langle f, \widetilde{e}^{\mu}_{\alpha}\rangle\ \widetilde{e}^{\mu}_{\alpha} = \widetilde{e}^{\mu}_{\alpha}(z) \int_{\mathbb{B}^n} f(w)\, \overline{\widetilde{e}^{\mu}_{\alpha}(w)}\, \mu(|w|)\, d\nu(w).$$

Corollary 3.4 *Let T_a be a bounded Toeplitz operator having radial symbol $a(r)$. Then the writing of the operator T_a in the form of operator with the Wick symbol (3.5) gives the spectral decomposition of the operator T_a,*

$$T_a = \sum_{|\alpha|=0}^{\infty} \gamma_{a,\mu}(|\alpha|) P^{\mu}_{\alpha}.$$

The eigenvalues $\gamma_{a,\mu}(|\alpha|)$ depend only on $|\alpha|$. Collecting the terms with the same $|\alpha|$ and using the formula

$$(z \cdot \overline{w})^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} z^\alpha \overline{w}^\alpha$$

we obtain

$$\widetilde{a}(z, \overline{w}) = K^{-1}(z, \overline{w}) \sum_{m=0}^{\infty} l(n,m) \ \gamma_{a,\mu}(m) \ (z \cdot \overline{w})^m,$$

where

$$l(n,m) = |S^{2n-1}|^{-1} \frac{(m+n-1)!}{m!(n-1)!} \ \lambda^2(n,m),$$

and

$$T_a = \sum_{m=0}^{\infty} \gamma_{a,\mu}(m) P^{\mu}_{(m)},$$

where

$$(P^{\mu}_{(m)} f)(z) = l(n,m) \int_{\mathbb{B}^n} f(w)(z \cdot \overline{w})^m \ \mu(|w|) \ d\nu(w)$$

is the orthogonal projection of $\mathcal{A}^2_\mu(\mathbb{B}^n)$ onto the subspace generated by all elements $\widetilde{e}^{\mu}_{\alpha}$ with $|\alpha| = m$, $m \in \mathbb{Z}_+$.

Corollary 3.5 *Let T_a be a bounded Toeplitz operator with radial symbol $a(r)$. Then the Wick symbol of the operator T_a is radial as well, and is given by the formula*

$$\widetilde{a}(r) = K^{-1}(z, \overline{z}) \sum_{m=0}^{\infty} l(n,m) \, \gamma_{a,\mu}(m) \, r^{2m},$$

where $K(z, \overline{z}) = \sum_{m=0}^{\infty} l(n,m) \, r^{2m}$.

In terms of Wick symbols the composition formula for Toeplitz operators is quite transparent:

Corollary 3.6 *Let T_a, T_b be the Toeplitz operators with the Wick symbols*

$$\begin{aligned}
\widetilde{a}(z, \overline{w}) &= K^{-1}(z, \overline{w}) \ \sum_{m=0}^{\infty} l(n,m) \ \gamma_{a,\mu}(m) \ (z \cdot \overline{w})^m, \\
\widetilde{b}(z, \overline{w}) &= K^{-1}(z, \overline{w}) \ \sum_{m=0}^{\infty} l(n,m) \ \gamma_{b,\mu}(m) \ (z \cdot \overline{w})^m,
\end{aligned}$$

respectively. Then the Wick symbol $\widetilde{c}(z,\overline{w})$ of the composition $T = T_a T_b$ is given by

$$\widetilde{c}(z,\overline{w}) = K^{-1}(z,\overline{w}) \sum_{m=0}^{\infty} l(n,m)\ \gamma_{a,\mu}(m)\gamma_{b,\mu}(m)\ (z\cdot\overline{w})^m.$$

The above corollary gives rise to a natural question: *when is the product of two Toeplitz operators a Toeplitz operator as well.* In the rest of this section we give a particular answer to this question considering, for the sake of simplicity, the case of a weightless space, i.e., the classical Bergman space $\mathcal{A}^2(\mathbb{B}^n) \equiv \mathcal{A}^2_1(\mathbb{B}^n)$ and the Toeplitz operators T_a acting on it.

As above, to each Toeplitz operator with radial symbol $a(r) \in L_1((0,1), r^{2n-1}dr)$ (perhaps unbounded, but in any case densely defined) there is assigned the operator (on l_2^+) of multiplication by the sequence

$$\gamma_a(m) = (m+n)\int_0^1 a(\sqrt{r})\ r^{m+n-1}\ dr = (m+n)\int_0^{\infty} a(\sqrt{e^{-t}})\ e^{-nt}\ e^{-mt}\ dt, \quad m = 0,1\ldots.$$

where obviously $a(\sqrt{e^{-t}})e^{-nt} \in L_1(\mathbb{R}_+)$. Now given two Toeplitz operators T_{a_1}, T_{a_2}, we will find the sufficient conditions under which there exists a radial function $a(r)$ such that $T_a = T_{a_1}T_{a_2}$, or, equivalently

$$\gamma_a(m) = \gamma_{a_1}(m)\gamma_{a_2}(m), \qquad m = 0,1,\ldots. \tag{3.8}$$

Let

$$A(t) = \begin{cases} a(\sqrt{e^{-t}})e^{-nt} & t > 0, \\ 0, & t \le 0. \end{cases}$$

The formal construction (inverse Fourier-Laplace transform)

$$(F^{-1}A)(z) \equiv \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} A(t)e^{izt}dt, \quad z \in \Pi \cup \mathbb{R}, \tag{3.9}$$

defines a holomorphic function in the upper half-plane Π ($\subset \mathbb{C}$) which coincides on the real axis with the inverse Fourier transform $(F^{-1}A)(\xi)$ of the function $A(t)$. Thus, in the above notation

$$\gamma_a(m) = (m+n)(F^{-1}A)(im) \tag{3.10}$$

Let $A_k(t)$ correspond to $a_k(\sqrt{r})$ as above. The convolution

$$A^0(t) = \int_{\mathbb{R}} A_1(t-s)A_2(s)ds \equiv \int_0^t A_1(t-s)A_2(s)ds$$

is supported on the positive real half-line, belongs to $L_1(\mathbb{R}_+)$, and its inverse Fourier transform is given by

$$(F^{-1}A^0)(z) = \sqrt{2\pi}(F^{-1}A_1)(z)\,(F^{-1}A_2)(z), \quad z \in \Pi \cup \mathbb{R},$$

where the expressions $(F^{-1}A^0)(z)$, $(F^{-1}A_k)(z)$, for $z \in \Pi$, are understood as integrals (3.9). By (3.10) the equality (3.8) is equivalent to

$$(F^{-1}A)(im) = \sqrt{2\pi}(m+n)(F^{-1}A_1)(im)\,(F^{-1}A_2)(im), \qquad m = 0, 1, \ldots. \tag{3.11}$$

Let the function

$$\sqrt{2\pi}(n - i\xi)(F^{-1}A_1)(\xi)\,(F^{-1}A_2)(\xi), \quad \xi \in \mathbb{R}, \tag{3.12}$$

belong to the Wiener ring W_0 of the (inverse) Fourier transforms of sumable functions (see [10] for numerous sufficient conditions for a function from $C_0(\mathbb{R})$ to be in W_0). Then there exists a function $A(t) \in L_1(\mathbb{R})$ whose Fourier transform coincides with (3.12). Moreover, we claim that the function $A(t)$ is supported on $\mathbb{R}_+$. It follows from the fact that $A(t)$ (as a regular functional on $C_0^\infty(\mathbb{R})$) coincides in the distributional sense with the functional $\sqrt{2\pi}(nI + \frac{d}{dx})A^0(x)$, which has the (distributional) support on $\mathbb{R}_+$. Thus, for that function $A(t)$ we have

$$F^{-1}A(z) = (n - iz)(F^{-1}A_1)(z)\,(F^{-1}A_2)(z), \quad z \in \Pi \cup \mathbb{R}.$$

In particular, the equality (3.11) is valid for our $A(t)$. Now if we set

$$a(\sqrt{r}) = A\,(\ln 1/r)$$

then obviously $a(\sqrt{r}) \in L_1((0,1), r^{n-1}dr)$. Finally, the function $a(r)$ defines the Toeplitz operator T_a for which

$$T_a = T_{a_1}T_{a_2}. \tag{3.13}$$

We summarize the above in the following theorem.

Theorem 3.7 *Let T_{a_1}, T_{a_2} be Toeplitz operators, not necessarily bounded, acting on the Bergman space $\mathcal{A}^2(\mathbb{B}^n)$. Let further $A_1(t) = a_1(\sqrt{e^{-t}})e^{-nt}$, $A_2(t) = a_2(\sqrt{e^{-t}})e^{-nt}$ as above. If the function (3.12) belongs to W_0, then there exists a Toeplitz operator T_a with the radial symbol $a(r)$ such that the equality (3.13) is satisfied.*

4 Weightless case

Here we continue consideration of the Toeplitz operators

$$T_a = B_{\mathbb{B}^n} a : \varphi \in \mathcal{A}^2(\mathbb{B}^n) \longmapsto B_{\mathbb{B}^n} a\varphi \in \mathcal{A}^2(\mathbb{B}^n)$$

with radial symbols $a = a(r)$, acting on the classical Bergman space $\mathcal{A}^2(\mathbb{B}^n)$; i.e., now $\mu(r) \equiv 1$. It turns out that this case is quite similar to the one-dimensional case studied in [6]. Therefore in this section we simply collect the corresponding results, omitting the proofs.

Remark 4.1 The sequence (3.1) for the weightless case is given by

$$\gamma_a(m) = \int_0^1 a(r^{\frac{1}{2(m+n)}})\, dr = (m+n)\int_0^1 a(\sqrt{r})\, r^{m+n-1}\, dr \quad m = 0, 1, \ldots, \tag{4.1}$$

and Theorem 3.1 has an obvious reformulation for this particular case.

Example 1. The general form of a radial function which is harmonic in $\mathbb{B}^n \setminus \{0\}$ is as follows,

$$h(r) = c_1 r^{2-2n} + c_2, \qquad c_1, c_2 \in \mathbb{C}.$$

We have

$$\gamma_h(m) = \int_0^1 h(r^{\frac{1}{2(m+n)}})\, dr = \frac{m+n}{m+1}\, c_1 + c_2,$$

that is, the Toeplitz operator T_h is *bounded* on $\mathcal{A}^2(\mathbb{B}^n)$, and its *discrete* spectrum is given by

$$\operatorname{sp} T_h = \left\{ \frac{m+n}{m+1}\, c_1 + c_2 \right\}_{m\in\mathbb{Z}_+}.$$

The Toeplitz operator T_h is compact if and only if $c_2 = -c_1$.

From now on we will assume that $a(r) \in L_1((0,1), r^{2n-1}dr)$. We will use the auxiliary function $b(r) = a(\sqrt{r})$. Hence, we have the condition $b(r) \in L_1((0,1), r^{n-1}dr)$. Following [6] introduce for $b \in L_1((0,1), r^{n-1}dr)$ the function

$$B(s) = \int_s^1 b(u) u^{n-1} du.$$

Then, integrating by parts we have

$$\gamma_a(m) = m(m+n)\int_0^1 B(s) s^{m-1} ds.$$

Theorem 4.2 ([6]) *If the function $B(s)$ when $s \to 1$ has the form*

$$B(s) = O(1-s) \tag{4.2}$$

then

$$\sup_{m \in \mathbb{Z}_+} |\gamma_a(m)| < \infty.$$

If

$$B(s) = o(1-s) \tag{4.3}$$

then

$$\lim_{m \to \infty} \gamma_a(m) = 0.$$

Theorem 4.3 ([6]) *Let $b \in L_1((0,1), r^{n-1}dr)$ and $b(u) \geq 0$ almost everywhere. Then the conditions (4.2), (4.3) are also necessary for $\gamma_a \in l_\infty$, $\gamma_a \in c_0$ respectively.*

Example 2. Let $a(r) = r^{2-2n}(1-r^2)^{-\beta} \sin(1-r^2)^{-\alpha}$. Then due to result of [6],

$$B(v) = \frac{\cos(1-v)^{-\alpha}}{\alpha}(1-v)^{\alpha-\beta+1} + O((1-v)^{\alpha-\beta-2}). \tag{4.4}$$

Hence the Toeplitz operator T_a with the above symbol a is bounded on $\mathcal{A}^2(\mathbb{B}^n)$ for $\alpha \geq \beta$, and moreover is compact for $\alpha > \beta$. These properties depend only on the correlation between α and β and thus the Toeplitz operator can be bounded and even compact for both bounded and unbounded (near the boundary $S^{2n-1} = \partial\mathbb{B}^n$) symbols of the above type.

Example 3. Consider the following family of radial nonnegative symbols $a_\alpha(r) = r^{2-2n}(1-r^2)^{\alpha-1},\ \alpha > 0$. We have

$$B_\alpha(s) = \frac{2}{\alpha}s(1-s)^\alpha + \frac{2}{\alpha(\alpha+1)}(1-s)^{\alpha+1}.$$

By Theorem 4.2 the operator T_{a_α} is bounded if and only if $\alpha \geq 1$, and compact if and only if $\alpha > 1$. That is, in this scale unbounded symbols generate unbounded Toeplitz operators. Moreover, as it will follow from Corollary 4.4, to generate bounded or compact Toeplitz operator its unbounded symbol must necessarily have sufficiently sophisticated oscillating behaviour near the unit sphere $S^{2n-1} = \partial\mathbb{B}^n$.

For a nonnegative symbol $a(r)$ introduce the function

$$m_a(u) = \inf_{r \in [u,1)} a(r),$$

which is obviously always monotone.

Corollary 4.4 ([6]) *If* $\lim_{u\to 1} m_a(u) = +\infty$ *(which is equivalent to* $\lim_{r\to 1} a(r) = +\infty$*), then the Toeplitz operator* T_a *is unbounded.*

Theorem 4.5 ([6]) *Let* $b(u) \in L_1((0,1), u^{2n-1}du)$. *Then*

$$\lim_{m\to\infty} (\gamma_a(m+1) - \gamma_a(m)) = 0. \tag{4.5}$$

From this theorem it is follows that the set of all limit points of the sequence $\gamma_a(m)$ forms a closed connected subset of $\mathbb{C}$. In particular, the sequence $\gamma_a(m)$ can not have a finite or countable set of limit points.

Corollary 4.6 *The essential spectrum of a bounded Toeplitz operator with a radial symbol* $a(r) \in L_1((0,1), r^{2n-1}dr)$ *is always connected.*

If the l_∞ sequence $\gamma_a(m)$ does not have a limit, then the essential spectrum of the corresponding Toeplitz operator can be quite rich. The following examples are based on the results of [6].

Example 4. Unit circle and unit interval.

1. Let $a_p(r) = \alpha_p r^{2-2n}(\ln r^{-2})^{ip}$, $\alpha_p = \{\int_0^1 (\ln \frac{1}{s})^{ip} ds\}^{-1}$, $p > 0$. Then $\gamma_{a_p}(m) = \exp\{-ip\ln(m+1)\}$ and
$$\operatorname{sp} T_{a_p} = \text{ess-sp}\, T_{a_p} = S^1.$$
2. If $c_p = r^{2-2n}\operatorname{Im} \alpha_p(\ln r^{-2})^{ip}$, then $\gamma_{c_p}(m) = -\sin(p\ln(m+1))$ and
$$\operatorname{sp} T_{c_p} = \text{ess-sp}\, T_{c_p} = [-1,1].$$

Example 5. Square.

Let $a(r) = c_1(r) + ic_2(r)$, then $\gamma_a(m) = -(\sin\ln(m+1) + i\sin\sqrt{2}\ln(m+1))$. Then

$$\operatorname{sp} T_a = \text{ess-sp}\, T_a = [0,1] \times [0,1].$$

5 Power weight case

Let us consider a partial, but most important case, when the weight is given by

$$\mu(z) = (1-|z|^2)^{\lambda-1}, \quad \lambda > 0.$$

We denote $\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n) = \mathcal{A}^2_{\mu}(\mathbb{B}^n)$, $\gamma_{a,(\lambda)}(m) = \gamma_{a,\mu}(m)$ for $\mu(z) = (1-|z|^2)^{\lambda-1}$. We obviously have

$$\gamma_{a,(\lambda)}(m) = 2B^{-1}(m+n,\lambda)\int_0^1 a(\sqrt{r})\ (1-r)^{\lambda-1}\ r^{m+n+1}\ dr,$$

where $B(z,w)$ is the Beta function.

As we will see the Toeplitz operator with positive symbol $a(|z|)$, being bounded on a certain $\mathcal{A}^2_{(\lambda_0)}(\mathbb{B}^n)$ is automatically bounded on all $\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n)$, $\lambda > 0$.

Theorem 5.1 *Let there exist a constant $M > 0$ such that $\operatorname{Re} a(|z|) + M \geq 0$ or $\operatorname{Re} a(|z|) - M \leq 0$ and analogously for $\operatorname{Im} a(|z|)$. Then the Toeplitz operator T_a is bounded or unbounded on each $\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n)$, $\lambda > 0$ simultaneously.*

Proof. Obviously, we can assume that $a(|z|) \geq 0$. The general case will follow by usual arguments. Let $\beta(z,w)$ denotes the distance in the Bergman metric between the point z and w of the unit ball, and let $E(z,l) = \{w \in \mathbb{B}^n : \beta(z,w) < l\}$ be the open Bergman metric ball centered at z with radius l. Denote by $|E(z,l)|$ the measure of $E(z,l)$. By the results of [13] the following quantities are equivalent for any fixed $l > 0$:

$$Q^1_l(a) \equiv \sup_{z\in\mathbb{B}^n} |E(z,l)|^{-n-\lambda} \int_{E(z,l)} a(|w|)(1-|w|^2)^{\lambda-1} d\nu(w) \tag{5.1}$$

and

$$Q^2(a) \equiv \sup_{f\in\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n)} \|f\|^{-2}_{\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n)} \int_{\mathbb{B}^n} a(|z|)|f(z)|^2(1-|z|^2)^{\lambda-1} d\nu(z). \tag{5.2}$$

Moreover, analysis of the corresponding results in [13] shows that there exist constants $C_1,\ C_2$ depending only on l (not on $a(r)$) such that $C_1 Q^2(a) \leq Q^1_l(a) \leq C_2 Q^2(a)$. The quantity (5.2) is equal to

$$\sup_{m\in\mathbb{N}\cup\{0\}} |\gamma_{a,(\lambda)}(m)| \equiv \sup_{|\alpha|\in\mathbb{N}^n\cup\{0\}} \int_{\mathbb{B}^n} a(|z|)|\widetilde{e}^{(\lambda)}_{\alpha,0}(z)|^2(1-|z|^2)^{\lambda-1} d\nu(z), \tag{5.3}$$

because $\gamma_{a,(\lambda)}(|\alpha|) = \langle T_a \widetilde{e}^{(\lambda)}_{\alpha,0}, \widetilde{e}^{(\lambda)}_{\alpha,0}\rangle_{(\lambda)} = \langle a(|z|)\widetilde{e}^{(\lambda)}_{\alpha,0}, \widetilde{e}^{(\lambda)}_{\alpha,0}\rangle_{(\lambda)}$, where the scalar product is taken in the space $\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n)$, and $\widetilde{e}^{(\lambda)}_{\alpha,0}$ are the elements of the standard ortonormal basis for $\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n)$:

$$\widetilde{e}^{(\lambda)}_{\alpha,0}(z) = |\mathbb{B}^n|^{-\frac{1}{2}} \sqrt{\frac{\Gamma(|\alpha|+n+\lambda)}{\Gamma(n+\lambda)\alpha!}}\ z^{\alpha}.$$

On the other hand, also by [13] for each $l > 0$ there exists a constant $C = C(l)$ such that

$$C^{-1} \le |E(z,l)|(1-|w|^2) \le C \tag{5.4}$$

for all $z \in \mathbb{B}^n$ and $w \in E(z,l)$. That is why, the expression under supremum sign in (5.1) is comparable with $|E(z,l)|^{-n-1} \int_{E(z,l)} a(|w|)d\nu(w)$ and hence (5.1) is comparable to

$$\sup_{z\in\mathbb{B}^n} |E(z,l)|^{-n-1} \int_{E(z,l)} a(|w|)d\nu(w), \tag{5.5}$$

which does not depend on $\lambda > 0$. $\square$

Corollary 5.2 *If the Toeplitz operator T_a with symbol $a(|z|)$ as in the previous theorem is bounded on some $\mathcal{A}^2_{(\lambda_0)}(\mathbb{B}^n)$, then*

$$\int_0^1 r^{2n-1}a(r)(1-r^2)^{\lambda-1}dr < \infty$$

for all $\lambda > 0$.

This says that a positive symbol of a bounded Toeplitz operator (being unbounded) can not have a "bad" (say power-like growth) behaviour near the point $r = 1$.

As a corollary, we show that compactness of a positive Toeplitz operator does not depend on λ as well.

Theorem 5.3 *Let $a(|z|)$ be as in Theorem 5.1. The Toeplitz operator T_a is compact or is not compact on each $\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n)$ simultaneously.*

Proof. Let the Toeplitz operator be compact on a fixed $\mathcal{A}^2_{(\lambda_0)}(\mathbb{B}^n)$. Consider the operators with the truncated symbols T_{a_N}, where $a_N(r) = a(r)$ for $0 \le r \le 1 - 1/N$ and $a_N(r) = 0$, for $1-1/N < r \le 1$. Each T_{a_N} is compact on any $\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n),\ \lambda > 0$. Using (3.2) it is easily seen that $\|T_a - T_{a_N}\|_{(\lambda_0)} \to 0$, $N \to \infty$. Here $\|\cdot\|_{(\lambda)}$ denotes the operator norm, as an operator on $\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n)$. But analysis of the proof of the previous theorem shows that for each fixed λ there exist constants $C_1,\ C_2$ depending only on $\lambda_0,\ \lambda$ such that

$$C_1\|T_a - T_{a_N}\|_{(\lambda_0)} \le \|T_a - T_{a_N}\|_{(\lambda)} \le C_2\|T_a - T_{a_N}\|_{(\lambda_0)}.$$

Hence T_a is compact on $\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n)$ (as the norm limit of compact operators). $\square$

The results of Theorems 5.1, 5.3 coincide with the corresponding result of [14] for the unit disk. As we will see now these results fail to be true for symbols unbounded from the both sides. Thus admitting such symbols is an important qualitative step.

Example 6. There exists a symbol $a = a(|z|)$ such that

– The Toeplitz operator T_a is bounded on $\mathcal{A}^2_{(1)}(\mathbb{B}^n) \equiv \mathcal{A}^2(\mathbb{B}^n)$, but unbounded on $\mathcal{A}^2_{(2)}(\mathbb{B}^n)$.

– The operator T_a is compact $\mathcal{A}^2_{(1)}(\mathbb{B}^n)$, and bounded but not compact on $\mathcal{A}^2_{(2)}(\mathbb{B}^n)$.

– The operator T_a is compact on $\mathcal{A}^2_{(1)}(\mathbb{B}^n)$, but unbounded on $\mathcal{A}^2_{(2)}(\mathbb{B}^n)$.

To construct such a symbol we will use the characterization given in Theorem 3.2. We have,

$$\gamma_{a,(1)}(m) \equiv \gamma_a(m) = (m+n)\int_0^1 a(\sqrt{r})r^{m+n-1}dr, \tag{5.6}$$

$$\gamma_{a,(2)}(m) = (m+n)(m+n+1)\int_0^1 a(\sqrt{r})(1-r)r^{m+n-1}dr. \tag{5.7}$$

Thus,

$$\gamma_{a,(2)}(m) = (m+n+1)\gamma_{a,(1)}(m) - (m+n)\gamma_{a,(1)}(m+1) \tag{5.8}$$

and boundedness, compactness of T_a on $\mathcal{A}^2_{(2)}(\mathbb{B}^n)$ is uniquely determined by the behaviour of $\gamma_{a,(1)}(m)$ when $m \to \infty$. Let us show that the following situations can be realized

– $\sup_{m\in\mathbb{N}\cup\{0\}} |\gamma_{a,1}(m)| < \infty$, but $\gamma_{a,(2)}(m) \to \infty$ when $m \to \infty$.

– $\gamma_{a,1}(m) \to 0$ when $m \to \infty$ and $\sup_{m\in\mathbb{N}\cup\{0\}} |\gamma_{a,(2)}(m)| < \infty$, but $\gamma_{a,(2)}(m)$ does not tend to 0 when $m \to \infty$.

– $\gamma_{a,1}(m) \to 0$ when $m \to \infty$, but $\gamma_{a,(2)}(m) \to \infty$ when $m \to \infty$.

Obviously, these three situations are exactly equivalent to the previous ones. Let us examine $\gamma_{a,1}(m)$. Changing the variable we get

$$\gamma_{a,1}(m) = (m+n)\int_0^\infty a(\sqrt{e^{-y}})e^{-(m+n)y}dy = -i(i(m+n))\int_0^\infty a(\sqrt{e^{-y}})e^{i(i(m+n))y}dy$$

and we can formally consider the above expression (up to the constant $-i$) as the Fourier transform, multiplied by the variable and then calculated at the point $i(m+n)$, of the function supported on the positive half-axis. It will be correct if we assume that $b(y) \equiv a(\sqrt{e^{-y}})e^{-ny}$ belongs to $L_2(0,\infty)$, which is the same that $a(r) \in L_2((0,1), r^{4n-1}dr)$.

Denote

$$F(z) = (n-iz)\int_0^\infty b(y)e^{izy}dy, \quad f(z) = F(z)/(n-iz). \tag{5.9}$$

By the Paley-Wiener theorem ([5]) there exists a one-by-one correspondence between the square integrable functions on the real axis supported on the positive half-axis and the functions from the Hardy space $H^2(\Pi)$ over the upper complex half-plane Π, i.e., with the functions φ which are analytic in Π and such that $g_y(x) = |\varphi(x+iy)|$ is a square integrable function on the real axis ($x \in \mathbb{R}$) for each fixed y with uniformly bounded L_2 norms. The correspondence is given by the (inverse) Fourier transform and is

$$\varphi(z) = \int_0^\infty \psi(t)e^{izt}dt, \quad z \in \Pi,$$

where $\varphi \in H^2(\Pi)$ if and only if $\psi \in L_2(\mathbb{R})$ and $\psi(t) = 0$ for almost everywhere $t < 0$.

Now, let us return to (5.9). Having fixed a function $f(z) \in H^2\Pi)$, the corresponding function $b(y)$ will belong to $L_2(0,\infty)$ and the function $a(r) = r^{-2n}b(2\ln 1/r)$ will be from $L_2((0,1), r^{4n-1}dr)$ and hence we can use it to define the corresponding Toeplitz operator T_a. Moreover, if $\gamma_{a,(\lambda)}$ is the sequence which corresponds to that Toeplitz operator acting on $\mathcal{A}^2_{(\lambda)}(\mathbb{B}^n)$, then

$$\begin{aligned}
\gamma_{a,(1)}(m) &= F(im) = (m+n)f(im), \\
\gamma_{a,(2)}(m) &= (m+n)(m+n+1)[f(im) - f(i(m+1))].
\end{aligned}$$

Thus we have to construct functions $f \in H^2(\Pi)$ which realize the all the situations mentioned above.

Let us introduce the function of one complex variable

$$f(z) = \exp\left\{\frac{i}{5\pi}\ln^2(z+ni)\right\}\ln^{-\nu}(z+ni), \quad \nu \geq 0. \tag{5.10}$$

Here we chouse the single-valued branch of the multi-valued analytic functions, being analytic function in $\frac{3\pi}{2} \leq \arg(z+ni) < \frac{7\pi}{2}$. This function is analytic in the upper half-plane and, moreover, belongs to $H^2(\Pi)$ since for $y \geq 0$,

$$|f(x+iy)| \leq C(x^2+(1+y)^2)^{-\frac{1}{5\pi}\arg(x+i(n+y))}, \quad \arg(x+i(n+y)) \in [2\pi, 3\pi].$$

Examine the function $f(z)$ at the points im. We have

$$f(im) = \exp\{-\frac{5\pi i}{4}\}\exp\{\frac{i}{5\pi}\ln^2(m+n)\}\frac{[\ln(m+n)+i\frac{5\pi}{2}]^{-\nu}}{m+n}.$$

The first conclusion is:

$$(m+n)f(im) \longrightarrow 0, \quad m\to\infty \quad \text{for all} \quad \nu > 0,$$

and

$$(m+n)|f(im)| \le C < \infty \qquad \text{for} \qquad \nu = 0.$$

Consider next

$$(m+n)(m+n+1)[f(im) - f(i(m+1))]e^{\frac{5\pi i}{4}} = \frac{\exp\{\frac{i}{5\pi}\ln^2(m+n)\}}{[\ln(m+n) + i\frac{5\pi}{2}]^\nu} +$$

$$(m+n)\exp\{\frac{i}{5\pi}\ln^2(m+n+1)\}\Big([\ln(m+n) + i\frac{5\pi}{2}]^{-\nu} - [\ln(m+n+1) + i\frac{5\pi}{2}]^{-\nu}\Big) +$$

$$(m+n)\frac{\exp\{\frac{i}{5\pi}\ln^2(m+n)\} - \exp\{\frac{i}{5\pi}\ln^2(m+n+1)\}}{[\ln(m+n) + i\frac{5\pi}{2}]^\nu} \equiv K(m) + L(m) + M(m).$$

Obviously, $K(m)$, $L(m)$ tend to 0 when $\nu > 0$, and are bounded when $\nu = 0$. Now, $M(k)$ is equivalent, when $m \to \infty$, to

$$M(m) \approx -\frac{i}{5\pi}\frac{\exp\{\frac{i}{5\pi}\ln^2(m+n)\}\ln(m+n)}{[\ln(m+n) + i\frac{5\pi}{2}]^\nu}. \qquad (5.11)$$

Thus, the sequence $M(m)$ is unbounded for $0 \le \nu < 1$, is bounded for $\nu \ge 1$ and tends to 0 (when $k \to \infty$) if and only if $\nu > 1$.

Finally, the cases $\nu = 0$, $\nu = 1$, and $0 < \nu < 1$ realize all three situations.

Remark 5.4 The results of this section can be obtained in the same way for another (comparable) type of a weight, i.e., for the weight $\mu(|z|) = \left(\ln \frac{1}{|z|^2}\right)^{\lambda-1}$, $\lambda > 0$. In this case

$$\gamma_{a,(\lambda)}(m) = 2(m+n)^\lambda \Gamma^{-1}(\lambda) \int_0^1 a(\sqrt{r}) \ \ln^{\lambda-1} 1/r \ r^{m+n-1} \, dr,$$

where $\Gamma(z)$ is the Gamma function.

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